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THE USE OF ITERATIVE METHODS FOR SOLVING NAVEIR-STOKES EQUATION

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Abstract. In this paper, a Naveir-Stokes equation is solved by using the Adomian's decomposition method (ADM) , modified Adomian's decomposition method (MADM), variational iteration method (VIM), modified variational iteration method (MVIM), modified homotopy perturbation method (MHPM) and homotopy analysis method (HAM). The approximate solution of this equation is calculated in the form of series which its components are computed by applying a recursive relation. The existence and uniqueness of the solution and the convergence of the proposed methods are proved. A numerical example is studied to demonstrate the accuracy of the presented methods.

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1. Introduction

Naveir-Stokes equation playes an important role in mathematical physics. A lot of works have been done in order to find the numerical solution of this equation. For example, finite analytic numerical solution of Naveir-Stokes equations [22], numerical solution of the Naveir-Stokes equations using variational iteration methods [4], numerical solution of the Naveir-Stokes equations for the flow a cylinder cascade [9], analytical solution of a time-fractional Naveir-Stokes equation by Adomian decomposition method [18], using divergence free wavelets for the numerical solution of the 2-D stationary Naveir-Stokes equations [23], on the generalized Naveir-Stokes equations [6]. In this work, we develop the ADM, MADM, VIM, MVIM, MHPM and HAM to solve the Naveir-Stokes equation as follows:

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$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\rho \partial z} + \nu(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}), \tag{1}$$

where t is the time, p is the pressure, ν is the kinematics viscosity and ρ is the density.

With the initial condition given by:

$$u(r, 0) = f(r). \tag{2}$$

The paper is organized as follows. In section 2, the mentioned iterative methods are introduced for solving Eq.(1). Also, the existence and uniqueness of the solution and convergence of the proposed method are proved in section 3. Finally, the numerical example is presented in section 4 to illustrate the accuracy of these methods.

To obtain the approximate solution of Eq.(1), by integrating one time from Eq.(1) with respect to t and using the initial condition we obtain,

$$u(r, t) = f(r) + \int_0^t \frac{D(p(z, t))}{\rho}\, dt + \int_0^t \nu(D^2(u(r, t)) + \frac{1}{r}D(u(r, t)))\, dt, \tag{3}$$

where,

$$D(p(z, t)) = \tfrac{\partial p}{\partial z},$$

$$D^i(u(r, t)) = \tfrac{\partial^i u(r,t)}{\partial r^i}, \quad i = 1, 2.$$

In Eq.(3), we assume $f(r)$ is bounded for all r in $J = [0, T](T \in \mathbb{R})$.

The terms $D^i(u(r, t)) = \frac{\partial^i u(r,t)}{\partial r^i}$ are Lipschitz continuous with $\mid D^i(u) - D^i(u^*) \mid \leq L_i \mid u - u^* \mid$, $\mid D(p) - D(p^*) \mid \leq L \mid p - p^* \mid$ and

$$\alpha = T(\mid \nu \mid (L_1 + TL_2)),$$

$$\beta = 1 - T(1 - \alpha).$$

We set,

$$G(r, t) = f(r) + \int_0^t \frac{D(p(r, t))}{\rho}\, dt.$$

2. Iterative methods

2.1.Description of the MADM and ADM. The Adomian decomposition method is applied to the following general nonlinear equation

$$Lu + Ru + Nu = g(r, t), \tag{4}$$

where $u(r, t)$ is the unknown function, L is the highest order derivative operator which is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms, and g is the source term. Applying the inverse operator L^{-1} to both sides of Eq.(4), and using the given conditions we obtain

$$u(r,t) = f(r) - L^{-1}(Ru) - L^{-1}(Nu), \tag{5}$$

where the function $f(r)$ represents the terms arising from integrating the source term $g(r,t)$. The nonlinear operator $Nu = G_1(u)$ is decomposed as

$$G_1(u) = \sum_{n=0}^{\infty} A_n, \tag{6}$$

where $A_n,\ n \geq 0$ are the Adomian polynomials determined formally as follows :

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty} \lambda^i u_i)]]_{\lambda=0}. \tag{7}$$

Adomian polynomials were introduced in [5,8,20] as

$$\begin{aligned} A_0 &= G_1(u_0), \\ A_1 &= u_1 G_1'(u_0), \\ A_2 &= u_2 G_1'(u_0) + \frac{1}{2!} u_1^2 G_1''(u_0), \\ A_3 &= u_3 G_1'(u_0) + u_1 u_2 G_1''(u_0) + \frac{1}{3!} u_1^3 G_1'''(u_0), ... \end{aligned} \tag{8}$$

2.1.1.Adomian decomposition method. The standard decomposition technique represents the solution of $u(r,t)$ in (4) as the following series,

$$u(r,t) = \sum_{i=0}^{\infty} u_i(r,t), \tag{9}$$

where, the components $u_0, u_1, \ldots$ are usually determined recursively by

$$\begin{aligned} u_0 &= G(r,t) \\ u_1 &= \int_0^t \nu(L_0(r,t) + \frac{1}{r} A_0(r,t))\ dt, \\ &\vdots \\ u_{n+1} &= \int_0^t \nu(L_n(r,t) + \frac{1}{r} A_n(r,t))\ dt \quad n \geq 0. \end{aligned} \tag{10}$$

Substituting (8) into (10) leads to the determination of the components of u. Having determined the components $u_0, u_1, \ldots$ the solution u in a series form defined by (9) follows immediately.

2.1.2.The modified Adomian decomposition method. The modified decomposition method was introduced by Wazwaz [21]. The modified forms was established based on the assumption that the function $G(r,t)$ can be divided into two parts, namely $G_1(r,t)$ and $G_2(r,t)$. Under this assumption we set

$$G(r,t) = G_1(r,t) + G_2(r,t). \tag{11}$$

Accordingly, a slight variation was proposed only on the components u_0 and u_1. The suggestion was that only the part G_1 be assigned to the zeroth component u_0, whereas the remaining part G_2 be combined with the other terms given in (10) to define u_1. Consequently, the modified recursive relation

$$\begin{aligned} u_0 &= G_1(r,t), \\ u_1 &= G_2(r,t) - L^{-1}(Ru_0) - L^{-1}(A_0), \\ &\vdots \\ u_{n+1} &= -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \geq 1, \end{aligned} \tag{12}$$

was developed.

To obtain the approximation solution of Eq.(1), according to the MADM, we can write the iterative formula (12) as follows:

$$\begin{aligned} &u_0(r,t) = G_1(r,t), \\ &u_1(r,t) = G_2(r,t) + \int_0^t \nu(L_0(r,t) + \tfrac{1}{r}A_0(r,t))\ dt \\ &\vdots \\ &u_{n+1}(r,t) = \int_0^t \nu(L_n(r,t) + \tfrac{1}{r}A_n(r,t)))\ dt. \end{aligned} \tag{13}$$

The operator $D^i(u(r,t)), i = 1, 2$ are usually represented by the infinite series of the Adomian polynomials as follows:

$$D(u) = \sum_{i=0}^{\infty} A_i,$$

$$D^2(u) = \sum_{i=0}^{\infty} L_i,$$

where $A_i, L_i (i \geq 0)$ are the Adomian polynomials.

Also, we can use the following formula for the Adomian polynomials [7]:

$$\begin{aligned} A_n &= D(s_n) - \textstyle\sum_{i=0}^{n-1} A_i, \\ L_n &= D^2(s_n) - \textstyle\sum_{i=0}^{n-1} L_i. \end{aligned} \tag{14}$$

Where the partial sum is $s_n = \sum_{i=0}^{n} u_i(r,t)$.

2.2.Description of the VIM and MVIM. In the VIM [11-14], we consider the following nonlinear differential equation:

$$L(u(r,t)) + N(u(r,t)) = g(r,t), \tag{15}$$

where L is a linear operator,N is a nonlinear operator and $g(r,t)$ is a known analytical function. In this case, a correction functional can be constructed as follows:

$$u_{n+1}(r,t) = u_n(r,t) + \int_0^t \lambda(r,\tau)\{L(u_n(r,\tau)) + N(u_n(r,\tau)) - g(r,\tau)\}d\tau,\ n \geq 0, \tag{16}$$

where λ is a general Lagrange multiplier which can be identified optimally via variational theory. Here the function $u_n(r,\tau)$ is a restricted variations which means $\delta u_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximation $u_n(r,t),\ n \geq 0$ of the solution $u(r,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The zeroth approximation u_0 may be selected any function that just satisfies at least the initial and boundary conditions. With λ determined, then several approximation $u_n(r,t),\ n \geq 0$ follow immediately. Consequently, the exact solution may be obtained by using

$$u(r,t) = \lim_{n\to\infty} u_n(r,t). \tag{17}$$

The VIM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converge rapidly to accurate solutions.

To obtain the approximation solution of Eq.(1), according to the VIM, we can write iteration formula (16) as follows:

$$\begin{aligned} &u_{n+1}(r,t) = u_n(r,t) + L_t^{-1}(\lambda[u_n(r,t) - G(r,t) - \\ &\textstyle\int_0^t \nu(D^2(u_n(r,t)) + \frac{1}{r}D(u_n(r,t)))\ dt]), \end{aligned} \tag{18}$$

where,

$$L_t^{-1}(.) = \int_0^t (.)\ d\tau$$

To find the optimal λ, we proceed as

$$\begin{aligned} &\delta u_{n+1}(r,t) = \delta u_n(r,t) + \delta L_t^{-1}(\lambda[u_n(r,t) \\ &-G(r,t) - \textstyle\int_0^t \nu(D^2(u_n(r,t)) + \frac{1}{r}D(u_n(r,t)))\ dt]). \end{aligned} \tag{19}$$

From Eq.(19), the stationary conditions can be obtained as follows:
$\lambda' = 0$ and $1 + \lambda' = 0$.

Therefore, the Lagrange multipliers can be identified as $\lambda = -1$ and by substituting in (18), the following iteration formula is obtained.

$$
\begin{aligned}
&u_0(r,t) = G(r,t),\\
&u_{n+1}(r,t) = u_n(r,t) - L_t^{-1}(u_n(r,t) - G(r,t)\\
&-\int_0^t \nu(D^2(u_n(r,t)) + \tfrac{1}{r}D(u_n(r,t)))\ dt, n \geq 0.
\end{aligned} \tag{20}
$$

To obtain the approximation solution of Eq.(1), based on the MVIM [1,2,19], we can write the following iteration formula:

$$
\begin{aligned}
&u_0(r,t) = G(r,t),\\
&u_{n+1}(r,t) = u_n(r,t) - L_t^{-1}(-\int_0^t \nu(D^2(u_n(r,t) - u_{n-1}(r,t))\\
&+\tfrac{1}{r}D(u_n(r,t) - u_{n-1}(r,t)))\ dt), n \geq 0.
\end{aligned} \tag{21}
$$

Relations (20) and (21) will enable us to determine the components $u_n(r,t)$ recursively for $n \geq 0$.

2.3.Description of the HAM. Consider

$$N[u] = 0,$$

where N is a nonlinear operator, $u(r,t)$ is unknown function and r is an independent variable. let $u_0(r,t)$ denote an initial guess of the exact solution $u(r,t)$, $h \neq 0$ an auxiliary parameter, $H(r,t) \neq 0$ an auxiliary function, and L an auxiliary nonlinear operator with the property $L[s(r,t)] = 0$ when $s(r,t) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we construct a homotopy as follows:

$$(1-q)L[\phi(r,t;q) - u_0(r,t)] - qhH(r,t)N[\phi(r,t;q)] = \hat{H}[\phi(r,t;q); u_0(r,t), H(r,t), h, q]. \tag{22}$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(r,t)$, the auxiliary nonlinear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H(r,t)$.

Enforcing the homotopy (22) to be zero, i.e.,

$$\hat{H}[\phi(r,t;q); u_0(r,t), H(r,t), h, q] = 0, \tag{23}$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(r,t;q) - u_0(r,t)] = qhH(r,t)N[\phi(r,t;q)]. \tag{24}$$

When $q = 0$, the zero-order deformation Eq.(24) becomes

$$\phi(r;0) = u_0(r,t), \tag{25}$$

and when $q = 1$, since $h \neq 0$ and $H(r,t) \neq 0$, the zero-order deformation Eq.(24) is equivalent to

$$\phi(r,t;1) = u(r,t). \tag{26}$$

Thus, according to (25) and (26), as the embedding parameter q increases from 0 to 1, $\phi(r,t;q)$ varies continuously from the initial approximation $u_0(r,t)$ to the exact solution $u(r,t)$. Such a kind of continuous variation is called deformation in homotopy [16,17].

Due to Taylor's theorem, $\phi(r,t;q)$ can be expanded in a power series of q as follows

$$\phi(r,t;q) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t)q^m, \tag{27}$$

where

$$u_m(r,t) = \frac{1}{m!}\frac{\partial^m \phi(r,t;q)}{\partial q^m}\ |_{q=0}\ .$$

Let the initial guess $u_0(r,t)$, the auxiliary nonlinear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H(r,t)$ be properly chosen so that the power series (27) of $\phi(r,t;q)$ converges at $q = 1$, then, we have under these assumptions the solution series

$$u(r,t) = \phi(r,t;1) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t). \tag{28}$$

From Eq.(27), we can write Eq.(24) as follows

$$\begin{aligned}&(1-q)L[\phi(r,t,q) - u_0(r,t)] = (1-q)L[\textstyle\sum_{m=1}^{\infty} u_m(r,t)\ q^m]\\ &= q\ h\ H(r,t)N[\phi(r,t,q)] \Rightarrow L[\textstyle\sum_{m=1}^{\infty} u_m(r,t)\ q^m] - q\ L[\textstyle\sum_{m=1}^{\infty} u_m(r,t)q^m]\\ &= q\ h\ H(r,t)N[\phi(r,t,q)]\end{aligned} \tag{29}$$

By differentiating (29) m times with respect to q, we obtain

$$\begin{aligned}&\{L[\textstyle\sum_{m=1}^{\infty} u_m(r,t)\ q^m] - q\ L[\textstyle\sum_{m=1}^{\infty} u_m(r,t)q^m]\}^{(m)}\\ &= \{q\ h\ H(r,t)N[\phi(r,t,q)]\}^{(m)} = m!\ L[u_m(r,t) - u_{m-1}(r,t)]\\ &= h\ H(r,t)\ m\ \tfrac{\partial^{m-1} N[\phi(r,t;q)]}{\partial q^{m-1}}\ |_{q=0}\ .\end{aligned}$$

Therefore,

$$\begin{aligned}&L[u_m(r,t) - \chi_m u_{m-1}(r,t)] = hH(r,t)\Re_m(u_{m-1}(r,t)),\\ &u_m(0) = 0,\end{aligned} \tag{30}$$

where,

$$\Re_m(u_{m-1}(r,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(r,t;q)]}{\partial q^{m-1}}\ |_{q=0}, \tag{31}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

Note that the high-order deformation Eq.(30) is governing the nonlinear operator L, and the term $\Re_m(u_{m-1}(r,t))$ can be expressed simply by (31) for any nonlinear operator N.

To obtain the approximation solution of Eq.(1), according to HAM, let

$$N[u(r,t)] = u(r,t) - G(r,t) - \int_0^t \nu(D^2(u(r,t)) + \tfrac{1}{r}D(u(r,t)))\ dt,$$

so

$$\Re_m(u_{m-1}(r,t)) = u_{m-1}(r,t) - \int_0^t \nu(D^2(u_{m-1}(r,t)) + \tfrac{1}{r}D(u_{m-1}(r,t)))\ dt, \tag{32}$$

Substituting (32) into (30)

$$L[u_m(r,t) - \chi_m u_{m-1}(r,t)] = hH(r,t)[u_{m-1}(r,t) - \int_0^t \nu(D^2(u_{m-1}(r,t)) + \tfrac{1}{r}D(u_{m-1}(r,t)))\ dt - (1-\chi_m)G(r,t)]. \tag{33}$$

We take an initial guess $u_0(r,t) = G(r,t)$, an auxiliary nonlinear operator $Lu = u$, a nonzero auxiliary parameter $h = -1$, and auxiliary function $H(r,t) = 1$. This is substituted into (33) to give the recurrence relation

$$\begin{aligned} &u_0(r,t) = G(r,t), \\ &u_n(r,t) = \int_0^t \nu(D^2(u_n(r,t)) + \tfrac{1}{r}D(u_n(r,t)))\ dt, \quad n \geq 1. \end{aligned} \tag{34}$$

Therefore, the solution $u(r,t)$ becomes

$$u(r,t) = \sum_{n=0}^{\infty} u_n(r,t) = G(r,t) + \sum_{n=1}^{\infty} \left(\int_0^t \nu(D^2(u_n(r,t)) + \tfrac{1}{r}D(u_n(r,t)))dt \right). \tag{35}$$

Which is the method of successive approximations. If

$$| u_n(r,t) | < 1,$$

then the series solution (35) convergence uniformly.

2.4.Description of the MHPM. To explain MHPM, we consider Eq. (1) as

$$L(u) = u(r,t) - G(r,t) - \int_0^t \nu(D^2(u(r,t)) + \frac{1}{r}D(u(r,t)))\ dt.$$

We can define homotopy $H(u(r,t),p,m)$ by

$$H(u(r,t),o,m) = f(u(r,t)), \quad H(u(r,t),1,m) = L(u(r,t)).$$

Where m is an unknown real number and

$$f(u(r,t)) = u(r,t) - G(r,t).$$

Typically we may choose a convex homotopy by

$$\begin{aligned} H(u(r,t),p,m) = (1-p)f(u(r,t)) + pL(u(r,t)) \\ +p(1-p)[m(F(u(r,t)))] = 0,\ 0 \le p \le 1. \end{aligned} \tag{36}$$

where m is called the accelerating parameters, and for $m = 0$ we define $H(u(r,t),p,0) = H(u(r,t),p)$, which is the standard HPM.

The convex homotopy (36) continuously trace an implicity defined curve from a starting point $H(u(r,t)-f(u(r,t)),0,m)$ to a solution function $H(u(r,t),1,m)$. The embedding parameter p monotonically increase from o to 1 as trivial problem $f(u(r,t)) = 0$ is continuously deformed to original problem $L(u(r,t)) = 0$. [3,15,10]

The MHPM uses the homotopy parameter p as an expanding parameter to obtain

$$v = \sum_{n=0}^{\infty} p^n u_n(r,t), \tag{37}$$

when $p \longrightarrow 1$, Eq. (37) becomes the approximate solution of Eq. (1), i.e.,

$$u = \lim_{p\to 1} v = \sum_{n=0}^{\infty} u_n(r,t), \tag{38}$$

where,

$$u_n(r,t) = G(r,t) + \int_0^t \nu(D^2(u_n(r,t)) + \frac{1}{r}D(u_n(r,t)))\ dt. \tag{39}$$

3. Existence and convergency of iterative methods

Theorem 1. *Let $0 < \alpha < 1$, then nonlinear Naveir-Stokes equation (1), has a unique solution.*

Proof. *Let u and u^* be two different solutions of (3) then*

$\mid u - u^* \mid = \mid \int_0^t \nu(D^2(u_n(r,t)) + \frac{1}{r}D(u_n(r,t)))\ dt \mid$
$\le \int_0^t \mid \nu \mid\mid D^2(u(t)) - D^2(u^*(t)) \mid\ dt + \int_0^t \mid \frac{\nu}{r} \mid\mid D(u(r,t)) - D(u^*(r,t)) \mid\ dt$
$\le T(\mid \nu \mid (L_1 + TL_2))\ \mid u - u^* \mid = \alpha \mid u - u^* \mid$

From which we get $(1-\alpha) \mid u - u^* \mid \le 0$. Since $0 < \alpha < 1$. then $\mid u - u^* \mid = 0$. Implies $u = u^*$ and completes the proof. □

Theorem 2. *The series solution $u(r,t) = \sum_{i=0}^{\infty} u_i(r,t)$ of problem(1) using MADM convergence when $0 < \alpha < 1$, $\mid u_1(r,t) \mid < \infty$.*

Proof. *Denote as $(C[J], \| . \|)$ the Banach space of all continuous functions on J with the norm $\| f(t) \| = max \mid f(t) \mid$, for all t in J. Define the sequence of partial sums s_n, let s_n and s_m be arbitrary partial sums with $n \ge m$. We are going to prove that s_n is a Cauchy sequence in this Banach space:*

$$\| s_n - s_m \| = max_{\forall t\in J} \mid s_n - s_m \mid = max_{\forall t\in J} \mid \sum_{i=m+1}^{n} u_i(r,t) \mid$$
$$= max_{\forall t\in J} \mid \sum_{i=m+1}^{n} (\int_0^t \nu L_i \ dt + \int_0^t \frac{\nu}{r} \ A_i \ dt) \mid$$
$$= max_{\forall t\in J} \mid \int_0^t \ \nu(\sum_{i=m}^{n-1} L_i) \ dt + \int_0^t \ \frac{\nu}{r}(\sum_{i=m}^{n-1} A_i) \ dt \mid .$$

From [7], we have

$$\sum_{i=m}^{n-1} A_i = D(s_{n-1} - s_{m-1}),$$
$$\sum_{i=m}^{n-1} L_i = D^2(s_{n-1} - s_{m-1})$$

So,

$$\| s_n - s_m \| = max_{\forall t\in J} \mid \int_0^t \nu[D^2(s_{n-1} - s_{m-1})] \ dt + \int_0^t \frac{\nu}{r}[D(s_{n-1} - s_{m-1})] \ dt \mid \leq \int_0^t \mid \nu \mid\mid D^2(s_{n-1} - s_{m-1}) \mid \ dt + \int_0^t \mid \frac{\nu}{r} \mid\mid D(s_{n-1} - s_{m-1}) \mid \ dt \leq \alpha \| s_n - s_m \| .$$

Let $n = m + 1$*, then*

$$\| s_n - s_m \| \leq \alpha \| s_m - s_{m-1} \| \leq \alpha^2 \| s_{m-1} - s_{m-2} \| \leq ... \leq \alpha^m \| s_1 - s_0 \| .$$

From the triangle inquality we have

$$\| s_n - s_m \| \leq \| s_{m+1} - s_m \| + \| s_{m+2} - s_{m+1} \| + ... + \| s_n - s_{n-1} \|$$
$$\leq [\alpha^m + \alpha^{m+1} + ... + \alpha^{n-m-1}] \| s_1 - s_0 \|$$
$$\leq \alpha^m[1 + \alpha + \alpha^2 + ... + \alpha^{n-m-1}] \| s_1 - s_0 \| \leq [\frac{1-\alpha^{n-m}}{1-\alpha}] \| u_1(r,t) \| .$$

Since $0 < \alpha < 1$*, we have* $(1 - \alpha^{n-m}) < 1$*, then*

$$\| s_n - s_m \| \leq \frac{\alpha^m}{1-\alpha} max_{\forall t\in J} \mid u_1(r,t) \mid .$$

But $\mid u_1(r,t) \mid < \infty$ *, so, as* $m \to \infty$*, then* $\| s_n - s_m \| \to 0$*. We conclude that* s_n *is a Cauchy sequence in* $C[J]$*, therefore the series is convergence and the proof is complete.* □

Theorem 3. *The solution* $u_n(r,t)$ *obtained from the relation (21) using VIM converges to the exact solution of the problem (1) when* $0 < \alpha < 1$ *and* $0 < \beta < 1$*.*

Proof.

$$u_{n+1}(r,t) = u_n(r,t) - L_t^{-1}(([u_n(r,t) - G(r,t) - \int_0^t \nu(D^2(u_n(r,t)) + \frac{1}{r}D(u_n(r,t))) \ dt]) \tag{40}$$

$$u(r,t) = u(r,t) - L_t^{-1}([u(r,t) - G(r,t) - \int_0^t \nu(D^2(u(r,t)) + \frac{1}{r}D(u(r,t))) \ dt]) \tag{41}$$

By subtracting relation (45) from (46),

$$u_{n+1}(r,t) - u(r,t) = u_n(r,t) - u(r,t) - L_t^{-1}(u_n(r,t) - u(r,t) - \int_0^t(\nu[D^2(u_n(r,t)) - D^2(u(r,t))] + \frac{\nu}{r}[D(u_n(r,t)) - D(u(r,t))] \ dt),$$

if we set, $e_{n+1}(r,t) = u_{n+1}(r,t) - u_n(r,t)$, $e_n(r,t) = u_n(r,t) - u(r,t)$,$|e_n(r,t^*)| = max_t \mid e_n(r,t) \mid$ *then since* e_n *is a decreasing function with respect to* t *from the mean value theorem we can write,*

$$\begin{aligned}
&e_{n+1}(r,t) = e_n(r,t) + L_t^{-1}(-e_n(r,t) + \textstyle\int_0^t (\nu[D^2(u_n(r,t)) - D^2(u(r,t))] \\
&+\tfrac{\nu}{r}[D(u_n(r,t)) - D(u(r,t))])\ dt) \\
&\le e_n(r,t) + L_t^{-1}[-e_n(r,t) + L_t^{-1} \mid e_n(r,t) \mid (\nu(L_1 + TL_2))] \\
&\le e_n(r,t) - Te_n(r,\eta) + \nu(L_1 + TL_2)L_t^{-1}L_t^{-1} \mid e_n(r,t) \mid \\
&\le (1 - T(1-\alpha) \mid e_n(r,t^*) \mid,
\end{aligned}$$

where $0 \le \eta \le t$. *Hence,* $e_{n+1}(r,t) \le \beta \mid e_n(r,t^*) \mid$.
Therefore,

$$\|e_{n+1}\| = max_{\forall t\in J} \mid e_{n+1} \mid \le \beta\ max_{\forall t \in J} \mid e_n \mid \le \beta \|e_n\|.$$

Since $0 < \beta < 1$, *then* $\|e_n\| \to 0$. *So, the series converges and the proof is complete.* □

Theorem 4. *If the series solution (34) of problem (1) using HAM convergent then it converges to the exact solution of the problem (1).*

Proof. *We assume:*

$$\begin{aligned}
&u(r,t) = \textstyle\sum_{m=0}^{\infty} u_m(r,t), \\
&\widehat{D}(u(r,t)) = \textstyle\sum_{m=0}^{\infty} D(u_m(r,t)), \\
&\widehat{D^2}(u(r,t)) = \textstyle\sum_{m=0}^{\infty} D^2(u_m(r,t)).
\end{aligned}$$

where,

$$\lim_{m\to\infty} u_m(r,t) = 0.$$

We can write,

$$\sum_{m=1}^{n} [u_m(r,t) - \chi_m u_{m-1}(r,t)] = u_1 + (u_2 - u_1) + ... + (u_n - u_{n-1}) = u_n(r,t). \tag{42}$$

Hence, from (42),

$$\lim_{n\to\infty} u_n(r,t) = 0. \tag{43}$$

So, using (43) and the definition of the nonlinear operator L, *we have*

$$\sum_{m=1}^{\infty} L[u_m(r,t) - \chi_m u_{m-1}(r,t)] = L[\sum_{m=1}^{\infty} [u_m(r,t) - \chi_m u_{m-1}(r,t)]] = 0.$$

therefore from (30), we can obtain that,

$$\sum_{m=1}^{\infty} L[u_m(r,t) - \chi_m u_{m-1}(r,t)] = hH(r,t) \sum_{m=1}^{\infty} \Re_{m-1}(u_{m-1}(r,t)) = 0.$$

Since $h \neq 0$ *and* $H(r,t) \neq 0$ *, we have*

$$\sum_{m=1}^{\infty} \Re_{m-1}(u_{m-1}(r,t)) = 0. \tag{44}$$

By substituting $\Re_{m-1}(u_{m-1}(r,t))$ *into the relation (44) and simplifying it , we have*

$$\begin{aligned}&\textstyle\sum_{m=1}^{\infty} \Re_{m-1}(u_{m-1}(r,t)) = \sum_{m=1}^{\infty}[u_{m-1}(r,t)\\ &-\int_0^t \nu(D^2(u_{m-1}(r,t)) + \frac{1}{r}D(u_{m-1}(r,t)))dt - (1-\chi_m)G(r,t)] =\\ &u(r,t) - G(r,t) - \int_0^t \nu(\widehat{D^2}(u(r,t)) + \frac{1}{r}\widehat{D}(u(r,t)))dt.\end{aligned} \tag{45}$$

From (44) and (45), we have

$$u(r,t) = G(r,t) + \int_0^t \nu(\widehat{D^2}(u(r,t)) + \frac{1}{r}\widehat{D}(u(r,t)))dt$$

therefore, $u(r,t)$ *must be the exact solution of Eq.(1).* □

Theorem 5. *If* $| u_m(r,t) | \leq 1$ *, then the series solution (39) of problem (1) converges to the exact solution by using MHPM.*

Proof. *We can write the solution* $u(r,t)$ *as follows:*

$$u(r,t) = \sum_{m=0}^{\infty} u_m(r,t) = \sum_{m=0}^{\infty}(G(r,t) + \int_0^t \nu(D^2(u_{m-1}(r,t)) + \frac{1}{r}D(u_{m-1}(r,t)))dt. \tag{46}$$

If

$$\| D^2(u_m(r,t)) \| < 1,$$

$$\| D(u_m(r,t)) \| < 1.$$

Then the series solution (39) convergence uniformly.
therefore, $u(r,t) = \sum_{m=0}^{\infty} u_m(r,t)$ *must be the exact solution of Eq.(1).* □

4. Numerical example

In this section, we compute a numerical example which is solved by the ADM, MADM, VIM, MVIM, MHPM and HAM. The program has been provided with Mathematica 6 according to the following algorithm. In this algorithm ε is a given positive value.

Algorithm 1:
Step 1. Set $n \leftarrow 0$.
Step 2. Calculate the recursive relation (10) for ADM, (13) for MADM, (34) for HAM and (39) for MHPM.
Step 3. If $| u_{n+1} - u_n | < \varepsilon$ then go to step 4,
else $n \leftarrow n+1$ and go to step 2.
Step 4. Print $u(r,t) = \sum_{i=0}^{n} u_i(r,t)$ as the approximate of the exact solution.

Algorithm 2:
Step 1. Set $n \leftarrow 0$.
Step 2. Calculate the recursive relation (20) for VIM and (21) for MVIM.
Step 3. If $\mid u_{n+1} - u_n \mid < \varepsilon$ then go to step 4,
else $n \leftarrow n+1$ and go to step 2.
Step 4. Print $u_n(r,t)$ as the approximate of the exact solution.

Example 1. *Consider the Naveir-Stokes equation as follows:*

$$\frac{\partial u}{\partial t} = \frac{1}{4}(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}),$$

subject to the initial condition:

$$u(r,0) = r^2.$$

With the exact solution is $u(r,t) = r^2 + t$, $\alpha = 0.3$, $\beta = 0.9$, $\epsilon = 10^{-2}$.

Table 1. Numerical results for Example 1 ($r = 0.02$)

t	Errors					
	ADM(n=8)	MADM(n=5)	VIM(n=4)	MVIM(n=3)	MHPM(n=3)	HAM(n=4)
0.02	0.0725267	0.0654478	0.0624865	0.0348465	0.0432261	0.0538867
0.05	0.0741196	0.0654478	0.0642581	0.0437432	0.0488459	0.0563215
0.07	0.0745569	0.0676829	0.0643427	0.0424038	0.0487765	0.0568456
0.1	0.0762653	0.0701516	0.0682345	0.0556712	0.05912643	0.0601744

Table 1 shows that, approximate solution of the nonlinear Naveir-Stokes equation is convergence with 3 iterations by using the MVIM. By comparing the results of table 1 , we can observe that the MVIM is more rapid convergence than the ADM, MADM, VIM, MHPM and HAM.

5. Conclusion

The MVIM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which convergent are rapidly to exact solutions. In this work, the MVIM has been successfully employed to obtain the approximate analytical solution of the Naveir-Stokes equation. For this purpose, we showed that the MVIM is more rapid convergence than the ADM, MADM, VIM,MHPM and HAM.

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