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THE CONSTRUCTION OF A NON-UNIMODAL GORENSTEIN SEQUENCE†

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Abstract. In this paper, we construct a Gorenstein Artinian algebra R/J with non-unimodal Hilbert function $h = (1, 13, 12, 13, 1)$ to investigate the algebraic structure of the ideal J in a polynomial ring R. For this purpose, we use a software system Macaulay 2, which is devoted to supporting research in algebraic geometry and commutative algebra.

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1. Introduction

Gorenstein algebras have been the subject of much research since they arise in many situations in commutative algebra and algebraic geometry. One of the most important problems in this area is whether a numerical sequence is the Hilbert function of a Gorenstein algebra.

R. Stanley [8] introduced level algebras as a generalization of Gorenstein algebras and investigated their nice properties via the trivial extension. The trivial extension or idealization of a commutative ring by a module is a classical method of constructing a new ring. This method has been exploited to construct Gorenstein algebras with non-unimodal Hilbert functions (cf. [1], [2], [3], [8]). As a typical example, one can see that there is an Artinian Gorenstein algebra over a field with h-vector $(1, 13, 12, 13, 1)$.

The goal of this paper is to compute minimal generators of homogeneous ideal J in a polynomial ring $R = K[x_1, x_2, \ldots, x_{13}]$ such that R/J is an Gorenstein algebra with h-vector $(1, 13, 12, 13, 1)$, and to understand the ideal structure of J. For this purpose, we will use a software system Macaulay 2, which is devoted to supporting research in algebraic geometry and commutative algebra. The

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minimal generators and minimal free resolution of J will be presented, and we investigate the structure of $(J,L)/(L)$ for general linear form L. Finally, we prove that the Hilbert function $h=(1,13,12,13,1)$ of R/J has a decomposition $h=\ell+b$, where $\ell=(1,12,3,4,0)$ and $b=(0,1,9,9,1)$ are the Hilbert functions of $R/(J,L)$ and $R/(J:L)(-1)$ respectively.

Our results are expected to be useful for understanding the structure of non-unimodal Gorenstein algebras.

2. Preliminaries

In this section we will introduce the basic notation and discuss some useful results concerning Gorenstein and level algebras.

2.1. Gorenstein and level Algebra. Let $R=K[x_1,\cdots,x_n]$ be a polynomial ring in n variables over a field K of characteristic zero, and $\mathfrak{m}=(x_1,\ldots,x_n)$ be the maximal ideal of R.

If we write $A=R/I$ for an Artinian K-algebra then the *socle* of $A=R/I$ is defined by

$$\operatorname{Soc}(A)=\operatorname{ann}(A)=(I:\mathfrak{m})/I\subset A.$$

We say that $A=\bigoplus_{i=0}^{\sigma}A_k$ is a level algebra if $\operatorname{Soc}(A))=\langle A_\sigma\rangle$. In this case, σ is called the socle degree of A and the vector space dimension of A_σ is called the *type* of A.

Definition 1. *A standard algebra $A=R/I$ is a Gorenstein algebra if A is a level algebra of type* 1.

2.2. Macaulay's Inverse System. There is a way to construct Gorenstein algebras, using Macaulay's Inverse Systems. Let $R=K[x_1,\ldots,x_n]$ and $S=K[y_1,\ldots,y_n]$ be the polynomial rings. For an element $\alpha=(a_0,\ldots,a_n)\in\mathbb{Z}^n_{\geq 0}$, we let $\mathbf{x}^\alpha$ denote the monomial $x_1^{a_1}\cdots x_n^{a_n}$.

If $\mathbf{x}^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}\in R$ is a monomial of degree d then we define

$$\mathbf{x}^\alpha\circ g(y_1,\ldots,y_n)=\frac{\partial g}{\partial\mathbf{y}^\alpha}=\frac{\partial g(y_1,\ldots,y_n)}{\partial y_1^{\alpha_1}\cdots\partial y_n^{\alpha_n}}.$$

If $f=\sum_{\alpha\in\mathbb{N}^\kappa}a_i\mathbf{x}^\alpha$ is a homogeneous polynomial in R then this action can be extended to $f\in R$ as follows:

$$f\circ g=\sum_{\alpha\in\mathbb{N}^n}a_i(\mathbf{x}^\alpha\circ g).$$

Definition 2. *Let $R=K[x_1,\cdots,x_n]$ and $S=K[y_1,\ldots,y_n]$ be the polynomial rings. If $I\subset R$ is a homogenous ideal, then we define the set*

$$I^\perp=\{g\in S\mid f\circ g=0,\quad\forall f\in I\}.$$

The following result shows the relationship between Artinian Gorenstein algebras and Macaulay's inverse system.

Theorem 3. *Let $A = R/I$ be an Artinian graded K-algebra. Then the followings are equivalent:*

(a) *A is a Gorenstein algebra.*
(b) *$I^{\perp}$ is generated by a homogeneous polynomial f of degree σ, where σ is the socle degree of A.*

Proof. See Theorem 1.1.13 in [4]. □

2.3. Trivial Extensions. The trivial extension of A by a module is a classical method of constructing new ring. R. Stanley has used this method to construct an example of a Gorenstein Artin algebra with a non-unimodal Hilbert function (see [8]).

For an Artinian K-algebra R/I, let ω_A be the canonical module of A. Let us consider the following set:

$$A \times \omega_A = \{\, (a, m) \,|\, a \in A, m \in M\}.$$

Then we can define a multiplication on $A \times \omega_A$ by

$$(a, m) * (a', m') = (aa', am' + a'm)$$

such that $A \times \omega_A$ becomes a commutative ring with an identity element $(1, 0)$. We say that this ring is a *trivial extension* of A and denote $TE(A)$.

Theorem 4. *Let $A = R/I$ be a level algebra of type c with the socle degree σ. Consider the shifted canonical module $\omega_A(-\sigma - 1)$ of A. Then $TE(A)$ is a Gorenstein Artinian K-algebra of socle degree $c + 1$. Moreover, suppose that there are polynomials $F_1, \dots, F_c \in S = K[y_1, \dots, y_n]$ of degree σ such that*

$$I = \{\, G \in R \,|\, G \circ F_i = 0, \quad \forall i = 1, \dots, c\}.$$

Then, we have $TE(A) = \bar{R}/J$ and $J = \{g \in \bar{R} \,|\, g \circ f = 0\}$, where

- *$\bar{R} = K[x_1, \dots, x_n, a_1, \dots, a_t]$ is a polynomial ring.*
- *$f = a_1F_1 + a_2F_2 + \cdots + a_cF_c \in \bar{S} = K[y_1, \dots, y_n, a_1, \dots, a_t]$ is a polynomial of degree $\sigma + 1$ in $\bar{R}$.*

In this case, the Hilbert function $\mathbf{H}(TE(A), i)$ is given by $\mathbf{H}(A, i) + \mathbf{H}(A, \sigma + 1 - i)$.

Proof. See [4] and [8]. □

3. Some construction of a non-unimodal Gorenstein Sequence

In this section, we compute a homogenous ideal J in a polynomial ring $\bar{R}$ such that $TE(A) = \bar{R}/J$ is a Gorenstein algebra with h-vector $(1, 13, 12, 13, 1)$. For this purpose, we use a software system Macaulay 2, which is devoted to supporting research in algebraic geometry and commutative algebra.

R. Stanley considered an Artinian level algebra $A = K[x_1, x_2, x_3]/(x_1, x_2, x_3)^4$ of type 10 with socle degree 3 and he has shown that there exists a non-unimodal Gorenstein sequence with the h-vector $(1, 13, 12, 13, 1)$ (see [8]). Note that the

h-vector of A is given by $(1,3,6,10)$. From Theorem 4, we see that $A \times \omega_A(-4)$ is a Gorenstein algebra with h-vector $(1,13,12,13,1)$ and

$$(x_1,x_2,x_3)^4 = \{\, f \in K[x_1,x_2,x_3] \,|\, f \circ g = 0, \quad \forall\, g \in K[y_1,y_2,y_3]_3\}.$$

Now we see that $(x_1,x_2,x_3)^4$ is the annihilator (under Macaulay's inverse system) of monomial basis of $K[y_1,y_2,y_3]_3$ in degree 3. By Theorem 4, we see that the ideal $J = \{G \ \in \bar{R} \,|\, G \circ F = 0\}$, where $\bar{R} = K[x_1,x_2,x_3,a_1,\ldots,a_{10}]$ and

$$\begin{aligned} F =& a_1y_1^3 + a_2y_1^2y_2 + a_3y_1^2y_3 + a_4y_1y_2^2 + a_5y_1y_2y_3 + \\ & a_6y_1y_3^2 + a_7y_2^3 + a_8y_2^2y_3 + a_9y_2y_3^2 + a_{10}y_3^3. \end{aligned}$$

Now, we have to compute the ideal $J = \{G \ \in \bar{R} \,|\, G \circ F = 0\}$. To do this, we need the following result.

Theorem 5. *Let $F \in K[x_1,\ldots,x_n]$ be a homogeneous polynomial of degree d. Let $H = \prod_{i=1}^n x_i^d$ and $K = (x_1^{d+1}, x_2^{d+1}, \ldots, x_n^{d+1})$. Then, we have*

$$\left(K \,:\, \frac{\partial H}{\partial F}\right) = \{G \ \in K[x_1,\ldots,x_n] \,|\, G \circ F = 0\}.$$

Proof. Note that it suffices to show that

$$G \circ F = \left(G\frac{\partial H}{\partial F}\right) \circ H, \tag{1}$$

since it follows directly from (1) that $G \circ F = 0$ if and only if a polynomial $G\dfrac{\partial H}{\partial F}$ is contained in $K = (x_1^{d+1}, x_2^{d+1}, \ldots, x_n^{d+1})$. For giving a proof (1), we use the following notations:

- $\mathbf{x}^\alpha = x_1^{a_1} \cdots x_n^{a_n}$ for an element $\alpha = (a_0,\ldots,a_n) \in \mathbb{Z}^n_{\geq 0}$.
- $\delta = (d,\ldots,d) \in \mathbb{Z}^n_{\geq 0}$, so $\mathbf{x}^\delta = H$.
- $\dfrac{\partial H}{\partial F} = \sum k_i \mathbf{x}^{\alpha_i}$, which means $F = \sum_i k_i \mathbf{x}^{\delta-\alpha_i}$.

Let $G = \sum_j \ell_j \mathbf{x}^{\beta_j}$. Then we have that

$$\begin{aligned} G \circ F =& G \circ (\sum k_i \mathbf{x}^{\delta-\alpha_i}) \\ =& \sum_{i,j} k_i \ell_j \mathbf{x}^{\delta-\alpha_i-\beta_j} \\ =& \left(G\frac{\partial H}{\partial F}\right) \circ H, \end{aligned}$$

which completes the proof. □

Using Macaulay 2, we define the polynomial ring $\bar{R}$ and F as follows:

```
i1   :   R = QQ[y1, y2, y3, a1,a2,a3,a4,a5,a6,a7,a8,a9,a10]

o1   =   R

o1   :   PolynomialRing

i2   :   F = a1*y1^3+a2*y1^2*y2+a3*y1^2*y3+a4*y1*y2^2+a5*y1*y2*y3+a6*...

           3       2           2             2                          2
o2   =   y1 a1 + y1 y2*a2 + y1 y3*a3 + y1*y2 a4 + y1*y2*y3*a5 + y1*y3 a6
         ---------------------------------------------------------------
             3       2             2       3
         + y2 a7 + y2 y3*a8 + y2*y3 a9 + y3 a10

o2   :   R

i3   :   H = product apply(generators R, v -> v^4)

           4  4  4  4  4  4  4  4  4  4  4  4   4
o3   =   y1 y2 y3 a1 a2 a3 a4 a5 a6 a7 a8 a9 a10

o3   :   R
```

Now, the expression `contract(m, n)` can be used to compute partial derivatives of a polynomial n by m (i.e., $m \circ n$).

```
i4   :   f1 := contract(F, H)

           4  4     4  4  4  4  4  4  4  4  4   3
o4   =   y1 y2 y3*a1 a2 a3 a4 a5 a6 a7 a8 a9 a10 +
         ---------------------------------------------------------------
           4  3  2  4  4  4  4  4  4  4  4  3   4
         y1 y2 y3 a1 a2 a3 a4 a5 a6 a7 a8 a9 a10 +
         ---------------------------------------------------------------
           4  2  3  4  4  4  4  4  4  4  3  4   4
         y1 y2 y3 a1 a2 a3 a4 a5 a6 a7 a8 a9 a10 +
         ---------------------------------------------------------------
           4     3  4  4  4  4  4  4  3  4  4   4
         y1 y2*y3 a1 a2 a3 a4 a5 a6 a7 a8 a9 a10 +
         ---------------------------------------------------------------
           3  4  2  4  4  4  4  4  3  4  4  4   4
         y1 y2 y3 a1 a2 a3 a4 a5 a6 a7 a8 a9 a10 +
         ---------------------------------------------------------------
           3  3  3  4  4  4  4  3  4  4  4  4   4
         y1 y2 y3 a1 a2 a3 a4 a5 a6 a7 a8 a9 a10 +
         ---------------------------------------------------------------
           3  2  4  4  4  4  3  4  4  4  4  4   4
         y1 y2 y3 a1 a2 a3 a4 a5 a6 a7 a8 a9 a10 +
         ---------------------------------------------------------------
           2  4  3  4  4  3  4  4  4  4  4  4   4
         y1 y2 y3 a1 a2 a3 a4 a5 a6 a7 a8 a9 a10 +
         ---------------------------------------------------------------
           2  3  4  4  3  4  4  4  4  4  4  4   4
         y1 y2 y3 a1 a2 a3 a4 a5 a6 a7 a8 a9 a10 +
         ---------------------------------------------------------------
              4  4  3  4  4  4  4  4  4  4  4   4
         y1*y2 y3 a1 a2 a3 a4 a5 a6 a7 a8 a9 a10
```

We define the ideal $K = (y_1^5, y_2^5, y_3^5, a_1^5, a_2^5, \ldots, a_{10}^5)$ and compute the desired ideal $\left(K : \dfrac{\partial H}{\partial F}\right) = \{G \in \bar{R} \mid G \circ F = 0\}$.

```
i5  :  K = matrix table(1, numgens R, (i, j) -> R_j^5)

o5  =  | y1^5 y2^5 y3^5 a1^5 a2^5 a3^5 a4^5 a5^5 a6^5 a7^5 a8^5 a9^5 a10^5 |

               1      13
o5  :  Matrix R <--- R

i6  :  mingens (ideal K : f1)
o6  =  a10^2 a9a10 a8a10 a7a10 a6a10 a5a10 a4a10 a3a10 a2a10 a1a10 y2a10 y1a10
       ------------------------------------------------------------------------
       a9^2 a8a9 a7a9 a6a9 a5a9 a4a9 a3a9 a2a9 a1a9 y2a9-y3a10 y1a9 a8^2 a7a8
       ------------------------------------------------------------------------
       a6a8 a5a8 a4a8 a3a8 a2a8 a1a8 y2a8-y3a9 y1a8 a7^2 a6a7 a5a7 a4a7 a3a7
       ------------------------------------------------------------------------
       a2a7 a1a7 y3a7 y2a7-y3a8 y1a7 a6^2 a5a6 a4a6 a3a6 a2a6 a1a6 y2a6
       ------------------------------------------------------------------------
       y1a6-y3a10 a5^2 a4a5 a3a5 a2a5 a1a5 y2a5-y3a6 y1a5-y3a9 a4^2 a3a4 a2a4
       ------------------------------------------------------------------------
       a1a4 y3a4 y2a4-y3a5 y1a4-y3a8 a3^2 a2a3 a1a3 y2a3 y1a3-y3a6 a2^2 a1a2
       ------------------------------------------------------------------------
       y3a2 y2a2-y3a3 y1a2-y3a5 a1^2 y3a1 y2a1 y1a1-y3a3 y3^4 y2y3^3 y1y3^3
       ------------------------------------------------------------------------
       y2^2y3^2 y1y2y3^2 y1^2y3^2 y2^3y3 y1y2^2y3 y1^2y2y3 y1^3y3 y2^4 y1y2^3
       ------------------------------------------------------------------------
       y1^2y2^2 y1^3y2 y1^4

               1      94
o6     Matrix R <--- R
```

Then we see that the h-vector of R/J is $(1, 13, 12, 13, 1)$ as we wished.

```
i7 :  J= ideal mingens (ideal K : f1);

o7 :  Ideal of R

i8 :  table(1, 5, (i, j) -> hilbertFunction(j, R/J))

o8 = {{1, 13, 12, 13, 1}}
```

Finally, let us compute the betti table of R/J.

```
i9  :  betti res J
```

o9 =

	0	1	2	3	4	5	6
total:	1	94	759	3145	8382	15620	21153
0:	1	.	.	.	.	.	.
1:		79	585	2220	5403	9150	11178
2:	.	.	.	.	.	.	.
3:	.	15	174	925	2979	6470	9975
4:	.	.	.	.	.	.	.

	7	8	9	10	11	12	13
total:	21153	15620	8382	3145	759	94	1
0:	.	.	.	.	.	.	.
1:	9975	6470	2979	925	174	15	.
2:	.	.	.	.	.	.	.
3:	11178	9150	5403	2220	585	79	.
4:	.	.	.	.	.	.	1

```
o9   :   BettiTally
```

This betti table shows that $A = R/J$ is a Gorenstein Artinian $\mathbb{Q}$-algebra, as we wished.

Remark 6. *Let R/J be an Artinian Gorenstein algebra with Hilbert function $h = (1, 13, 12, 13, 1)$ and let L be a general linear form in R. Without loss of generality, we may assume that $L = y_3$. Then, we have a decomposition of Hilbert function $h = (1, 13, 12, 13, 1)$;*

$$\begin{array}{cccccccc} h & : & 1 & 13 & 12 & 13 & 1 \\ \ell & : & 1 & 12 & \ell_2 & \ell_3 & 0 \\ b & : & & 1 & b_2 & b_3 & 1 \end{array}$$

where ℓ and b are Hilbert functions of $R/(J, y_3)$ and $R/(J : y_3)$, respectively (see [5]). Then, there are two possibilities for ℓ and b, namely,

$$\begin{array}{ccccccc} \ell & : & 1 & 12 & 3 & 4 & 0 \\ b & : & & 1 & 9 & 9 & 1\,, \end{array}$$

or

$$\begin{array}{ccccccc} \ell & : & 1 & 12 & 4 & 5 & 0 \\ b & : & & 1 & 8 & 8 & 1\,. \end{array}$$

Which one does correspond to our case? Using Macaulay 2, we can compute the restriction of the ideal J with the linear form y_3.

```
i10  :   S=QQ[y1,y2,a1,a2,a3,a4,a5,a6,a7,a8,a9,a10];

i11  :   ringhomomorphism = map(S, R,{y1,y2,0,a1,a2,a3,a4,a5,a6,a7,a8,a9,a10});

o11  :   RingMap S <--- R

i12  :   J1= mingens ringhomomorphism J

o12  =   a10^2 a9a10 a8a10 a7a10 a6a10 a5a10 a4a10 a3a10 a2a10 a1a10 y2a10 y1a10
         -----------------------------------------------------------------------
         a9^2 a8a9 a7a9 a6a9 a5a9 a4a9 a3a9 a2a9 a1a9 y2a9 y1a9 a8^2 a7a8 a6a8
         -----------------------------------------------------------------------
         a5a8 a4a8 a3a8 a2a8 a1a8 y2a8 y1a8 a7^2 a6a7 a5a7 a4a7 a3a7 a2a7 a1a7
         -----------------------------------------------------------------------
         y2a7 y1a7 a6^2 a5a6 a4a6 a3a6 a2a6 a1a6 y2a6 y1a6 a5^2 a4a5 a3a5 a2a5
         -----------------------------------------------------------------------
         a1a5 y2a5 y1a5 a4^2 a3a4 a2a4 a1a4 y2a4 y1a4 a3^2 a2a3 a1a3 y2a3 y1a3
         -----------------------------------------------------------------------
         a2^2 a1a2 y2a2 y1a2 a1^2 y2a1 y1a1 y2^4 y1y2^3 y1^2y2^2 y1^3y2 y1^4
                1      80
o12  :   Matrix S <--- R

i13  :   table(1, 5, (i, j) -> hilbertFunction(j, S/J1))

o13  =   {{1, 12, 3, 4, 0}}
```

From the computation by using Macaulay 2, we see that our case corresponds to the former case. However, we do not know if there would be a Gorenstein algebra satisfying the latter case. Hence there is a natural question here.

Question 7. *Can we construct a Gorenstein Artinian algebra $\bar{R}/J$ with the following decomposition of h-vector*

$$\begin{array}{ccccccc} h & : & 1 & 13 & 12 & 13 & 1 \\ \ell & : & 1 & 12 & 4 & 5 & 0 \\ b & : & 0 & 1 & 8 & 8 & 1 \end{array}$$

where ℓ and b are Hilbert functions of $R/(J,L)$ and $R/(J:L)(-1)$, respectively.

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