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FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPS UNDER E.A. PROPERTY IN FUZZY METRIC SPACES

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Abstract. In this paper, we prove a common fixed point theorem for a pair of weakly compatible maps under E.A. property.

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1. Introduction

It proved a turning point in the development of mathematics when the notion of fuzzy set was introduced by Zadeh [28] which laid the foundation of fuzzy mathematics. Fuzzy set theory has applications in applied sciences such as neural network theory, stability theory, mathematical programming, modeling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication etc. There are many view points of the notion of the metric space in fuzzy topology, see, e.g., Erceg [4], Deng [2], Kaleva and Seikkala [12], Kramosil and Michalek [13], George and Veeramani [5]. For the reader convenience we recall some terminology from the theory of fuzzy metric spaces.

Definition 1.1. A binary operation $*$ on $[0,1]$ is a t-norm (in the sense of Schweizer and Sklar, [10]) if it satisfies the following conditions:
(i) $*$ is associative and commutative;
(ii) $a * 1 = a$ for every $a \in [0,1]$;
(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.

Basic examples are the Lukasiewicz t-norm $T_L, T_L(a,b) = \text{Max}(a+b-1, 0)$, t-norms $T_P, T_P(a,b) = ab$, and the t-norm $T_M, T_M(a,b) = \text{Min}\{a,b\}$.

A t-norm T is said to be of Hadžić-type (denoted $T \in H$) if the family, $(x_T^{(n)})_{n\in N}$, where $(x_T^{(n)})$ is defined for every $x \in [0,1]$ by $(x_T^{(n)}) = x$ if $n = 1$ and $= 0$ and $T((x_T^{(n-1)}, x)$ if $n \geq 2$, is equicontinuous at $x = 1$.).

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Definition 1.2. The 3-tuple $(X, M, *)$ is called a fuzzy metric space ((in the sense of Kramosil and Michalek)) if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:

(FM-1) $M(x, y, 0) = 0$,
(FM-2) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous for all for all $x, y, z \in X$ and $s, t > 0$.

Note that $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$ and $M(x, y, t) = 0$ with $t = 0$. Since $*$ is a continuous t-norm, it follows from (FM-4) that the limit of the sequence in FM-space is uniquely determined.

Definition 1.3. A sequence $(x_n)_{n \in N}$ in X is said to be convergent to $x \in X$ if $\lim_{n \to \infty} M(x_n, x, t) = 1$ for all $t > 0$. The sequence $(x_n)_{n \in N}$ is called G-Cauchy sequence if $\lim_{n \to \infty} M(x_n, x_{n+p}, t) = 1$ for all $t > 0$ and $p \in N$.

A sequence $\{x_n\}$ is said to be M-Cauchy sequence if for every $\lambda \in (0, 1)$ and $t > 0$ there exists a positive integer N such that $M(x_n, x_m, t) > 1 - \lambda$ whenever $n, m \geq N$.

A fuzzy metric space is called M-complete (G-complete) if every M-Cauchy sequence (G-Cauchy sequence) is convergent.

Fixed point theory in fuzzy metric spaces has been developing since the paper of Grabiec [6]. Subramanyam [25] gave a generalization of Jungk's [8] theorem for commuting mappings in the setting of fuzzy metric spaces. In the recent literature of metric fixed point theory, weaker conditions of commutativity are described as: weakly commuting mappings ([25]), compatible mappings ([8]) R-weakly commuting maps ([19]), R-weakly commutativity of type (A_g), R-weakly commutativity of type (A_f) ([11], [21], [26]) and several others, with their correspondents in fuzzy metric spaces (see [18], [27]), have been utilizing. It is to be noted that all such mappings commute at their coincidence points. In 1996, Jungck [10] introduced the notion of weakly compatible as follow:

Definition 1.4. Two maps f and g are said to be weakly compatible if they commute at their coincidence points.

In 1999, Vasuki [26] introduced the notion of weakly commuting as follow:

Definition 1.5. Two self-mappings f and g of a fuzzy metric space $(X, M, *)$ are said to be weakly commuting if $M(fgx, gfx, t) \geq M(fx, gx, t)$, for each $x \in X$ and for each $t > 0$.

In 1994, Mishra [18] generalized the notion of weakly commuting to compatible mappings in fuzzy metric spaces akin to the concept of compatible mapping in metric spaces, see [8].

Definition 1.6. Let f and g mappings from a fuzzy metric space $(X, M, *)$ into itself. A pair of map $\{f, g\}$ is said to be compatible if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$ and for all $t > 0$.

In 1994 Pant [19] introduced the concept of R-weakly commuting maps in metric spaces. Later on, Vasuki [26] initiated the concept of non compatible of mapping in fuzzy metric spaces and introduced the notion of R-weakly commuting mappings in fuzzy metric spaces and proved some common fixed point theorems for these mappings.

Definition 1.7. Let f and g be self mappings on a fuzzy metric space $(X, M, *)$. The mappings f and g are said to be non compatible if

$\lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$ and for all $t > 0$.

Definition 1.8. A pair of self-mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be R-weakly commuting if there exists some $R > 0$ such that

$$M(fgx, gfx, t) \geq M(fx, gx, t/R),$$

of course, R-weak commutativity implies weak commutativity only when $R \leq 1$.

Later on, Pathak *et al.* [20] improved the notion of R-weakly commuting mappings in metric spaces by introducing the notions of R-weakly commutativity of type (Ag) and R-weakly commutativity of type (A_f).

In 2006, Imdad and Ali [11] initially introduced the notion of R-weakly commutativity of type (A_g) and R-weakly commutativity of type (A_f) in fuzzy metric with inspiration from Pathak *et al.* [20] and further they introduced the notion of of R-weakly commuting mappings of type (P).

Definition 1.9. A pair of self-mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be

(i) R-weakly commuting mappings of type (A_g) if there exists some $R > 0$ such that $M(gfx, ffx, t) \geq M(fx, gx, t/R)$.

(ii) R-weakly commuting mappings of type (A_f) if there exists some $R > 0$ such that $M(fgx, ggx, t) \geq M(fx, gx, t/R)$,

(iii) R-weakly commuting mappings of type (P) if there exists some $R > 0$ such that $M(ffx, ggx, t) \geq M(fx, gx, t/R)$, for all $x \in X$ and $t > 0$.

Aamri and Moutawakil [1] generalized the notion of non compatible mapping in metric space by E.A. property.

It was pointed out in [11], that property E.A. buys containment of ranges without any continuity requirements besides minimizes the commutativity conditions of the maps to the commutativity at their points of coincidence. Moreover, E. A. property allows replacing the completeness requirement of the space with a more natural condition of closeness of the range. Some common fixed point theorems in probabilistic or fuzzy metric spaces by E.A.property under weak compatibility have been recently obtained in ([3], [13],[20], [28]). The aim of this paper is to strengthen these results and to emphasize the role of E.A. property in the existence of common fixed point.

Definition 1.10 ([1])**.** Let A and S be two self-maps of a metric space (X, d) then they are said to satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t, \text{ for some } \ t \in X.$$

Now in a similar mode we state E.A. property in fuzzy metric spaces as follow:

Definition 1.11. A pair of self-mappings (f, g) of a fuzzy metric spaces $(X, M, *)$ is said to satisfy E. A property, if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} M(fx_n, gx_n, t) = 1 \text{ for some } \ t \in X.$$

Example 1.1. Let $X = [2, +\infty)$. Define $f, g : X \to X$ by $gx = x + 1$ and $fx = 2x + 1$, for all $x \in X$. Suppose that the E.A. property holds. Then, there exists a sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$. Therefore, $\lim_{n\to\infty} x_n = z - 1$ and $\lim_{n\to\infty} x_n = \dfrac{z-1}{2}$. Thus, $z = 1$, which is a contradiction, since 1 is not contained in X. Hence f and g do not satisfy E.A property.

Notice that weakly compatible and E.A. property are independent to each other.

Example 1.2. Let $X = [0, 1]$ with the usual metric space d i.e., $d(x, y) = |x-y|$. Define $M(x, y, t) = \left(\dfrac{t}{t + d(x, y)}\right)$ for all x, y in X and for all $t > 0$ and also define.

$$fx = \begin{cases} 1 - x & \text{if } \ x \in \left[0, \frac{1}{2}\right] \\ 0 & \text{if } \ x \in \left(\frac{1}{2}, 1\right] \end{cases} \qquad gx = \begin{cases} \frac{1}{2} & \text{if } \ x \in \left[0, \frac{1}{2}\right] \\ \frac{3}{4} & \text{if } \ x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

Consider the sequence $\{x_n\} = \left\{\dfrac{1}{2} - \dfrac{1}{n}\right\}$, $n \geq 2$, we have $\lim_{n\to\infty} f\left(\dfrac{1}{2} - n\right) = \dfrac{1}{2} = \lim_{n\to\infty} g\left(\dfrac{1}{2} - \dfrac{1}{n}\right)$. Thus, the pair (f, g) satisfies E.A. property. Further, f and g are weakly compatible since $x = \dfrac{1}{2}$ is their unique coincidence point and $fg\left(\dfrac{1}{2}\right) = f\left(\dfrac{1}{2}\right) = g\left(\dfrac{1}{2}\right) = gf\left(\dfrac{1}{2}\right)$. We further observe that $\lim_{n\to\infty} d\left(fg\left(\dfrac{1}{2} - \dfrac{1}{n}\right), gf\left(\dfrac{1}{2} - \dfrac{1}{n}\right)\right) \neq 0$, showing that $\lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$, therefore, the pair (f, g) is non-compatible.

Example 1.3. Let $X = R^+$ and d be the usual metric on X. Define $M(x, y, t) = \left(\frac{t}{t + d(x, y)}\right)$ for all $x, y \in X$ and for all $t > 0$ and also define $f, g : X \to X$ by $fx = 0$, if $0 < x \leq 1$ and $fx = 1$, if $x > 1$ or $x = 0$; and $gx = [x]$, the greatest integer that is less than or equal to x, for all $x \in X$. Consider a sequence $\{x_n\} = \left\{1 + \frac{1}{n}\right\}$, $n \geq 2$ in $(1, 2)$, then we have $\lim_{n\to\infty} fx_n = 1 = \lim_{n\to\infty} gx_n$. Similarly for the sequence $\{y_n\} = \left\{1 - \frac{1}{n}\right\}$, $n \geq 2$ in $(0, 1)$, we have $\lim_{n\to\infty} fy_n = 0 = \lim_{n\to\infty} gy_n$. Thus the pair (f, g) satisfies E.A. property. However, f and g are not weakly compatible as as each $u_1 \in (0, 1)$ and $u_2 \in (1, 2)$ are coincidence points of f and g, where they do not commute. Moreover, they commute at $x = 0, 1, 2 \ldots$ but none of these points are coincidence points of f and g. Further, we note that pair (f, g) is non compatible. Thus we can conclude that, E.A. property does not imply weak compatibility. Here, we notice that weakly compatible and E.A. property are independent to each other.

Lemma 1.1 ([18])**.** *Let $(X, M, *)$ be a fuzzy metric space. If there exists $k \in (0, 1)$ such that $M(x, y; kt) \geq M(x, y; t)$ for all $x, y \in X$ and $t > 0$, then $x = y$.*

2. Main results

The following fuzzy version of a theorem of Pant ([19], Theorem 1) appears in [26].

Theorem 2.1. *Let f and g be R-weakly commuting self mappings of a fuzzy metric spaces $(X, M, *)$, satisfying the following:*

(a-I) $f(X) \subset g(X)$

(a-II) $M(fx, fy, t) \geq r(M(gx, gy, t))$, *for all x, y in X and $t > 0$,*

where $r : [0, 1] \to [0, 1]$ is a continuous function such that $r(t) > t$ for each $0 < t < 1, r(0) = 0$ and $r(1) = 1$.

(a-III) *If there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_n \to x$ and $y_n \to y$, and $t > 0$, then $M(x_n, y_n, t) \to M(x, y, t)$.*

If one of the mappings f and g is continuous, then the mappings f and g have a unique common fixed point.

Now we prove our main result for weakly compatible maps under E.A. property as follows:

Theorem 2.2. *Let f and g be self maps of a fuzzy metric spaces $(X, M, *)$, satisfying $M(x, y, t) > 0$ for all x, y in X and $t > 0$ such that conditions* (a-II) *and* (a-III) *and the following holds:*

(a-IV) *f and g satisfy the E. A. property,*

(a-V) *$g(X)$ is a closed subspace of X.*

Then f and g have a unique common fixed point in X provided f and g are weakly compatible maps.

Proof. Since f and g satisfy the E. A. property therefore, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u \in X$. As $g(X)$ is a closed subspace of X, therefore every convergent sequence of points of $g(X)$ has a limit point in $g(X)$.

Therefore, $u = \lim_{n\to\infty} gx_n = ga = \lim_{n\to\infty} fx_n$ for some $a \in X$. This implies $u = ga \in g(X)$.

Now we show that $fa = ga$. From (a-II), we have, $M(fa, fxn, t) \geq r(M(ga, gx_n, t))$. Proceeding limit as $n \to \infty$, we have

$M(fa, u, t) \geq (M((u, u, t)) = r(1) = 1$, this implies that $u = ga = fa$.

Thus a is the coincidence point of f and g.

Since f and g are weakly compatible, therefore, $fu = fga = gfa = gu$.

Now we show that $fu = u$. From (a-II), we have

$M(fu, fa, t) \geq r(M(gu, ga, t))$, which in turns implies that $fu = u$. Hence u is the unique common fixed point of f and g.

Uniqueness follows easily from (a-II).

Consider the mapping $\phi : [0,1]^5 \to [0,1]$, which is upper semicontinuous, non-decreasing in each coordinate variable and such that

$$\phi(1, t, 1, t, 1) \geq t, \phi(1, 1, t, t, 1) \geq t, \phi(1, 1, 1, t, t) \geq t(t \in [0,1]).$$

Now we prove a common fixed point theorem for pairs of mappings using control function under E.A. property provided maps are weakly compatible. For some similar results in metric spaces we refer for nice survey in the papers [10] and [21].

Theorem 2.3. *Let A, B, S and T be self maps of a fuzzy metric spaces $(X, M, *)$ satisfying the following conditions:*

(i) $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,

(ii) $M(Ax, By, kt)$
$\geq \phi(M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Sx, By, t), M(Ax, Ty, t))$, *for all x, y in X and $t > 0$, where $k\varepsilon(0, 1)$,*

(iii) *pairs (A, S) or (B, T) satisfy E.A. property,*

(iv) *pairs (A, S) and (B, T) are weakly compatible.*

If the range of one of A, B, S and T is a closed subset of X, then A, B, S and T have a unique common fixed point in X.

Proof. Suppose that (B, T) satisfies the E.A. property. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Since $B(X) \subset S(X)$, therefore, there exists a sequence $\{y_n\} \in X$ such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Sy_n = z$. Hence $\lim_{n\to\infty} Sy_n = z$.

Now we shall show that $\lim_{n\to\infty} Ay_n = z$. Suppose that $\lim_{n\to\infty} Ay_n = l$.

Therefore, from (ii) we have

$$M(Ay_n, Bx_n, kt) \geq \phi(M(Sy_n, Tx_n, t), M(Ay_n, Sy_n, t), M(Bx_n, Tx_n, t), M(Sy_n, Bx_n, t), M(Ay_n, Tx_n, t)).$$

Proceeding limit as $n \to \infty$,

$M(l, z, kt) \geq \phi(l, M(l, z, t), 1, 1, M(l, z, t)) \geq M(l, z, t)$, using (φ) and by Lemma 1.1, we have $l = z$.

Therefore, we have $\lim_{n\to\infty} Ay_n = z$.

Suppose that $S(X)$ is a closed subspace of X. Then $z = Su$ for some $u \in X$. Subsequently, we have $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = z = Su$.

Now, we shall show that $Au = z$.

From (ii) we have

$$\begin{aligned} M(Au, Bx_n, kt) &\geq \phi(M(Su, Tx_n, t), M(Au, Su, t), M(Bx_n, Tx_n, t), \\ &\qquad M(Su, Bx_n, t), (Au, Tx_n, t)). \end{aligned}$$

Letting limit as $n \to \infty$,

$M(Au, z, kt) \geq \phi(1, M(Au, z, t), 1, 1, M(Au, z, t)) \geq M(Au, z, t)$, using (φ) and Lemma 1.1 ,we have

$$Au = Su = z.$$

Since $A(X) \subset T(X)$, so there exists $v \in X$ such that $z = Au = Tv$.

Now, we claim that $z = Bv$. Then From (ii) we have

$$\begin{aligned} M(Au, Bv, kt) &\geq \phi(M(Su, Tv, t), M(Au, Su, t), M(Bv, Tv, t), \\ &\qquad M(Su, Bv, t), M(Au, Tv, t)). \end{aligned}$$

Or $M(z, Bv, kt) \geq \phi(1, 1, M(Bv, z, t), M(z, Bv, t), 1) \geq M(z, Bv, t)$, using (φ) and Lemma 1.1, we have $z = Bv$.

Thus we have $Au = Su = Tv = Bv = z$.

Since the pair (A, S) is weak compatible,therefore, $ASu = SAu$ i.e, $Az = Sz$.

Now we show that $Az = z$.

$$\begin{aligned} M(Az, Bv, kt) &\geq \phi(M(Sz, Tv, t), M(Az, Sz, t), M(Bv, Tv, t), \\ &\qquad M(Sz, Bv, t), M(Az, Tv, t)). \end{aligned}$$

Or $M(Az, z, kt) \geq \phi(M(Az, z, t), 1, 1, M(Az, z, t), M(Az, z, t)) \geq M(Az, z, t)$, using (φ) and Lemma 1.1 $Az = Sz = z$.

The weak compatibility of B and T implies that $BTv = TBv$,i.e., $Bz = Tz$.

Now we shall further show that z is the common fixed point of B.

From (ii), one obtain

$$\begin{aligned} M(Az, Bz, kt) &\geq \phi(M(Sz, Tz, t), M(Az, Sz, t), M(Bz, Tz, t), \\ &\qquad M(Sz, Bz, t), M(Az, Tz, t)). \end{aligned}$$

Or $M(Az, Bz, kt) \geq (M(Az, Bz, t), 1, 1, M(Az, Bz, t), 1)$, using (φ) and by Lemma 1.1, $Bz = z$.

Hence $Az = Bz = Sz = Tz = z$ and z is a common fixed point of A, B, S and T.

Corollary 2.1. *Let A, B, S and T be self maps of a fuzzy metric space $(X, M, *)$ with continuous t-norm satisfying* (i), *(*iii), (iv) *and the following:*

(v) $M(Ax, By, kt) \geq \min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Sx, By, t), M(Ax, Ty, t)\}$ *for all x, y in X and $t > 0$, where $k\varepsilon(0, 1)$.*

If the range of one of A, B, S and T is a closed subset of X, then A, B, S and T have a unique common fixed point in X.

Proof. Take in the above Theorem $\phi(x_1, x_2, x_3, x_4, x_5) = \min\{x_1, x_2, x_3, x_4, x_5\}$

Example 2.1. Let $X = [0, 2]$ equipped with the Euclidian distance and the fuzzy metric spaces induced by (X, d) i.e., $M(x, y, t) = \dfrac{t}{t + d(x, y)}$. Clearly $(X, M, *)$ is a fuzzy metric space with $* = \min\{a, b\}$. Define the self maps A, B, S and $T : X \to X$ by

$$Ax = \begin{cases} 0 & \text{if } x = 0, \\ 0.25 & \text{if } x > 0. \end{cases} \qquad Bx = \begin{cases} 0 & \text{if } x = 0, \\ 0.45 & \text{if } x > 0. \end{cases}$$

$$Sx = \begin{cases} 0 & \text{if } x = 0, \\ 0.40 & \text{if } 0 < x \leq 0.6, \\ x - 0.45 & \text{if } x > 0.6. \end{cases} \qquad Tx = \begin{cases} 0 & \text{if } x = 0, \\ 0.25 & \text{if } 0 < x \leq 0.6, \\ x - 0.25 & \text{if } x > 0.6. \end{cases}$$

$$AX = 0.025,\ BX = 0.045,\ SX = 0 \cup (0.15, 1.55),\ TX = 0 \cup 0.25 \cup (0.35, 1.75).$$

Let us consider the sequence $x_n = 0.60 + 1/n$, then $Ax_n \to 0.25$, $Bx_n \to 0.45$, $Cx_n \to 0.15$, $Tx_n \to 0.35$, $ASx_n \to 0.25$, $SAx_n \to 0.40$, $BTx_n \to 0.45$, $Bx_n \to 0.25$. Pairs (A, S) and (B, T) are non compatible. If we take $k = 0.6$, and $t = 1$, then A, B, S and T satisfy all the conditions of the Theorem 2.3 and 0 is the unique common fixed point of A, B, S and T. Moreover, A, B, S and T are discontinuous at the fixed point 0.

Next we consider a function $\psi : [0, 1] \to [0, 1]$ satisfying the conditions

$$(*) \qquad \begin{cases} \psi & \text{if continuous and nondecreasing on [0,1]}, \\ \psi(t) > t & \text{for all } t \text{ in (0,1)}. \end{cases}$$

We note that $\psi(1) = 1$ and $\psi(t) \geq t$ for all t in $[0, 1]$, that is,
$\psi(M(x, y, t)) \geq \psi(M(x, y, t))$ holds for every $t > 0$ and for all x, y in X.

Theorem 2.4. *Let A, B, S and T be self maps of a fuzzy metric space $(X, M, *)$ with continuous t-norm $*$ satisfying* (i), (iii), (iv) *and the following:*

(vi) $M(Ax, By, t) \geq \psi(\min\{M(Sx, Ty, t)M(Ax, Sx, t)M(By, Ty, t), M(Sx, By, t), M(Ax, Ty, t)\})$ *with $M(x, y, t) > 0$ for all x, y in X and $t > 0$.*

If the range of one of A, B, S and T is a closed subset of X, then A, B, S and T have a unique common fixed point in X.

Proof. Suppose that (B, T) satisfies the E.A property. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Since $BX \subset SX$ there exists a sequence $\{y_n\} \in X$ such that $Bx_n = Sy_n = z$. Hence $\lim_{n\to\infty} Sy_n = z$.

We shall show that $\lim_{n\to\infty} Ay_n = z$.

From (vi) we have

$$M(Ay_n, Bx_n, t) \geq \psi(\min\{M(Sy_n, Tx_n, t), M(Ay_n, Sy_n, t), M(Bx_n, Tx_n, t), M(Sy_n, Bx_n, t), M(Ay_n, Tx_n, t)\}).$$

Proceeding limit as $n \to \infty$, one obtain, $\lim_{n\to\infty} Ay_n = z$.

Suppose that $S(X)$ is a closed subspace of X. Then $z = Su$ for some $u \in X$. Subsequently we have

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = z = Su.$$

Now, we shall show that $Au = Su$. From (vi) we have

$$M(Au, Bx_n, t) \geq \psi(\min\{M(Su, Tx_n, t), M(Au, Su, t), M(Bx_n, Tx_n, t), M(Su, Bx_n, t), M(Au, Tx_n, t)\}).$$

Letting limit as $n \to \infty$, we get

$$M(Au, z, t) \geq \psi(\min\{M(z, z, t), M(Au, z, t), M(z, z, t), M(z, z, t), M(Au, z, t)\}),$$

using $(*)$, we have, $Au = Su = z$.

Since $AX \subset TX$, so there exists $v \in X$ such that $z = Au = Tv$.

Now, we claim that $z = Tv = Bv$.

From (vi) we have

$$M(Au, Bv, t) \geq \psi(\min\{M(Su, Tv, t), M(Au, Su, t), M(Bv, Tv, t), M(Su, Bv, t), M(Au, Tv, t)\}),$$

or

$$M(z, Bv, t) = \psi(\min\{M(z, z, t), M(z, z, t), M(Bv, z, t), M(z, Bv, t), M(z, z, t)\}),$$

using $(*)$, we have, $z = Bv$. Thus we have $Au = Su = Tv = Bv = z$. Since the pair (A, S) is weak compatible which implies $ASu = SAu$ i.e, $Az = Sz$.

From (vi),

$$M(Az, Bv, t) \geq \psi(\min\{M(Sz, Tv, t), M(Az, Sz, t), M(Bv, Tv, t), M(Sz, Bv, t), M(Az, Tv, t)\})$$

using $(*)$, we have, $Az = Sz = z$.

The weak compatibility of B and T implies that $BTv = TBv$, i.e., $Bz = Tz$.

Now we shall show that z is the common fixed point of A, B, T and S.

Suppose that $Bz \neq z$. Then using (vi) one obtain

$$M(Az, Bz, t) \geq \psi(\min\{M(Sz, Tz, t), M(Az, Sz, t), M(Bz, Tz, t), M(Sz, Bz, t), M(Az, Tz, t)\}),$$

using $(*)$, we have, $Bz = z$.

Hence $Az = Bz = Sz = Tz = z$ and z is a common fixed point of A, B, S and T.

Uniqueness follows easily.

Theorem 2.5. *Let A, B, S and T be self maps of a fuzzy metric space $(X, M, *)$ satisfying* (i)*,* (ii)*,* (iv) *and the following conditions:.*

(vii) *pairs (A, S) and (B, T) satisfy a common E.A.property*

If the range of S and T is a closed subset of X, then A, B, S and T have a unique common fixed point in X.

Proof. Suppose that (A, S) and (B, T) satisfy a common E.A. property. Then there exists a sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z \text{ for some } z \in X.$$

Since $S(X)$ and $T(X)$ are closed subsets of X, we obtain $z = Su = Tv$ for some u, v in X. From (vi),

$$\begin{aligned} M(Au, By_n, t) \quad \geq \quad & \psi(\min\{M(Su, Ty_n, t), M(Au, Su, t), M(By_n, Ty_n, t), \\ & M(Su, By_n, t), M(Au, Ty_n, t)\}) \end{aligned}$$

Letting $n \to \infty$ and using $(*)$, we have, $z = Au = Su = Tv$.

The rest of the proof follows from the Theorem 2.4.

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