J. Appl. Math. & Informatics Vol. **29**(2011), No. 1 - 2, pp. 119 - 133
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A FULL-NEWTON STEP INFEASIBLE INTERIOR-POINT ALGORITHM FOR LINEAR PROGRAMMING BASED ON A SELF-REGULAR PROXIMITY†

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Abstract. This paper proposes an infeasible interior-point algorithm with full-Newton step for linear programming. We introduce a special self-regular proximity to induce the feasibility step and also to measure proximity to the central path. The result of polynomial complexity coincides with the best-known iteration bound for infeasible interior-point methods, namely, $O(n \log n/\varepsilon)$.

AMS Mathematics Subject Classification : 65K05, 90C51.
Key words and phrases : linear programming, infeasible interior-point methods, full-Newton step, polynomial complexity, self-regular.

1. Introduction

We are concerned with the (LP) problem given in the following standard form:

$$(P) \qquad \begin{array}{l} \min \ c^T x \\ s.t. \ Ax = b, \quad x \geq 0, \end{array}$$

and its associated dual problem:

$$(D) \qquad \begin{array}{l} \max \ b^T y \\ s.t. \ A^T y + s = c, \quad s \geq 0, \end{array}$$

where $c, x, s \in \mathbb{R}^n$, $b, y \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$ is of full row rank.

For a comprehensive learning about interior-point methods (IPMs), we refer to Roos *et al.* [7]. In Roos [8], a full-Newton step infeasible interior-point algorithm for linear programming (LP) was presented and he also proved that the complexity of the algorithm coincides with the best known iteration bound

Received April 14, 2010. Revised June 8, 2010. Accepted June 21, 2010. *Corresponding author. †This work was in part supported by the Fundamental Research Funds for the Central Universities.

for infeasible IPMs. In Liu and Sun [1], Mansouri and Roos [3], they defined the feasibility step by special search directions, respectively. Such directions can be seen as parameterized affine scaling directions.

Recently Peng *et al.* [5] introduced a new class of primal-dual IPMs based on self-regular proximities. These methods do not use the classic Newton directions. Instead they use a direction that can be characterized as a steepest descent direction (in a scaled space) for a so-called self-regular barrier function. Each such barrier function is determined by a simple univariate self-regular function, called its kernel function. Salahi [9] extended the method in Peng and Terlaky [6] to infeasible IPMs. For both of them, the center path neighborhood are defined by the proximity function and they don't utilize any inner iteration to get centered.

Inspired by Salahi [9] and Liu and Sun [2], we develop an infeasible IPMs with full-Newton steps for (LP). The search direction of the feasibility step is induced by a proximity function. We also use a norm-based proximity to define the central neighborhood. We can get the same result of polynomial complexity, that is, $n \log n/\varepsilon$, which is the best currently for infeasible IPMs.

Throughout the paper $\| \cdot \|$ denotes the l_2-norm. We use Φ to denote the proximity function though $\Phi(v)$ and $\Phi(x, s; \mu)$ have different domains.

2. The statement of algorithm

As usual for infeasible IPMs we assume that the initial iterates (x^0, y^0, s^0) are as follows:

$$x^0 = s^0 = \zeta e, \quad y^0 = 0, \quad \mu^0 = \zeta^2,$$

where e is the all-one vector of length n, μ^0 is the initial dual gap and $\zeta > 0$ is such that

$$\|x^* + s^*\|_\infty \leq \zeta,$$

for some optimal solution (x^*, y^*, s^*) of (P) and (D). It is not trivial to find such initial feasible interior point. One method to overcome this difficult is to use the homogeneous self-dual embedding model by introducing artificial variables. The embedding technique was presented first by Ye *et al.* [10] and described in detail in Part I of Roos *et al.* [7].

It is generally agreed that the total number of inner iterations required by the algorithm is an appropriate measure for its efficiency and this number is referred to as the iteration complexity of the algorithm. Using $(x^0)^T s^0 = n\zeta^2$, the total number of iterations in the algorithm of Roos [8] is bounded above by

$$24 n \log \frac{\max\{n\zeta^2, \|r_b^0\|, \|r_c^0\|\}}{\varepsilon}, \tag{1}$$

where r_b^0 and r_c^0 is the initial residual vectors:

$$\begin{aligned} r_b^0 &= b - Ax^0, \\ r_c^0 &= c - A^T y^0 - s^0. \end{aligned}$$

Up to a constant factor, the iteration bound (1) was first obtained by Mizuno [4] and it is the best known iteration bound for infeasible IPMs.

Now we recall the main ideas underlying the algorithm in Roos [8]. For any ν with $0 < \nu \leq 1$ we consider the perturbed problem (P_ν), defined by

$$(P_\nu) \qquad \min\{(c - \nu r_b^0)^T x : \; Ax = b - \nu r_b^0, \; x \geq 0\},$$

and its dual problem (D_ν), which is given by

$$(D_\nu) \qquad \max\{(b - \nu r_b^0)^T y : \; A^T y + s = c - \nu r_c^0, \; s \geq 0\}.$$

Note that if $\nu = 1$ then $x = x^0$ yields a strictly feasible solution of (P_ν), and $(y, s) = (y^0, s^0)$ a strictly feasible solution of (D_ν). Due to the choice of the initial iterates we may conclude that if $\nu = 1$ then (P_ν) and (D_ν) each have a strictly feasible solution, which means that both perturbed problems then satisfy the well known interior-point condition (IPC).

Lemma 1. ([8, Lemma 1.1]) *The perturbed problems (P_ν) and (D_ν) satisfy the IPC for each $\nu \in (0, 1]$, if and only if the original problems (P) and (D) are feasible.*

Assuming that (P) and (D) are feasible, it follows from Lemma 1 that the problems (P_ν) and (D_ν) satisfy the IPC, for each $\nu \in (0, 1]$. And then their central paths exist. This means that the system

$$\begin{aligned} b - Ax &= \nu r_b^0, \quad x \geq 0, \\ c - A^T y - s &= \nu r_c^0, \quad s \geq 0, \\ xs &= \mu e \end{aligned}$$

has a unique solution for every $\mu > 0$, where xs denotes a Hadamard (componentwise) product of two vectors x and s. If $\nu \in (0, 1]$ and $\mu = \nu\zeta^2$ we denote this unique solution in the sequel as $(x(\nu), y(\nu), s(\nu))$. As a consequence, $x(\nu)$ is the μ-center of (P_ν) and $(y(\nu), s(\nu))$ the μ-center of (D_ν). Due to this notation we have, by taking $\nu = 1$,

$$(x(1), y(1), s(1)) = (x^0, y^0, s^0) = (\zeta e, 0, \zeta e).$$

One measures proximity of iterates (x, y, s) to the μ-center of the perturbed problems (P_ν) and (D_ν) by the quantity $\delta(x, s; \mu)$, which is defined as follows:

$$\delta(x, s; \mu) := \delta(v) := \frac{1}{2}\|v - v^{-1}\|, \qquad \text{where} \qquad v := \sqrt{\frac{xs}{\mu}}. \tag{2}$$

Initially one has $x = s = \zeta e$ and $\mu = \zeta^2$, whence $v = e$ and $\delta(x, s; \mu) = 0$. In the sequel assuming that at the start of each iteration, $\delta(x, s; \mu)$ is smaller than or equal to a (small) threshold value $\tau > 0$. So this is certainly true at the start of the first iteration.

For the feasibility step in Roos [8] they used search directions $\Delta^f x$, $\Delta^f y$ and

$\Delta^f s$ that are (uniquely) defined by the system

$$A\Delta^f x = \theta\nu r_b^0, \tag{3}$$
$$A^T\Delta^f y + \Delta^f s = \theta\nu r_c^0, \tag{4}$$
$$s\Delta^f x + x\Delta^f s = \mu e - xs. \tag{5}$$

Now we describe one main iteration of the algorithm in Roos [8]. Suppose that for some $\nu \in (0, 1]$ one has $x,\ y$ and s satisfying the feasibility conditions (3)-(4), and such that

$$x^T s = n\mu \quad \text{and} \quad \delta(x, s; \mu) \leq \tau,$$

where $\mu = \nu\zeta^2$. Each main iteration consists of a so-called feasibility step, a μ-update, and a few centering steps, respectively. First, we find new iterates $x^f,\ y^f$ and s^f that satisfy (3) and (4) with ν replaced by ν^+. As we will see, by taking θ small enough this can be realized by one feasibility step, to be described below soon. So, as a result of the feasibility step we obtain iterates that are feasible for (P_{ν^+}) and (D_{ν^+}). Then we reduce ν to $\nu^+ = (1-\theta)\nu$, with $\theta \in (0, 1)$, and apply a limited number of centering steps with respect to the μ^+-centers of (P_{ν^+}) and (D_{ν^+}). The centering steps keep the iterates feasible for (P_{ν^+}) and (D_{ν^+}), their purpose is to get iterates $x^+,\ y^+$ and s^+ such that $(x^+)^T s^+ = n\mu^+$, where $\mu^+ = \nu^+\zeta^2$ and $\delta(x^+, s^+; \mu^+) \leq \tau$. This process is repeated until the duality gap and the norms of the residual vectors are less than some prescribed accuracy parameter ε.

It can easily be understood that if (x, y, s) is feasible for the perturbed problems (P_ν) and (D_ν) then after the feasibility step the iterates satisfy the feasibility conditions for (P_{ν^+}) and (D_{ν^+}), provided that they satisfy the nonnegativity conditions. Assuming that before the step $\delta(x, s; \mu) \leq \tau$ holds, and by taking θ small enough, it can be guaranteed that after the step the iterates

$$x^f = x + \Delta^f x, \quad y^f = y + \Delta^f y, \quad s^f = s + \Delta^f s$$

are nonnegative and moreover $\delta(x^f, s^f; \mu^+) \leq 1/\sqrt{2}$, where $\mu^+ = (1-\theta)\mu$. So, after the μ-update the iterates are feasible for (P_{ν^+}) and (D_{ν^+}) and μ is such that $\delta(x^f, s^f; \mu) \leq 1/\sqrt{2}$.

In the centering steps, starting at the iterates $(x, y, s) = (x^f, y^f, s^f)$ and targeting at the μ-centers, the search directions $\Delta x, \Delta y, \Delta s$ are the usual primal-dual Newton directions, (uniquely) defined by

$$A\Delta x = 0,$$
$$A^T\Delta y + \Delta s = 0,$$
$$s\Delta x + x\Delta s = \mu e - xs.$$

Denoting the iterates after a centering step as $x^+,\ y^+$ and s^+, we recall the following results from Roos [7].

Lemma 2. *If $\delta := \delta(x, s; \mu) \leq 1$, then the primal-dual Newton step is feasible, i.e., x^+ and s^+ are nonnegative, and $(x^+)^T s^+ = n\mu$. Moreover, if $\delta := \delta(x, s; \mu) \leq 1/\sqrt{2}$, then $\delta(x^+, s^+; \mu) \leq \delta^2$.*

The centering steps serve to get iterates that satisfy $x^T s = n\mu^+$ and $\delta := \delta(x, s; \mu) \leq \tau$, where τ is (much) smaller than $1/\sqrt{2}$. By using Lemma 2, the required number of centering steps can easily be obtained. Because after the μ-update we have $\delta = \delta(x^f, s^f; \mu^+) \leq 1/\sqrt{2}$, and hence after k centering steps the iterates (x, y, s) satisfy

$$\delta(x, s; \mu^+) \leq (\frac{1}{\sqrt{2}})^{2^k}.$$

From this one easily deduces that no more than

$$\log_2(\log_2 \frac{1}{\tau^2}) \tag{6}$$

centering steps are needed.

Defining

$$d_x^f := \frac{v\Delta^f x}{x}, \quad d_s^f := \frac{v\Delta^f s}{s}, \tag{7}$$

with v as defined in (2). The system which defines the search directions $\Delta^f x$, $\Delta^f y$ and $\Delta^f s$, can be expressed in terms of the scaled search directions d_x^f and d_s^f as follows:

$$\begin{aligned} \bar{A} d_x^f &= \theta \nu r_b^0, \\ \bar{A}^T \frac{\Delta^f y}{\mu} + d_s^f &= \theta \nu v s^{-1} r_c^0, \\ d_x^f + d_s^f &= v^{-1} - v, \end{aligned}$$

where

$$\bar{A} = A V^{-1} X, \quad V = \operatorname{diag}(v), \quad X = \operatorname{diag}(x).$$

Note that the right-hand side of the third equation in the system is the negative gradient induced by the logarithmic barrier function

$$\Psi(v) := \sum_{i=1}^{n} \psi(v_i), \quad v_i = \sqrt{\frac{x_i s_i}{\mu}},$$

whose kernel function is

$$\psi(t) = \frac{1}{2}(t^2 - 1) - \log t.$$

In this paper the feasibility step is a slight modification of the classic primal-dual Newton direction. The feasibility direction is defined by a new system as follows

$$\begin{aligned} \bar{A}d_x^f &= \theta\nu r_b^0, \\ \bar{A}^T\frac{\Delta^f y}{\mu} + d_s^f &= \theta\nu v s^{-1} r_c^0, \\ d_x^f + d_s^f &= -\nabla\Phi(v), \end{aligned}$$

where the kernel function of $\Phi(v)$ is

$$\phi(t) := \frac{1}{2}(t - \frac{1}{t})^2.$$

Since $\phi'(t) = t - 1/t^3$, the third equation in the system can be written as

$$d_x^f + d_s^f = v^{-3} - v. \tag{8}$$

Note that $\|\nabla\Phi(v)\| = 0$ if and only if $v = e$, thus $\|\nabla\Phi(v)\|$ is also a suitable proximity. This norm-based proximity is used to define the central neighborhood. Now we give a more formal description of the algorithm in Figure 1.

Primal-Dual Infeasible IPMs

Input:
  Accuracy parameter $\varepsilon > 0$;
  barrier update parameter θ, $0 < \theta < 1$;
  threshold parameter $\tau > 0$.
begin
  $x := \zeta e$; $y := 0$; $s := \zeta e$; $\nu = 1$;
  while $\max\{x^T s, \|b - Ax\|, \|c - A^T y - s\|\} \geq \varepsilon$ **do**
  begin
    feasibility step: $(x, y, s) := (x, y, s) + (\Delta^f x, \Delta^f y, \Delta^f s)$;
    μ-update: $\mu := (1 - \theta)\mu$;
    centering steps:
    while $\|\nabla\Phi(v)\| \geq \tau$ **do**
      $(x, y, s) := (x, y, s) + (\Delta x, \Delta y, \Delta s)$;
    end while
  end
end

Figure 1: Algorithm

Before we go to the next section, we give several lemmas, which are needed for the analysis of the algorithm.

Lemma 3. ([3, Lemma A.1]) *For $i = 1, \ldots, m$, let $f_i : \mathbb{R}_+ \to \mathbb{R}$ denote a convex function. Then, for any nonzero vector $z \in \mathbb{R}^{\ltimes}_+$, the following inequality*

$$\sum_{i=1}^{n} f_i(z_i) \leq \frac{1}{e^T z} \sum_{j=1}^{n} z_j \left(f_j(e^T z) + \sum_{i \neq j} f_i(0) \right)$$

holds.

The next lemma focuses on the effect of the feasible search direction induced by the self-regular proximity function.

Lemma 4. *If $\Phi(v) := \Phi(x, s; \mu) \leq 2$, then the primal-dual Newton step is feasible, i.e., x^+ and s^+ are nonnegative, and $(x^+)^T s^+ = n\mu$. Moreover, if $\Phi(v) := \Phi(x, s; \mu) \leq 1$, then $\Phi(x^+, s^+; \mu) \leq (\frac{1}{\sqrt{2}}\Phi(v))^2$.*

Proof. The result can easily be obtained by the special relation between δ and Φ, see Lemma 2. □

The following lemma quantifies the effect on the proximity measure if v is replaced by $\tilde{v} = \sqrt{1-\theta}v$.

Lemma 5. *Let (x, s) be a positive primal-dual pair and $\mu > 0$ such that $x^T s = n\mu$. Moreover let $\Phi(v) = \Phi(x, s; \mu)$ and $\tilde{v} := \sqrt{1-\theta}v$. Then*

$$\Phi(\tilde{v}) = \frac{1}{1-\theta}\Phi(v) + \frac{\theta^2 n}{1-\theta}.$$

Proof.

$$\begin{aligned}
\Phi(\tilde{v}) &= \frac{1}{2}\|\sqrt{1-\theta}v - \frac{1}{\sqrt{1-\theta}}v^{-1}\|^2 \\
&= \frac{1}{2}\|\frac{1}{\sqrt{1-\theta}}(v - v^{-1}) + (\sqrt{1-\theta}v - \frac{1}{\sqrt{1-\theta}}v)\|^2 \\
&= \frac{1}{2}\frac{1}{1-\theta}\|v - v^{-1}\|^2 + \frac{\theta^2 n}{1-\theta} \\
&= \frac{1}{1-\theta}\Phi(v) + \frac{\theta^2 n}{1-\theta},
\end{aligned}$$

where the third equality is due to $\|v\|^2 = n$ and the orthogonality between v and $v^{-1} - v$. □

The following result tells us that the relation between the norm-based proximity $\|\nabla\Phi(v)\|$ and the proximity function $\Phi(v)$.

Lemma 6. *One has $\frac{1}{2}\|\nabla\Phi(v)\|^2 \geq \Phi(v)$.*

Proof. Since

$$\frac{1}{2}(t^{-3} - t)^2 - \frac{1}{2}(t^{-1} - t)^2 = \frac{1}{2}(t^{-2} - 1)^2(t^{-2} + 2) \geq 0, \ \forall\, t > 0,$$

thus the lemma follows. □

3. Analysis of the feasibility step

Let x, y and s denote the iterates at the start of an iteration, and assume that $\|\nabla\Phi(v)\| \leq \tau$. Recall that in the first iteration we have $\|\nabla\Phi(v)\| = 0$.

3.1.Effect of the feasibility step. According Lemma 4, we need to show that $\Phi(x^f, s^f; \mu^+) \leq 1$ after the feasibility step, i.e., that the new iterates are positive and within the region where the Newton process targeting at the μ^+-centers of (P_{ν^+}) and (D_{ν^+}) is quadratically convergent.

Now using (8) and $xs = \mu v^2$ we may write

$$\begin{aligned} x^f s^f &= xs + (s\Delta^f x + x\Delta^f s) + \Delta^f x \Delta^f s \\ &= \mu^2 (xs)^{-1} + \Delta^f x \Delta^f s = \mu(v^{-2} + d_x^f d_s^f). \end{aligned} \tag{9}$$

Lemma 7. *The new iterates are certainly strictly feasible if and only if* $v^{-2} + d_x^f d_s^f > 0$.

Proof. Note that if x^f and s^f are positive then (9) makes clear that $v^{-2} + d_x^f d_s^f > 0$. Following Lemma 4.1 in Mansouri and Roos [3], the converse can be proved. Thus the lemma follows. □

Using (7) we may also write

$$x^f = x + \Delta^f x = x + \frac{x d_x^f}{v} = \frac{x}{v}(v + d_x^f), \tag{10}$$

$$s^f = s + \Delta^f s = s + \frac{s d_s^f}{v} = \frac{s}{v}(v + d_s^f). \tag{11}$$

Lemma 8. *The new iterates are certainly strictly feasible if*

$$\|d_x^f\|^2 < \frac{1}{\rho(\Phi(v))} \quad \textit{and} \quad \|d_s^f\|^2 < \frac{1}{\rho(\Phi(v))}, \tag{12}$$

where

$$\rho(\Phi(v)) := (\Phi(v) + 1) + \sqrt{(\Phi(v) + 1)^2 - 1}.$$

Proof. It is clear from (10) that x^f is strictly feasible if and only if $v + d_x^f > 0$. This certainly holds if $\|d_x^f\| < \min(v_i)$. Since $\Phi(v) = \frac{1}{2}\|v - v^{-1}\|^2$, the minimal value t that an entry of v can attain will satisfy $t \leq 1$ and $t^2 + 1/t^2 = 2\Phi(v) + 2$. The last equation implies $t^4 - 2(\Phi(v) + 1)t^2 + 1 = 0$, which gives $t^2 = (\Phi(v) + 1) - \sqrt{(\Phi(v) + 1)^2 - 1} = 1/\rho(\Phi(v))$. This proves the first inequality in (12). The second inequality can be obtained in the same way. □

The proof of Lemma 8 makes clear that the elements of the vector v satisfy

$$\frac{1}{\rho(\Phi(v))} \leq v_i^2 \leq \rho(\Phi(v)), \quad i = 1, \ldots, n. \tag{13}$$

In the sequel we denote

$$\omega_i := \omega_i(v) := \frac{1}{2}\sqrt{|d_{xi}^f|^2 + |d_{si}^f|^2},$$

and

$$\omega := \omega(v) := \|(\omega_1, \dots, \omega_n)\|.$$

This implies

$$(d_x^f)^T d_s^f \le \|d_x^f\|\|d_s^f\| \le \frac{1}{2}(\|d_x^f\|^2 + \|d_s^f\|^2) \le 2\omega^2,$$
$$|d_{xi}^f d_{si}^f| = |d_{xi}^f||d_{si}^f| \le \frac{1}{2}(|d_{xi}^f|^2 + |d_{si}^f|^2) \le 2\omega_i^2 \le 2\omega^2, \quad 1 \le i \le n.$$

Lemma 9. *Assuming* $v^{-2} + d_x^f d_s^f > 0$, *one has*

$$2\Phi(v^f) \le \frac{2}{1-\theta}\Phi(v) + \frac{\theta^2 n}{1-\theta} + \frac{2\omega^2}{1-\theta} + \frac{2(1-\theta)\rho(\Phi(v))^4\omega^2}{1-2\rho(\Phi(v))^2\omega^2}.$$

Proof.

$$(v^f)^2 = \frac{\mu(v^{-2} + d_x^f d_s^f)}{\mu^+} = \frac{v^{-2} + d_x^f d_s^f}{1-\theta},$$

hence according to (2),

$$\begin{aligned} 2\Phi(v^f) &= \sum_{i=1}^n \Big((v_i^f)^2 + (v_i^f)^{-2} - 2\Big) \\ &= \sum_{i=1}^n \left(\frac{v_i^{-2} + d_{xi}^f d_{si}^f}{1-\theta} + \frac{1-\theta}{v_i^{-2} + d_{xi}^f d_{si}^f} - 2\right) \\ &\le \sum_{i=1}^n \left(\frac{v_i^{-2} + 2\omega_i^2}{1-\theta} + \frac{1-\theta}{v_i^{-2} - 2\omega_i^2} - 2\right). \end{aligned}$$

For each i we define the function

$$f_i(z_i) = \frac{v_i^{-2} + z_i}{1-\theta} + \frac{1-\theta}{v_i^{-2} - z_i} - 2, \quad i = 1, \dots, n.$$

One can easily verify that if $v_i^{-2} - z_i > 0$ then $f_i(z_i)$ is convex in z_i. Taking $z_i = 2\omega_i^2$, we can require

$$v_i^{-2} - 2\omega_i^2 > 0.$$

By using (13), this certainly holds if

$$2\omega^2 < \frac{1}{\rho(\Phi(v))}. \tag{14}$$

We therefore may use Lemma 3 and give

$$\begin{aligned}
2\Phi(v^f) &\leq \sum_{j=1}^{n} f_j(\omega_j) \leq \frac{1}{2\omega^2}\sum_{i=1}^{n} 2\omega_j^2 \left(f_j(2\omega^2) + \sum_{i\neq j} f_i(0) \right) \\
&= \frac{1}{2\omega^2}\sum_{j=1}^{n} \left[2\omega_j^2 \left(\left(\frac{v_j^{-2} + 2\omega^2}{1-\theta} + \frac{1-\theta}{v_j^{-2} - 2\omega^2} - 2 \right) \right.\right. \\
&\quad \left.\left. + \sum_{i\neq j} \left(\frac{v_i^{-2}}{1-\theta} + \frac{1-\theta}{v_i^{-2}} - 2 \right) \right) \right].
\end{aligned}$$

Using Lemma 5, we obtain

$$\begin{aligned}
\sum_{i\neq j} \left(\frac{v_i^{-2}}{1-\theta} + \frac{1-\theta}{v_i^{-2}} - 2 \right) &= \sum_{i=1}^{n} \left(\frac{v_i^{-2}}{1-\theta} + \frac{1-\theta}{v_i^{-2}} - 2 \right) - \left(\frac{v_j^{-2}}{1-\theta} + \frac{1-\theta}{v_j^{-2}} - 2 \right) \\
&= \frac{2}{1-\theta}\Phi(v) + \frac{2\theta^2 n}{1-\theta} - \left(\frac{v_j^{-2}}{1-\theta} + \frac{1-\theta}{v_j^{-2}} - 2 \right).
\end{aligned}$$

Then

$$\begin{aligned}
& 2\Phi(v^f) \\
\leq\ & \frac{2}{1-\theta}\Phi(v) + \frac{2\theta^2 n}{1-\theta} \\
& + \frac{1}{2\omega^2}\sum_{j=1}^{n} 2\omega_j^2 \left(\frac{v_j^{-2} + 2\omega^2}{1-\theta} + \frac{1-\theta}{v_j^{-2} - 2\omega^2} - 2 - \left(\frac{v_j^{-2}}{1-\theta} + \frac{1-\theta}{v_j^{-2}} - 2 \right) \right) \\
=\ & \frac{2}{1-\theta}\Phi(v) + \frac{2\theta^2 n}{1-\theta} + \frac{2\omega^2}{1-\theta} + \frac{1}{2\omega^2}\sum_{j=1}^{n} 2\omega_j^2 \frac{(1-\theta)2\omega^2}{v_j^{-2}(v_j^{-2} - 2\omega^2)} \\
\leq\ & \frac{2}{1-\theta}\Phi(v) + \frac{2\theta^2 n}{1-\theta} + \frac{2\omega^2}{1-\theta} + \frac{(1-\theta)2\omega^2}{\frac{1}{\rho(\Phi(v))^2}\left(\frac{1}{\rho(\Phi(v))^2} - 2\omega^2\right)} \\
=\ & \frac{2}{1-\theta}\Phi(v) + \frac{2\theta^2 n}{1-\theta} + \frac{2\omega^2}{1-\theta} + \frac{2(1-\theta)\rho(\Phi(v))^4\omega^2}{1 - 2\rho(\Phi(v))^2\omega^2}.
\end{aligned}$$

□

We conclude this section by presenting a value that we don't allow ω to exceed. We observe that because we need to have $\Phi(v^f) \leq 1$, it follows from Lemma 9 that it suffices if

$$\frac{2}{1-\theta}\Phi(v) + \frac{2\theta^2 n}{1-\theta} + \frac{2\omega^2}{1-\theta} + \frac{2(1-\theta)\rho(\Phi(v))^4\omega^2}{1 - 2\rho(\Phi(v))^2\omega^2} \leq 2. \tag{15}$$

At this stage we decide to choose

$$\tau = \frac{1}{4}, \quad \theta = \frac{\alpha}{4\sqrt{n}}, \quad \alpha \leq 1. \tag{16}$$

Since the left-hand side of (15) is monotonically increasing with respect to ω^2, then, for $n \geq 1$ and $\|\nabla\Phi(v)\| \leq \tau$ (Lemma 6 implies $\Phi(v) \leq \frac{1}{2}\|\nabla\Phi(v)\|^2 \leq \frac{1}{2}\tau^2$), together with (14), one can verify that

$$\omega \leq \frac{1}{2\sqrt{2}} \quad \Rightarrow \quad \Phi(v^f) \leq 1. \tag{17}$$

Lemma 6 has showed the relation between $\|\nabla\Phi(v)\|$ and $\Phi(v)$, but it isn't enough to know when $\|\nabla\Phi(v)\| \leq \tau$ can occur.

Lemma 10. *If* $\Phi(v) \leq \frac{1}{256}$, *then* $\|\nabla\Phi(v)\| \leq \frac{1}{4}$.

Proof. Assume that $\tau_0 = 1/256$, then

$$\frac{1}{2}(t - t^{-1})^2 \leq \tau_0. \tag{18}$$

Thus the above inequality reduces to

$$\begin{cases} h_1(t) := t^2 - \sqrt{2\tau_0}t - 1 \leq 0, & t \geq 1, \\ h_2(t) := -t^2 - \sqrt{2\tau_0}t + 1 \leq 0, & t < 1. \end{cases}$$

Note the function $h_1(t)$ is monotone increasing with respect to $t \geq 1$ and $h_2(t)$ is monotone decreasing with respect to $t < 1$, and they both attain a minimum at $t = 1$. Then we can easily get, for

$$\frac{\sqrt{\tau_0 + 2} - \sqrt{\tau_0}}{\sqrt{2}} \leq t \leq \frac{\sqrt{\tau_0 + 2} + \sqrt{\tau_0}}{\sqrt{2}}, \tag{19}$$

the inequality (18) holds. Note that we set $\tau_0 = 1/256$, the inequality (19) reduces to

$$0.9569 \leq t \leq 1.0454. \tag{20}$$

Similarly, now we consider

$$\begin{cases} u_1(t) := 4t^4 - t^3 - 4 \leq 0, & t \geq 1, \\ u_2(t) := -4t^4 - t^3 + 4 \leq 0, & t < 1. \end{cases} \tag{21}$$

We can verify, for all t in (20), the inequality (21) holds, which implies $|t^{-3} - t| \leq \frac{1}{4}$. □

3.2.Upper bound for $\omega(v)$. Let us denote the null space of the matrix $\bar{A}$ as $\mathcal{L}$. So,

$$\mathcal{L} := \{\xi \in \mathbb{R}^{\ltimes} : \ \bar{\mathbb{A}}\xi = \nvdash\}.$$

Obviously, the affine space $\{\xi \in \mathbb{R}^{\ltimes} : \ \bar{\mathbb{A}}\xi = \theta\nu\searrow^{\nvdash}\}$ equals $d_x^f + \mathcal{L}$. The row space of $\bar{A}$ equals the orthogonal complement $\mathcal{L}^\perp$ of $\mathcal{L}$, and $d_s^f \in \theta\nu v s^{-1} r_c^0 + \mathcal{L}^\perp$. We recall a lemma from Roos [8] except that we use $v^{-3} - v$ instead of $v^{-1} - v$.

Lemma 11. *Let q be the (unique) point in the intersection of the affine spaces $d_x^f + \mathcal{L}$ and $d_s^f + \mathcal{L}^\perp$. Then*

$$2\omega(v) \leq \sqrt{\|q\|^2 + (\|q\| + \|\nabla\Phi(v)\|)^2}.$$

Recall from (17) that in order to guarantee that $\Phi(v^f) \leq 1$ we want to have $\omega \leq 1/(2\sqrt{2})$. Due to Lemma 11 this will certainly hold if $\|q\|$ satisfies

$$\|q\|^2 + (\|q\| + \|\nabla\Phi(v)\|)^2 \leq \frac{1}{2}. \tag{22}$$

Now still from Roos [8], we can get

$$\sqrt{\mu}\|q\| \leq \theta\nu\zeta\sqrt{e^T(\frac{x}{s} + \frac{s}{x})}. \tag{23}$$

To further proceed we need upper and lower bounds for the elements of the vectors x and s.

3.3.Bounds for x/s and s/x. Recall that x is feasible for (P_ν) and (y, s) for (D_ν) and, moreover $\|\nabla\Phi(v)\| \leq \tau$, i.e., these iterates are close to the μ-centers of (P_ν) and (D_ν). Based on this information we need to estimate the sizes of the entries of the vectors x/s and s/x. Since $\tau_0 = 1/256$, namely, $\delta(v) \leq 1/(32\sqrt{2})$ according to the relation between $\delta(v)$ and $\Phi(v)$, we can again use a result from Roos [8], namely, Corollary A.10, which gives

$$\sqrt{\frac{x}{s}} \leq \frac{10x(\mu,\nu)}{\sqrt{\mu}}, \quad \sqrt{\frac{s}{x}} \leq \frac{10s(\mu,\nu)}{\sqrt{\mu}}.$$

Note that Corollary A.10 in Roos [8] is not dependent on $\Psi(v)$ except the constants τ' and $\chi(\tau')$, that is to say, the result still holds true after we use the new direction and the new proximity function. Substitution into (23) yields

$$\sqrt{\mu}\|q\| \leq \theta\nu\zeta\sqrt{100e^T(\frac{x(\mu,\nu)^2}{\mu} + \frac{s(\mu,\nu)^2}{\mu})}.$$

This implies

$$\mu\|q\| \leq 10\theta\nu\zeta\sqrt{\|x(\mu,\nu)\|^2 + \|s(\mu,\nu)\|^2}.$$

By using $\mu = \mu^0\nu = \zeta^2\nu$ and $\theta = \alpha/(4\sqrt{n})$, we obtain the following upper bound for the norm of q:

$$\|q\| \leq \frac{5\alpha}{2\zeta\sqrt{n}}\sqrt{\|x(\mu,\nu)\|^2 + \|s(\mu,\nu)\|^2}.$$

Define

$$\kappa(\zeta,\nu) = \frac{\sqrt{\|x(\mu,\nu)\|^2 + \|s(\mu,\nu)\|^2}}{\zeta\sqrt{n}}, \quad 0 < \nu \leq 1, \quad \mu = \mu^0\nu,$$

and

$$\bar{\kappa}(\zeta) = \max_{0<\nu\leq 1} \kappa(\zeta,\nu),$$

we may have

$$\|q\| \leq \frac{5}{2}\alpha\bar{\kappa}(\zeta).$$

We find in (22) that in order to have $\Phi(v^f) \leq 1$, we should have $\|q\|^2 + (\|q\| + \|\nabla\Phi(v)\|)^2 \leq 1/2$. Note that $\|\nabla\Phi(v)\| \leq \tau = \frac{1}{4}$, it suffices if q satisfies $\|q\|^2 + (\|q\| + \frac{1}{4})^2 \leq 1/2$. This holds if and only if $\|q\| \leq 1/4$. We conclude that if we take

$$\alpha = \frac{1}{10\bar{\kappa}(\zeta)}, \tag{24}$$

we will certainly have $\Phi(v^f) \leq 1$. Following Roos [8], we can prove that $\bar{\kappa}(\zeta) = 2\sqrt{n}$.

4. Iteration bound

In the previous sections we have found that if at the start of an iteration the iterates satisfy $\|\nabla\Phi(v)\| \leq \tau$, with τ as defined in (16), then after the feasibility step, with θ as in (16), and α as in (24), the iterates satisfy $\Phi(x, s; \mu^+) \leq 1$.

Before we enter the central neighborhood, we decrease $\Phi(v)$ from 1 to 1/256. According to (6), at most

$$\log_2(\log_2 256^2) = 4$$

centering steps suffice to get iterates that satisfy $\|\nabla\Phi(x, s; \mu^+)\| \leq \tau$. So each iteration consists of at most 4 so-called 'inner' iterations, in each of which we need to compute a new search direction. In each main iteration both the duality gap and the norms of the residual vectors are reduced by the factor $1-\theta$. Hence, using $(x^0)^T s^0 = n\zeta^2$, the total number of iterations is bounded above by

$$\frac{1}{\theta} \log \frac{\max\{n\zeta^2, \|r_b^0\|, \|r_c^0\|\}}{\varepsilon}.$$

Since

$$\theta = \frac{\alpha}{4\sqrt{n}} = \frac{1}{40\sqrt{n}\bar{\kappa}(\zeta)},$$

the total number of inner iterations is therefore bounded above by

$$320n \log \frac{\max\{n\zeta^2, \|r_b^0\|, \|r_c^0\|\}}{\varepsilon}.$$

5. Numerical implementation

We test the algorithm in this section. The code is written in MATLAB. The computer is Lenove PC (Intel(R) Core(TM)2 Duo CPU T9300 2.5Hz). We select several small-size problems from NETLIB to show the algorithm is feasible. In the list, the initial situation, $\max\{n\zeta^2, \|r_b^0\|, \|r_c^0\|\}$ is denoted by **Init.** and the number of outer iterations is denoted by **Iter.**. In fact this is a small-update (short-step) interior-point algorithm with θ as above. So lots of iterations are needed for finding solutions.

Prob.	m	n	Init.	Iter.	Time
E226	223	472	4.9303E3	37781	2.0139E3
AFIRO	27	51	832.8759	3712	3.7628
ADLITTLE	56	138	1.0616E4	11462	69.4210
BANDM	305	472	1.4999E3	35535	2.1406E3
BEACONFD	173	295	6.4029E3	23918	777.4179
BLEND	74	114	236.0117	7733	48.7910
CAPRI	271	482	7.8728E3	39484	2.1993E3
RECIPE	91	204	2.7486E3	15847	218.6644

6. Concluding remarks

In this paper we introduce a self-regular proximity in the infeasible interior-point algorithm with full-Newton step for linear programming. We also use a norm-based proximity to define the central neighborhood. Extensions to second-order cone programming and semidefinite programming seem to be within reach. We only discuss a special self-regular proximity in this paper, our future work will focus on more general self-regular proximities.

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