J. Appl. Math. & Informatics Vol. **29**(2011), No. 1 - 2, pp. 191 - 200
Website: http://www.kcam.biz

MIXED BOUNDARY VALUE PROBLEMS FOR SECOND ORDER DIFFERENTIAL EQUATIONS WITH DIFFERENT DEVIATED ARGUMENTS

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Abstract. This paper deals with second order differential equations with different deviated arguments $\alpha(t)$ and $\beta(t, \mu(t))$. We investigate the existence of solutions of such problems with nonlinear mixed boundary conditions. To obtain corresponding results we apply the monotone iterative technique and the lower-upper solutions method. Two examples demonstrate the application of our results.

AMS Mathematics Subject Classification : 34B15, 34K10.
Key words and phrases : Nonlinear mixed boundary conditions, Different deviated arguments, Monotone iterative technique, Lower-upper solutions method.

1. Introduction

In this paper, we study the following boundary value problems:

$$\begin{cases} x''(t) = f(t, x(t), x(\alpha(t)), \lambda x(\beta(t, \mu(t)))) \equiv Fx(t), \\ g_1(x(0), x(\delta)) = \int_0^T D_1(s)x(s)ds, \\ g_2(x(T), x(\gamma)) = \int_0^T D_2(s)x(s)ds, \end{cases} \tag{1}$$

where $t \in J = [0, T](T > 0)$, $f \in C(J \times R^3, R)$, $\alpha \in C(J, J)$, $\mu \in C(J, R)$, $\beta \in C(J \times R, J)$, $\lambda \in [0, \infty)$, $D_i(i = 1, 2) \in C(J, [0, \infty))$, $0 < \delta, \gamma < T$, $g_1, g_2 \in C(R \times R, R)$.

Note that equation from (1.1) has a very general form, and (1.1) includes a number of differential equations. In addition to, boundary value conditions from (1.1) include second-order two, three, multi-point and integral boundary conditions. It is important to indicate that (1.1) is different from corresponding problems investigated in the papers published earlier.

Received March 25, 2010. Revised June 30, 2010. Accepted July 5, 2010. *Corresponding author.

The theory of differential equations with deviated argument is an important and significant branch of nonlinear analysis. It is worthwhile mentioning that differential equations with deviated arguments appear often in investigations connected with mathematical physics, mechanics, engineering, economics and so on (see [11, 15, 16]. Recently, many authors have considered theory of differential equations with deviated arguments [1, 2, 7, 8, 9, 10], One basic problem of which is to establish convenient conditions guaranteeing the existence of solutions of those equations. To obtain existence results we apply the monotone iterative method combined with lower and upper solutions; for details, see [1, 2, 3, 4, 5, 6, 12, 13, 14]. There exists a vast literature devoted to the applications of this method. In this paper, we apply this technique to second order differential equations with different deviated arguments $\alpha(t)$ and $\beta(t,\mu(t))$. As far as I am concerned, no paper has considered a class of nonlinear mixed integral boundary value problem of second-order differential equations with different deviated arguments $\alpha(t)$ and $\beta(t,\mu(t))$. The purpose of this paper is to improve and generalize the results mentioned to some degree.

We organize the rest of this paper as follows: in Section 2, we present some useful Lemmas. Then in Section 3, the main result and proof are given. Furthermore, we give two examples which demonstrate the application of our results.

2. Several Lemmas

The following lemmas play an important role in this paper.

Lemma 1.(*Comparison theorem*) Assume that $\alpha \in C(J,J)$, $\alpha(t) \leq t$ on J, and $p = p(t) \in C^2(J,R)$ satisfies

$$\begin{cases} p''(t) \geq M(t)p(t) + N(t)p(\alpha(t)), \\ p(0) \leq 0, \quad p(T) \leq 0, \end{cases} \tag{2}$$

where $M(t), N(t)$ are non-negative bounded integrable functions on J, and $M(t) > 0 (t \in (0,T))$.

If

$$\int_0^T [M(t) + N(t)]tdt \leq 1. \tag{3}$$

Then $p(t) \leq 0, \ \forall t \in J$.

Proof. Assume that $p(t) \leq 0, \forall t \in J$ is not true. Then, there exists a $t_0 \in (0,T)$ such that $p(t_0) = \max_{t\in J} p(t) = \lambda > 0$, which implies $p''(t_0) \leq 0$ and $p'(t_0) = 0$.

If $p(t) \geq 0, t \in [0,t_0]$. Then, by (2.1), we have

$$p''(t_0) \geq M(t_0)\lambda + N(t_0)p(\alpha(t_0)) > 0,$$

which contradicts $p''(t_0) \leq 0$.

If there exists a $t_1 \in [0, t_0)$ such that $p(t_1) = \min_{t\in[0,t_0]} p(t) = -b < 0$. Then, for $\forall s \in [0, t_0]$, we have

$$\begin{aligned} p'(s) &= -[p'(t_0) - p'(s)] = -\int_s^{t_0} p''(t)dt \le -\int_s^{t_0} [M(t)p(t) + N(t)p(\alpha(t))]dt \\ &\le b\int_s^{t_0} [M(t) + N(t)]dt, \end{aligned}$$

and

$$\begin{aligned} b &= -p(t_1) < p(t_0) - p(t_1) = \int_{t_1}^{t_0} p'(s)ds \le \int_{t_1}^{t_0} b\int_s^{t_0} [M(t) + N(t)]dtds \\ &\le b\int_0^T \int_s^T [M(t) + N(t)]dtds = b\int_0^T [M(t) + N(t)]tdt. \end{aligned}$$

Therefore, we have $\int_0^T [M(t) + N(t)]tdt > 1$, which contradicts (2.2).
Lemma 1 is proved. $\square$
For $\forall\sigma \in C(J, R)$, we consider the linear problem:

$$\begin{cases} x''(t) = M(t)x(t) + N(t)x(\alpha(t)) + \sigma(t), \ t \in J, \\ x(0) = x_0, \ x(T) = x_1, \ x_0, x_1 \in R, \end{cases} \tag{4}$$

where $M(t), N(t)$ are non-negative bounded integrable functions on J, and $M(t) > 0 (t \in (0, T))$.

Lemma 2. Assume that $\alpha \in C(J, J),\ \alpha(t) \le t$ on J, and non-negative bounded integrable functions $M(t), N(t)$ satisfy (2.2). Then, (2.3) has a unique solution $x \in C^2(J, R)$.
Proof. First, we prove that (2.3) has at most one solution in $C^2(J, R)$. Let $x, y \in C^2(J, R)$ be two different solutions of (2.3). Put $p = x-y$, then p satisfies

$$\begin{cases} p''(t) = M(t)p(t) + N(t)p(\alpha(t)), \\ p(0) = 0, \ p(T) = 0, \end{cases}$$

By Lemma 1, we know $p(t) \le 0, \forall t \in J$, which implies $x(t) \le y(t), \forall t \in J$. If let $p = y-x$, then by using the same way as above, we can get $y(t) \le x(t), \forall t \in J$. It show that $x(t) = y(t), \forall t \in J$, which implies that (2.3) has at most one solution in $C^2(J, R)$.

Next, we prove that (2.3) has a solution in $C^2(J, R)$. Obviously, $x(t) \in C^2(J, R)$ is a solution of (2.3) if and only if $x(t) \in C(J, R)$ is a solution of the following integral equation:

$$\begin{aligned} &x(t) = x_0 + \tfrac{t}{T}(x_1 - x_0) - \tfrac{t}{T}\int_0^T \int_0^r [M(s)x(s) + N(s)x(\alpha(s)) + \sigma(s)]dsdr \\ &+ \int_0^t \int_0^r [M(s)x(s) + N(s)x(\alpha(s)) + \sigma(s)]dsdr \\ &\equiv (Ax)(t). \end{aligned} \tag{5}$$

Indeed, the operator A is bounded and continuous. It is easy to show that A is a compact map. By Schauder fixed point theorem, we know that A has at least one fixed point, which implies that (2.4) has a solution $x(t) \in C(J, R)$.

Therefore, (2.3) has a unique solution in $C^2(J,R)$.
This completes the proof of Lemma 2. $\square$

3. Main Results

Let us introduce the following definition.

Definition 1. Let $y_0, z_0 \in C^2(J,R)$. Then y_0 is said to be a lower solution of (1.1) if

$$\begin{cases} y_0''(t) \geq f(t, y_0(t), y_0(\alpha(t)), \lambda y_0(\beta(t,\mu(t)))) \equiv F y_0(t), \\ g_1(y_0(0), y_0(\delta)) \leq \int_0^T D_1(s) y_0(s) ds, \\ g_2(y_0(T), y_0(\gamma)) \leq \int_0^T D_2(s) y_0(s) ds. \end{cases} \tag{6}$$

and z_0 is said to be an upper solution of (1.1) if the above inequalities are reversed.

We list for convenience the following assumptions.

(H_1) : $f \in C(J \times R^3, R)$, $\mu \in C(J,R)$, $\beta \in C(J \times R, J)$, $\lambda \in [0,\infty)$, $D_i (i = 1,2) \in C(J,[0,\infty))$, $0 < \delta, \gamma < T$, $\alpha \in C(J,J)$, $\alpha(t) \leq t$ on J.

(H_2) : There exist non-negative bounded integrable functions $M(t), N(t)$ and $M(t) > 0 (t \in (0,T))$ which satisfy (2.2), such that

$$f(t, \bar{u}, \bar{v}, \bar{w}) - f(t,u,v,w) \leq M(t)(\bar{u} - u) + N(t)(\bar{v} - v),$$

for $y_0(t) \leq u \leq \bar{u} \leq z_0(t)$, $y_0(\alpha(t)) \leq v \leq \bar{v} \leq z_0(\alpha(t))$, $\lambda y_0(\beta(t,\mu(t))) \leq w \leq \bar{w} \leq \lambda z_0(\beta(t,\mu(t)))$.

(H_3) : g_1, g_2 are non-increasing with respect to the second variable and there exist constants $a > 0, b > 0$ such that

$$g_1(\bar{u}_1, v_1) - g_1(u_1, v_1) \leq a(\bar{u}_1 - u_1), \ g_2(\bar{u}_2, v_2) - g_2(u_2, v_2) \leq b(\bar{u}_2 - u_2),$$

for $y_0(0) \leq u_1 \leq \bar{u}_1 \leq z_0(0)$, $y_0(\delta) \leq v_1 \leq z_0(\delta)$, $y_0(T) \leq u_2 \leq \bar{u}_2 \leq z_0(T)$, $y_0(\gamma) \leq v_2 \leq z_0(\gamma)$.

Theorem 1. Assume that conditions $(H_1) \sim (H_3)$ hold. Let $y_0, z_0 \in C^2(J,R)$ be lower and upper solutions of (1.1), respectively, and $y_0(t) \leq z_0(t), t \in J$. Then (1.1) has extremal solutions in the sector $[y_0, z_0] = \{w \in C^2(J,R) : y_0(t) \leq w(t) \leq z_0(t), t \in J\}$.

Proof. Let

$$\begin{cases} y_{n+1}''(t) &= F y_n(t) + M(t)[y_{n+1}(t) - y_n(t)] + N(t)[y_{n+1}(\alpha(t)) - y_n(\alpha(t))], \\ y_{n+1}(0) &= -\frac{1}{a} g_1(y_n(0), y_n(\delta)) + y_n(0) + \frac{1}{a}\int_0^T D_1(s) y_n(s) ds, \\ y_{n+1}(T) &= -\frac{1}{b} g_2(y_n(T), y_n(\gamma)) + y_n(T) + \frac{1}{b}\int_0^T D_2(s) y_n(s) ds; \end{cases} \tag{7}$$

$$\begin{cases} z_{n+1}''(t) &= F z_n(t) + M(t)[z_{n+1}(t) - z_n(t)] + N(t)[z_{n+1}(\alpha(t)) - z_n(\alpha(t))], \\ z_{n+1}(0) &= -\frac{1}{a} g_1(z_n(0), z_n(\delta)) + z_n(0) + \frac{1}{a}\int_0^T D_1(s) z_n(s) ds, \\ z_{n+1}(T) &= -\frac{1}{b} g_2(z_n(T), z_n((\gamma)) + z_n(T) + \frac{1}{b}\int_0^T D_2(s) z_n(s) ds; \end{cases} \tag{8}$$

for $n = 0, 1, \cdots$. By (2.3), note that y_1, z_1 are well defined, Firstly, we prove that

$$y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t), \ \forall t \in J. \tag{9}$$

Let $p(t) = y_0(t) - y_1(t)$, from (3.1), (3.2) and (H_2), we have that

$$\begin{aligned} p''(t) &\geq Fy_0(t) - Fy_0(t) - M(t)[y_1(t) - y_0(t)] - N(t)[y_1(\alpha(t)) - y_0(\alpha(t))] \\ &= M(t)p(t) + N(t)p(\alpha(t)), \end{aligned}$$

and

$$p(0) = y_0(0) - y_1(0) = y_0(0) + \frac{1}{a}g_1(y_0(0), y_0(\delta)) - y_0(0) - \frac{1}{a}\int_0^T D_1(s)y_0(s)ds \leq 0,$$

$$p(T) = y_0(T) - y_1(T) = y_0(T) + \frac{1}{b}g_2(y_0(T), y_0(\gamma)) - y_0(T) - \frac{1}{b}\int_0^T D_2(s)y_0(s)ds \leq 0.$$

Hence, by Lemma 1, we have $y_0(t) \leq y_1(t)$. Similarly, we can show that $z_1(t) \leq z_0(t), t \in J$. Now, we let $p(t) = y_1(t) - z_1(t)$. Then, by (H_2) and (H_3), we have

$$\begin{aligned} p''(t) &= Fy_0(t) - Fz_0(t) + M(t)[y_1(t) - y_0(t)] + N(t)[y_1(\alpha(t)) - y_0(\alpha(t))] \\ &\quad - M(t)[z_1(t) - z_0(t)] - N(t)[z_1(\alpha(t)) - z_0(\alpha(t))] \\ &\geq M(t)[y_1(t) - z_1(t)] + N(t)[y_1(\alpha(t)) - z_1(\alpha(t))] \\ &= M(t)p(t) + N(t)p(\alpha(t)), \end{aligned}$$

and

$$\begin{aligned} p(0) &= \frac{1}{a}\int_0^T D_1(s)(y_0 - z_0)(s)ds + \frac{1}{a}[g_1(z_0(0), z_0(\delta)) - g_1(y_0(0), y_0(\delta))] \\ &\quad + y_0(0) - z_0(0) \\ &\leq \frac{1}{a}[g_1(z_0(0), y_0(\delta)) - g_1(y_0(0), y_0(\delta))] + y_0(0) - z_0(0) \\ &\leq z_0(0) - y_0(0) + y_0(0) - z_0(0) = 0, \\ p(T) &= \frac{1}{b}\int_0^T D_2(s)(y_0 - z_0)(s)ds + \frac{1}{b}[g_2(z_0(T), z_0(\gamma)) - g_2(y_0(T), y_0(\gamma))] \\ &\quad + y_0(T) - z_0(T) \\ &\leq z_0(T) - y_0(T) + y_0(T) - z_0(T) = 0. \end{aligned}$$

Thus, by Lemma 1, we can get $y_1(t) \leq z_1(t), \ \forall t \in J$. So, we prove (3.4).

Now, we prove that y_1, z_1 are lower and upper solutions of (1.1), respectively. By (H_2), we have

$$\begin{aligned} y_1'' &= Fy_0(t) - Fy_1(t) + Fy_1(t) + M(t)[y_1(t) - y_0(t)] + N(t)[y_1(\alpha(t)) - y_0(\alpha(t))] \\ &\geq Fy_1(t) - M(t)[y_1(t) - y_0(t)] - N(t)[y_1(\alpha(t)) - y_0(\alpha(t))] \\ &\quad + M(t)[y_1(t) - y_0(t)] + N(t)[y_1(\alpha(t)) - y_0(\alpha(t)) \\ &= Fy_1(t), \end{aligned}$$

and by (3.2) and (H_3), we have

$$\begin{aligned}
&\frac{1}{a}\int_0^T D_1(s)y_0(s)ds\\
=\ & y_1(0)-y_0(0)+\frac{1}{a}[g_1(y_0(0),y_0(\delta))-g_1(y_1(0),y_1(\delta))+g_1(y_1(0),y_1(\delta))]\\
\geq\ & y_1(0)-y_0(0)-y_1(0)+y_0(0)+\frac{1}{a}g_1(y_1(0),y_1(\delta))\\
=\ & \frac{1}{a}g_1(y_1(0),y_1(\delta)),
\end{aligned}$$

which implies $g_1(y_1(0),y_1(\delta)) \leq \int_0^T D_1(s)y_0(s)ds \leq \int_0^T D_1(s)y_1(s)ds$. Similarly, we have $g_2(y_1(T),y_1(\gamma)) \leq \int_0^T D_2(s)y_1(s)ds$. Hence, y_1 is a lower solution of (1.1). Obviously, it is easy to show z_1 is an upper solution of (1.1).

By induction, for $\forall t \in J$, we can obtain the relation

$$y_0(t) \leq y_1(t) \leq \cdots \leq y_n(t) \leq z_n(t) \leq \cdots \leq z_1(t) \leq z_0(t),\ n=1,2,\cdots.$$

Using the standard argument, we have

$$\lim_{n\to\infty} y_n(t) = y(t),\ \lim_{n\to\infty} z_n(t) = z(t)$$

uniformly on $t \in J$, and the limit functions y, z satisfy (1.1). Moreover, $y, z \in [y_0, z_0]$.

Now we show that y is the minimal solution of (1.1) and z is the maximal solution of (1.1). To show it, we assume that u is any solution of problem (1.1) such that $u \in [y_0, z_0]$. Let $y_n(t) \leq u(t) \leq z_n(t), \forall t \in J$, for some positive integer n. Put $p = y_{n+1} - u$. We have

$$\begin{aligned}
p''(t) &= Fy_n(t)+M(t)[y_{n+1}(t)-y_n(t)]+N(t)[y_{n+1}(\alpha(t))-y_n(\alpha(t))]-Fu(t)\\
&\geq -M(t)[u(t)-y_n(t)]-N(t)[u(\alpha(t))-y_n(\alpha(t))]\\
&\quad +M(t)[y_{n+1}(t)-y_n(t)]+N(t)[y_{n+1}(\alpha(t))-y_n(\alpha(t))]\\
&= M(t)p(t)+N(t)p(\alpha(t)),\\
p(0) &= \frac{1}{a}\int_0^T D_1(s)(y_n-u)(s)ds+\frac{1}{a}[g_1(u(0),u(\delta))-g_1(y_n(0),y_n(\delta))]+y_n(0)-u(0)\\
&\leq \frac{1}{a}[g_1(u(0),u(\delta))-g_1(y_n(0),u(\delta))]+y_n(0)-u(0)\\
&\leq u(0)-y_n(0)+y_n(0)-u(0)=0,
\end{aligned}$$

Similarly, we have $p(T) \leq 0$. By Lemma 1, we have $y_{n+1}(t) \leq u(t),\ \forall t \in J$. Obviously, it is easy to show $u(t) \leq z_{n+1}(t),\ \forall t \in J$. That is $y_{n+1}(t) \leq u(t) \leq z_{n+1}(t),\ \forall t \in J$.

Now, if $n \to \infty$, then

$$y_0(t) \leq y(t) \leq u(t) \leq z(t) \leq z_0(t),\ \forall t \in J.$$

That is, y, z are extremal solutions of (1.1) in the sector $[y_0, z_0]$.

The proof of the theorem is complete. □

Remark 1. In the special case where f does not contain the deviating argument $x(\beta(t,\mu(t)))$, by setting $D_1(t) = D_2(t) \equiv 0$, $g_1(x(0), x(\delta)) = x(0)$ and $g_2(x(T), x(\gamma)) = x(T) - rx(\gamma)$, Theorem 1 develops and generalizes Theorem 3 and Theorem 5 of T.Jankowski [3]. It is worth pointing out that we obtain the solutions of (1.1) in Theorem 1. But, in Theorem 5 of T.Jankowski [3], the author do not obtain the solutions of (1.1).

Remark 2. Using the same approach as in Theorem 1, we can also improve Theorem 7 of T.Jankowski [3] and Theorem 2 of W.Szatanik [14], and obtain the solutions of the corresponding problems. But, in Theorem 7 of T.Jankowski [3] and Theorem 2 of W.Szatanik [14], the authors do not obtain the solutions of the corresponding problems.

Corollary 1. Let all assumptions of Theorem 1 hold with $D_i(i = 1,2) = 0$ on J. Assume that $g_i(i = 1,2)$ do not depend on the last argument. In addition assume that

(H_4) : There exist bounded integrable functions $0 < \bar{M}(t) \le M(t)$, $0 \le \bar{N}(t) \le N(t)$ such that

$$f(t,\bar{u},\bar{v},\bar{w}) - f(t,u,v,w) \ge \bar{M}(t)(\bar{u}-u) + \bar{N}(t)(\bar{v}-v),$$

for $y_0(t) \le u \le \bar{u} \le z_0(t)$, $y_0(\alpha(t)) \le v \le \bar{v} \le z_0(\alpha(t))$, $\lambda y_0(\beta(t,\mu(t))) \le w \le \bar{w} \le \lambda z_0(\beta(t,\mu(t)))$.

(H_5) : there exist constants $0 < \bar{a} \le a, 0 < \bar{b} \le b$ such that

$$g_1(\bar{u}_1) - g_1(u_1) \ge \bar{a}(\bar{u}_1 - u_1),\ g_2(\bar{u}_2) - g_2(u_2) \ge \bar{b}(\bar{u}_2 - u_2),$$

for $y_0(0) \le u_1 \le \bar{u}_1 \le z_0(0)$, $y_0(T) \le u_2 \le \bar{u}_2 \le z_0(T)$.
Then (1.1) has a unique solution in the sector $[y_0, z_0]$.

Proof. From Theorem 1, we know that $y, z \in [y_0, z_0]$, and $y(t) \le z(t)$, $\forall t \in J$. We need to show that $y = z$. Put $p = z - y$. Then, by (H_4) and (H_5), we have

$$\begin{aligned}
p''(t) &= z''(t) - y''(t) \\
&\ge \bar{M}(t)[z(t) - y(t)] + \bar{N}(t)[z(\alpha(t)) - y(\alpha(t))] \\
&= \bar{M}(t)p(t) + \bar{N}(t)p(\alpha(t)), \\
p(0) &= \frac{1}{\bar{a}}[g_1(y(0)) - g_1(z(0))] + z(0) - y(0) \\
&\le y(0) - z(0) + z(0) - y(0) = 0, \\
p(T) &= \frac{1}{\bar{b}}[g_2(y(T)) - g_2(z(T))] + z(T) - y(T) \\
&\le y(T) - z(T) + z(T) - y(T) = 0,
\end{aligned}$$

By Lemma 1, we have $p(t) \le 0$, $\forall t \in J$, which implies $z(t) \le y(t)$, $\forall t \in J$. It proves that $y(t) = z(t)$, $\forall t \in J$. Therefore, (1.1) has a unique solution in the sector $[y_0, z_0]$.

This completes the proof of Corollary 1. □

Remark 3. The significance of Corollary 1 lies in obtaining the unique solution.

4. Applications

Example 1. Consider the following boundary value problems:

$$\begin{cases} x''(t) = \frac{t}{15}[x(t)-t]^3 + \frac{1}{10}x^2(t^2) - \frac{1}{10}x(te^{-t^3}) \equiv Fx(t),\ t \in J = [0,1], \\ g_1(x(0), x(\frac{1}{3})) \equiv x(0) - \frac{1}{2}x(\frac{1}{3}) = \int_0^1 sx(s)ds, \\ g_2(x(1), x(\frac{1}{5})) \equiv x(1) - \frac{1}{3}x(\frac{1}{5}) = \int_0^1 s^2x(s)ds, \end{cases} \tag{10}$$

where $\alpha(t) = t^2,\ \beta(t,\mu(t)) = te^{-t^3},\ \delta = \frac{1}{3}.\ \gamma = \frac{1}{5},\ \lambda = \frac{1}{10},\ D_1(t) = t,\ D_2(t) = t^2$.

Note that condition (H_1) holds.

Take $y_0(t) = 0,\ z_0(t) = 1,\ \forall t \in J$. Then

$$\begin{cases} Fy_0(t) = -\frac{t^4}{15} \le 0 = y_0''(t), \\ g_1(y_0(0), y_0(\frac{1}{3})) = g_1(0,0) = 0 = \int_0^1 sy_0(s)ds, \\ g_2(y_0(1), y_0(\frac{1}{5})) = g_2(0,0) = 0 = \int_0^1 s^2y_0(s)ds, \end{cases}$$

and

$$\begin{cases} Fz_0(t) = \frac{t}{15}(1-t)^3 \ge 0 = z_0''(t), \\ g_1(z_0(0), z_0(\frac{1}{3})) = g_1(1,1) = \frac{1}{2} = \int_0^1 sz_0(s)ds, \\ g_2(z_0(1), z_0(\frac{1}{5})) = g_2(1,1) = \frac{2}{3} > \frac{1}{3} = \int_0^1 s^2z_0(s)ds. \end{cases}$$

Therefore, y_0, z_0 are lower and upper solutions of problem (4.1).

For $M(t) = \frac{t}{5},\ N(t) = \frac{1}{5},\ a = b = 1$, it is difficult to verify that conditions $(H_2), (H_3)$ hold. Hence, all assumptions of Theorem 1 hold. By Theorem 1, (4.1) has extremal solutions in the sector $[y_0, z_0]$.

Remark 4. By Theorem 1, we obtain the conclusion of Example 1, which can not be obtained by the corresponding results in [3, 14].

Example 2. Consider the following boundary value problems:

$$\begin{cases} x''(t) = \frac{1}{5}x(t) + \frac{t}{15}[x(t)-t]^3 + \frac{1}{4}x(\alpha(t)) + \frac{t^2}{20}[x(\alpha(t))-t]^5 \equiv Fx(t),\ t \in J = [0,1], \\ 0 = \mu_1[x(0) + x^2(0)] - d_1 \equiv g_1(x(0)), \\ 0 = \mu_2[x(1) + x^3(1)] - d_2 \equiv g_2(x(1)), \end{cases} \tag{11}$$

where $\mu_i \ge 0, d_i \ge 0, 2\mu_i - d_i \ge 0 (i = 1,2),\ \alpha \in C(J,J)$ and $\alpha(t) \le t$ on J.

Take $y_0(t) = 0,\ z_0(t) = 1,\ \forall t \in J$. Then,

$$\begin{cases} Fy_0(t) = -\frac{t^4}{15} - \frac{t^7}{20} \le 0 = y_0''(t), \\ g_1(y_0(0)) = g_1(0) = -d_1 \le 0, \\ g_2(y_0(1)) = g_2(0) = -d_2 \le 0, \end{cases}$$

and

$$\begin{cases} Fz_0(t) = \frac{1}{5} + \frac{t}{15}(1-t)^3 + \frac{1}{4} + \frac{t^2}{20}(1-t)^5 > 0 = z_0''(t), \\ g_1(z_0(0)) = g_1(1) = 2\mu_1 - d_1 \ge 0, \\ g_2(z_0(1)) = g_2(1) = 2\mu_2 - d_2 \ge 0. \end{cases}$$

Therefore, y_0, z_0 are lower and upper solutions of problem (4.2).

Put $M(t) = \frac{1+t}{5},\ N(t) = \frac{1+t^2}{4},\ a = 3\mu_1,\ b = 4\mu_2$, it is easy to see that conditions $(H_2), (H_3)$ hold. Hence, all assumptions of Theorem 1 hold. By Theorem 1, (4.2) has extremal solutions in the sector $[y_0, z_0]$.

Now, we show that all assumptions of Corollary 1 hold. For $\bar{M}(t) = \frac{1}{5},\ \bar{N}(t) = \frac{1}{4},\ \bar{a} = \mu_1,\ \bar{b} = \mu_2$, obviously, $(H_4), (H_5)$ hold. By Corollary 1, (4.2) has a unique solution in the sector $[y_0, z_0]$.

Remark 5. By Corollary 1, we obtain the unique solution of (4.2), so, our result improves the corresponding results in [3].

Acknowledgment

The authors thanks the referee for his/her careful reading of the manuscript and useful suggestions.

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