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ALMOST SURE CONVERGENCE FOR WEIGHTED SUMS OF NEGATIVELY ASSOCIATED RANDOM VARIABLES UNDER h-INTEGRABILITY

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Abstract. We establish strong laws of large numbers for weighted sums of arrays of negatively associated random variables under the condition of h-integrability and suitable conditions on the array of weights.

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1. Introduction

Strong law of large numbers for a sequence of random variables plays a important role in probability theory and mathematical statistics. The classical notion of uniform integrability of a sequence $\{X_n, n \geq 1\}$ of integrable random variables is defined through the condition $\lim_{a\to\infty} \sup_{n\geq 1} E|X_n|I(X_n| > a) = 0$, where a a positive constant. Landers and Rogge (1987) proved that the uniform integrability condition is sufficient in order that a sequence of pairwise independent random variables verifies the law of large numbers. Chandra(1989) introduced Cesàro uniform integrability which is weaker than uniform integrability : A sequence $\{X_n, n \geq 1\}$ of integrable random variables is said to be Cesàro uniformly integrable if $\lim_{a\to\infty} \sup_{n\geq 1} \frac{1}{n} \sum_{k=1}^{n} E|X_k|I(X_k| > a) = 0$.

Ordóñez Cabrera(1994) introduced uniform integrability concerning the weights, which is weaker than uniform integrability, and leads to Cesàro uniform integrability as a special case. Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of random variables and $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ an array of constants with $\sum_{k=1}^{n} |a_{nk}| \leq C$ for all $n \geq 1$ and some constant $C > 0$.

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The array $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ is $\{a_{nk}\}$-uniformly integrable if

$$\lim_{a\to\infty} \sup_{n\geq 1} \sum_{k=1}^{n} |a_{nk}| E|X_{nk}| I(|X_{nk}| > a) = 0.$$

Under the condition of $\{a_{nk}\}$-uniform integrability, Ordónez Cabrera(1994) obtained law of large numbers for weighted sums of pairwise independent random variables; the condition of pairwise independence can be even dropped, at the price of slightly strengthening the conditions on the weights. Ordónez Cabrera and Volodin(2005) considered the notion of h-integrability for an array of random variables concerning an array of constant weights and proved that this concept is weaker than Cesàro uniform integrability and $\{a_{nk}\}$-uniform integrability. Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of random variables and $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ an array of constants with $\sum_{k=1}^{n} |a_{nk}| < C$ for all $n \geq 1$ and some constant $C > 0$. Let moreover $\{h(n), n \geq 1\}$ be an increasing sequence of positive constants with $h(n) \uparrow \infty$ as $n \uparrow \infty$. The array $\{X_{nk}\}$ is said to be h-integrable with respect to the array $\{a_{nk}\}$ of constants if the following conditions hold:

$$\sup_{n\geq 1} \sum_{k=1}^{n} |a_{nk}| E\,|X_{nk}| < \infty \ and \ \lim_{n\to\infty} \sum_{k=1}^{n} |a_{nk}| E\,|X_{nk}|\, I\,[\,|X_{nk}| > h(n)] = 0.$$

We will prove strong laws of large numbers for weighted sums of an array of negatively associated random variables under suitable conditions on the weights and h-integrability concerning the weights.

2. Results

In the first theorem, we obtain a strong law of large numbers for the weighted sums of rowwise negatively associated random variables with a certain technique of truncation under the condition of h-integrability concerning the array $\{a_{nk}\}$ of constants.

Definition 2.1(Joag-Dev, Proschan, 1983) A finite family $\{X_1, \cdots, X_n\}$ is said to be negatively associated(NA) if for any disjoint subsets $A, B \subset \{1, \cdots, n\}$ and any real coordinatewise nondecreasing functions f on R^A, g on R^B, $Cov(f(X_i, i \in A), g(X_j, j \in B)) \leq 0$ and an infinite family of random variables is NA if every finite subfamily is NA.

The following lemma gives a certain technique of truncation that preserves the dependence property.

Lemma 2.2 Let $\{X_n, n \geq 1\}$ be a sequence of negatively associated random variables. Then, for any sequences $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ of constants such that $a_n < b_n$ for all $n \geq 1$, the sequence $\{Y_n, n \geq 1\}$ is still a sequence of negatively associated random variables, where

$$Y_n = a_n\, I\,[X_n < a_n] + X_n\, I\,[a_n \leq X_n \leq b_n] + b_n\, I\,[X_n > b_n]. \tag{2.1}$$

Lemma 2.3(Matula, 1992) Let $\{X_n\}$ be a sequence of negatively associated random variables with $EX_k = 0$ and $\sigma_k^2 = EX_k{}^2$, $k \geq 1$. Then, for $\varepsilon > 0$,

$$P\{\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_i| \geq \varepsilon\} \leq 8\varepsilon^{-2} \sum_{k=1}^{n} \sigma_k{}^2. \tag{2.2}$$

Theorem 2.4 Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of rowwise negatively associated random variables with $EX_{nk} = 0$ and $EX_{nk}^2 < \infty$ for all $1 \leq k \leq n$ and $n \geq 1$ and $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of positive constants with $a_{nk} \leq 1$, $\sum_{k=1}^{n} a_{nk} \leq C$ for all $n \geq 1$ and some constant $C > 0$. Let moreover $\{h(n), n \geq 1\}$ be an increasing sequence of positive constants with $h(n) \uparrow \infty$ as $n \uparrow \infty$. Suppose that

(a) $\{X_{nk}\}$ is h-integrable concerning the array $\{a_{nk}\}$ of constants,
(b) $h(n) \geq Cn^{\alpha}$ for some $\alpha > \frac{1}{2}$.
(c) $h^2(n) \sum_{k=1}^{n} a_{nk}{}^2 = O((\log n)^{-1-\delta})$, for some $\delta (0 < \delta < 1)$

Then, for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P(\max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk} X_{nk}| > \varepsilon) < \infty \tag{2.3}$$

and

$$\sum_{k=1}^{n} a_{nk} X_{nk} \to 0 \ a.s. \ as \ n \to \infty. \tag{2.4}$$

Proof. For each $1 \leq k \leq n$, $n \geq 1$, truncate at the level $h(n)$ and put

$$Y_{nk} = X_{nk} I(|X_{nk}| \leq h(n)) - h(n) I(X_{nk} < -h(n)) + h(n) I(X_{nk} > h(n)). \tag{2.5}$$

Noting that Y_{nk}'s are NA and that $EX_{nk} I(|X_{nk}| \leq h(n)) = -EX_{nk} I(|X_{nk}| > h(n))$ in view of the fact that $EX_{nk} = 0$, we have

$$\begin{aligned}
&P(\max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk} X_{nk}| > \varepsilon) \\
&\leq P(\max_{1\leq j\leq n} |X_{nj}| > h(n)) + P(\max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk} Y_{nk}| > \varepsilon) \\
&\leq P(\max_{1\leq j\leq n} |X_{nj}| > h(n)) + P(\max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk}(Y_{nk} - EY_{nk})| > \varepsilon - \max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk} EY_{nk}|) \\
&\leq \sum_{k=1}^{n} P(|X_{nk}| > h(n)) + P(\max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk}(Y_{nk} - EY_{nk})| > \varepsilon - \max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk} EY_{nk}|). \qquad (2.6)
\end{aligned}$$

By assumption (a) we also have

$$
\begin{aligned}
&\max_{1\le j\le n}|\sum_{k=1}^{j} a_{nk}EY_{nk}| \\
= \;& \max_{1\le j\le n}|\sum_{k=1}^{j} a_{nk}E\{X_{nk}I(|X_{nk}|\le h(n)) \\
& -h(n)I(X_{nk}<-h(n))+h(n)I(X_{nk}>h(n))\}| \\
& \le \sum_{k=1}^{n} a_{nk}\{E|X_{nk}|I(|X_{nk}|>h(n))\} \\
& +\sum_{k=1}^{n} a_{nk}h(n)EI(|X_{nk}|>h(n)) \\
\le \;& 2\sum_{k=1}^{n} a_{nk}E|X_{nk}|I(|X_{nk}|>h(n)) \to 0,\ as\ n\to\infty.
\end{aligned}
\tag{2.7}
$$

Hence, it follows from (2.6) and (2.7) that for n large enough

$$
P(\max_{1\le j\le n}|\sum_{k=1}^{j} a_{nk}X_{nk}|>\varepsilon)
\tag{2.8}
$$

$$
\le \sum_{k=1}^{n} P(|X_{nk}|>h(n)) + P(\max_{1\le j\le n}|\sum_{k=1}^{j} a_{nk}(Y_{nk}-EY_{nk})|>\frac{\varepsilon}{2}).
$$

It therefore remains to show that

$$
\sum_{n=1}^{\infty} n^{-1}\sum_{k=1}^{n} P(|X_{nk}|>h(n))<\infty
\tag{2.9}
$$

and

$$
\sum_{n=1}^{\infty} n^{-1}P(\max_{1\le j\le n}|\sum_{k=1}^{j} a_{nk}(Y_{nk}-EY_{nk})|>\frac{\varepsilon}{2})<\infty.
\tag{2.10}
$$

It follows from Chebyshev inequality and assumptions $EX_{nk}^2<\infty$ for all $1\le k\le n,\ n\ge 1$ and (b) that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1}\sum_{k=1}^{n} P(|X_{nk}|>h(n)) \\
&\le \sum_{n=1}^{\infty} n^{-1}\frac{\sum_{k=1}^{n} E|X_{nk}|^2}{h^2(n)} \\
&\le C\sum_{n=1}^{\infty} n^{-2\alpha}<\infty.
\end{aligned}
\tag{2.11}
$$

It also follows from Lemma 2.3 and assumption (c) that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} P(\max_{1\le j\le n} |\sum_{k=1}^{j} a_{nk}(Y_{nk}-EY_{nk})| > \frac{\varepsilon}{2}) \\
\le\ & 32 \sum_{n=1}^{\infty} n^{-1}\varepsilon^{-2} \sum_{k=1}^{n} ({a_{nk}}^2 E{Y_{nk}}^2) \\
\le\ & C \sum_{n=1}^{\infty} n^{-1} \sum_{k=1}^{n} {a_{nk}}^2 E[{X_{nk}}^2 I(|X_{nk}| \le h(n)) + h^2(n) I(|X_{nk}| > h(n))] \\
\le\ & C \sum_{n=1}^{\infty} n^{-1} \sum_{k=1}^{n} {a_{nk}}^2 h^2(n) \\
\le\ & C \sum_{n=1}^{\infty} n^{-1} (\log n)^{-1-\delta} < \infty. \qquad (2.12)
\end{aligned}$$

Thus by (2.11) and (2.12) we obtain (2.3).

Next, by (2.3) we have

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{-1} P(\max_{1\le j\le n} |\sum_{k=1}^{j} a_{nk}X_{nk}| > \varepsilon) \\
&= \sum_{i=0}^{\infty} \sum_{n=2^i}^{2^{i+1}-1} n^{-1} P(\max_{1\le j\le n} |\sum_{k=1}^{j} a_{nk}X_{nk}| > \varepsilon) \qquad (2.13) \\
&\ge \frac{1}{2} \sum_{i=1}^{\infty} P(\max_{1\le j\le 2^i} |\sum_{k=1}^{j} a_{2^i,k}X_{2^i,k}| > \varepsilon).
\end{aligned}$$

By Borel-Cantelli Lemma and (2.13) we have

$$P(\max_{1\le j\le 2^i} |\sum_{k=1}^{j} a_{2^i,k}X_{2^i,k}| > \varepsilon \ i.o.) = 0$$

and hence,

$$\max_{1\le j\le 2^i} |\sum_{k=1}^{j} a_{2^i,k}X_{2^i,k}| \to 0 \ a.s. \ as \ i \to \infty. \qquad (2.14)$$

From (2.14) and the fact that

(2.15)

$$\lim_{n\to\infty} |\sum_{k=1}^{n} a_{nk}X_{nk}| \le \lim_{i\to\infty} \max_{2^{i-1}\le n\le 2^i} |\sum_{k=1}^{n} a_{nk}X_{nk}| \le \lim_{i\to\infty} \max_{1\le j\le 2^i} |\sum_{k=1}^{j} a_{2^i,k}X_{2^i,k}|$$

the desired result (2.4) follows.

Next, we consider Theorem 2.4 under negative quadrant dependence and φ-mixing assumption, respectively.

Definition 2.5(Lehmann, 1966) A sequence of random variables $\{X_n, n \geq 1\}$ is said to be pairwise negative quadrant dependent(NQD) if for all $i \neq j$ and all $x_i, x_j,\ P(X_i > x_i, X_j > x_j) \leq P(X_i > x_i)P(X_j > x_j)$.

The following lemma is an extension of the well-known Rademacher - Mension inequality.

Lemma 2.6(Chandra, Ghosal, 1996) Let $Y_1, \cdots, Y_n$ be square integrable random variables and let there exist $a_1^2, \cdots, a_n^2$ satisfying $E(Y_{m+1}+\cdots+Y_{m+p})^2 \leq a_{m+1}^2 + \cdots + a_{m+p}^2$ for all $m, p \geq 1,\ m+p \leq n$. Then we have

$$E(\max_{1\leq k\leq n}(\sum_{i=1}^{k} Y_i)^2 \leq ((\log n/\log 3)+2)^2 \sum_{i=1}^{n} a_i^2.$$

Theorem 2.7 Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of rowwise pairwise NQD random variables with $EX_{nk} = 0,\ EX_{nk}^2 < \infty$ for all $1 \leq k \leq n$ and $n \geq 1$ and $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of positive constants with $a_{nk} \leq 1$ and $\sum_{k=1}^{n} a_{nk} \leq C$ for all $n \geq 1$ and some constant $C > 0$. Let moreover $\{h(n), n \geq 1\}$ be an increasing sequence of positive constants with $h(n) \uparrow \infty$ as $n \uparrow \infty$. Suppose that

(a) $\{X_n\}$ is h-integrable concerning the array $\{a_{nk}\}$ of constants,

(b) $h(n) \geq Cn^{\alpha}$ for some $\alpha > \frac{1}{2}$,

(d) $h^2(n)\sum_{k=1}^{n} {a_{nk}}^2 = O((\log n)^{-3-\delta})$ for some $\delta(0 < \delta < 1)$.

Then, (2.4) holds.

Proof. The proof is similar to that of Theorem 2.4 and thus it is sufficient to show (2.4). Define Y_{nk} as in (2.5). Then $\{Y_{nk}\}$ is still pairwise NQD. Hence by assumption (a), (2.6) and (2.7) we obtain

$$\begin{aligned} &P(\max_{1\leq j\leq n}|\sum_{k=1}^{j} a_{nk}X_{nk}| > \varepsilon) \\ &\leq \sum_{k=1}^{n} P(|X_{nk}| > h(n)) + P(\max_{1\leq j\leq n}|\sum_{k=1}^{j} a_{nk}(Y_{nk} - EY_{nk})| > \frac{\varepsilon}{2}). \end{aligned} \tag{2.16}$$

Note that $a_n(Y_{nk} - EY_{nk})$'s satisfy the conditions of Lemma 2.6. As in the proof of (2.11) it follows from Markov inequality and assumption (b) that

$$\sum_{n=1}^{\infty} n^{-1}\sum_{k=1}^{n} P(|X_{nk}| > h(n)) \leq C\sum_{n=1}^{\infty} n^{-2\alpha} < \infty \tag{2.17}$$

and as in the proof of (2.12) it also follows from Lemma 2.6 and assumption (d) that

$$\sum_{n=1}^{\infty} n^{-1}P(\max_{1\leq j\leq n}|\sum_{k=1}^{j} a_{nk}(Y_{nk} - EY_{nk})| > \frac{\varepsilon}{2})$$

$$\begin{aligned}
(2.18) \qquad & \leq (\frac{\varepsilon}{2})^{-2} \sum_{n=1}^{\infty} n^{-1}((\log n/\log 3)+2)^2 \sum_{k=1}^{n} {a_{nk}}^2 E{Y_{nk}}^2 \\
& \leq (\frac{\varepsilon}{2})^{-2} \sum_{n=1}^{\infty} n^{-1}((\log n/\log 3)+2)^2 \sum_{k=1}^{n} {a_{nk}}^2 h^2(n) \\
& \leq C \sum_{n=1}^{\infty} n^{-1}(\log n)^{-1-\delta} < \infty.
\end{aligned}$$

Hence by (2.17) and (2.18) we obtain (2.3) and the proof is complete.

Definition 2.8 Let $\{X_n, -\infty < n < \infty\}$ be a sequence of random variables. Let $\mathcal{B}^k$ be the σ-algebra generated by $\{X_n, n \leq k\}$, and $\mathcal{B}_k$ the σ-algebra generated by $\{X_n, n \geq k\}$. We say that $\{X_n, -\infty < n < \infty\}$ is φ-mixing if there exists a non-negative sequence $\{\varphi(i), i \geq 1\}$ with $\lim_{i\to\infty} \varphi(i) = 0$, such that, for each $-\infty < k < \infty$ and for each $i \geq 1$,

$$(2.19) \qquad |P(E_2 \mid E_1) - P(E_2)| \leq \varphi(i) \ for\ E_1 \in \mathcal{B}^k, E_2 \in \mathcal{B}_{k+i}.$$

Lemma 2.9(Billingsley, 1968) Let ζ be a $\mathcal{B}^k$-measurable random variable, and η be a $\mathcal{B}_{k+i}$-measurable random variable, with $|\zeta| \leq C_1$ and $|\eta| \leq C_2$. Then

$$(2.20) \qquad |Cov(\zeta, \eta)| \leq 2C_1C_2\varphi(i).$$

Theorem 2.10 Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of mean zero and square integrable random variables such that for each $n \geq 1$, $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ is a φ_n-mixing sequence of random variables satisfying

$$(2.21) \qquad \limsup_{n\to\infty} \sum_{i=1}^{n} \varphi_n(i) < \infty.$$

Let $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of non-negative constants such that $a_{nk} \leq 1$, $\sum_{k=1}^{n} a_{nk} \leq C$ for all $n \geq 1$ and some constant $C > 0$, and $a_{nj} \leq a_{ni}$ if $i < j$ for all $n \geq 1$. Let moreover $\{h(n), n \geq 1\}$ be a sequence of increasing to infinity positive constant. Suppose that conditions (a), (b) and (d) in Theorem 2.7 are satisfied. Then (2.4) holds.

Proof. The proof is similar to that of Theorem 2.7, only we can use usual truncation technique. Hence, for each $n \geq 1$, $1 \leq k \leq n$, let

$$(2.22) \qquad Y_{nk} = X_{nk} I(|X_{nk}| \leq h(n)).$$

Note that $EX_{nk}I(|X_{nk}| \leq h(n)) = -EX_{nk}I(|X_{nk}| > h(n))$ in view of the fact $EX_{nk} = 0$ and that

$$\max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk}EY_{nk}| = \max_{1\leq j\leq n} |\sum_{k=1}^{j} a_{nk}EX_{nk}I(|X_{nk}| \leq h(n))|$$

$$\leq \sum_{k=1}^{n} a_{nk} E|X_{nk}| I(|X_{nk}| > h(n)) \to 0 \ as \ n \to \infty \tag{2.23}$$

by assumption (a). Hence by (2.6) and (2.23) we have (2.16).

It remains to show that (2.17) and (2.18)

By Chebyshev inequality and assumption (b), (2.17) follows. To apply Lemma 2.6 we need to show that

$$\limsup_{n\to\infty} \sum_{\substack{k,j=1 \\ k<j}}^{n} a_{nk} a_{nj} Cov(Y_{nk}, Y_{nj}) \leq 0. \tag{2.24}$$

By Lemma 2.9 and assumption (d) we have

$$\begin{aligned}
\sum_{\substack{k,j=1 \\ k<j}}^{n} a_{nk} a_{nj} Cov(Y_{nk}, Y_{nj}) &= \sum_{i=1}^{n} \sum_{k=1}^{n-i} a_{nk} a_{n(k+i)} Cov(Y_{nk}, Y_{n(k+i)}) \\
&\leq 2h^2(n) \sum_{i=1}^{n} \sum_{k=1}^{n-i} {a_{nk}}^2 \varphi_n(i) \\
&\leq 2h^2(n) \sum_{k=1}^{n} {a_{nk}}^2 \sum_{i=1}^{n} \varphi_n(i) \to 0
\end{aligned}$$

which yields (2.24).

Hence, by Lemma 2.6 and assumption (d) (2.18) follows, that is, we obtain (2.3) and the proof is complete.

Corollary 2.11 Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of random variables such that for each $n \geq 1$, $\{X_{nk}, 1 \leq k \leq n\}$ is a $m(n)$-dependent sequence of random variables with $\limsup_{n\to\infty} m(n) < \infty$. Let the other conditions of Theorem 2.10 be satisfied. Then, (2.4) holds.

Proof. We only have to note that we can consider $\varphi_n(i) = 0$ for $i > m(n)$ and $\varphi_n(i) = 1$ for $i \leq m(n)$, and so $\sum_{i=1}^{n} \varphi_n(i) \leq m(n)$ for all $n \geq 1$.

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