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MAXIMUM CURVES OF TRANSCENDENTAL ENTIRE FUNCTIONS OF THE FORM $E^{p(z)}$†

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Abstract. The function $f(z) = e^{p(z)}$ where $p(z)$ is a polynomial of degree n has $2n$ Julia lines. Julia lines of $e^{p(z)}$ divide the complex plane into $2n$ equal sectors with the same vertex at the origin. In each sector, $e^{p(z)}$ has radial limits of 0 or infinity. Main results of the paper are concerned with maximum curves of $e^{p(z)}$. We deal with some properties of maximum curves of $e^{p(z)}$ and we give some examples of the maximum curves of functions of the form $e^{p(z)}$.

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1. Introduction

For a transcendental entire function $f(z)$, we define maximum function $M(r, f)$ and maximum curve $M_c(f)$ as follows:

$$M(r, f) := \max_{|z|=r}(|f(z)|), \quad M_c(f) := \{z : |f(z)| = M(|z|, f)\}.$$

We call all curves where $\frac{\partial}{\partial\theta}|f(re^{i\theta})| = 0$ the beta curves of $f(z)$ as in T. Tylor [7]. Every maximum curve of $f(z)$ is a beta curve. For example, real line is the beta curve and positive real line is the maximum curve of $f(z) = e^z$. And its maximum function is $M(r, e^z) = e^r$.

We call $\phi \in [0, 2\pi)$ a Julia line of the entire funtion f, if in every sector

$$\{z : |\arg z - \phi| < \delta\}, \quad \delta > 0,$$

the function f assumes each complex value infinitely often, with at most one exception.

In this paper, we mainly consider behaviors and maximum curves of transcendental entire functions of the form $e^{p(z)}$ where $p(z)$ is a nonconstant polynomial.

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2. Behaviors of $e^{p(z)}$

Let $\{\theta_j\}$ be an increasing sequence of $[0, 2\pi)$ and $S_j := \{z : \theta_j < \arg z < \theta_{j+1}\}$ be open sectors with the common vertex at the origin. We know that if the open set $\bigcup_{j=1}^{\infty}(\theta_j, \theta_{j+1})$ is everywhere dense on $[0, 2\pi)$, then there exists an entire function f such that

$$\lim_{r\to\infty} f(re^{i\theta}) = c_j$$

for each j where $re^{i\theta} \in S_j$ and c_j are preassigned values from $\mathbb{C} \cup \{\infty\}$ (see D. Gaier [3]).

But the function $f(z) = e^{p(z)}$ takes 0 or infinity as its radial limits on the equally devided open sectors and the number of open sectors depends on the degree of the polynomial $p(z)$.

Throughout the paper, $p(z)$ denote polynomial of degree n:

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad (a_n \neq 0,\ n \geq 1) \tag{1}$$

where

$$a_k = s_k e^{i\alpha_k},\ z = re^{i\theta} \quad (k = 0, 1, 2, \cdots, n). \tag{2}$$

For each $j = 0, 1, 2, \cdots, 2n-1$, we write

$$\tau_j = -\frac{\alpha_n}{n} + (2j-1)\frac{\pi}{2n}, \tag{3}$$

and

$$L_j = \{re^{\tau_j} : r > 0\}. \tag{4}$$

We divide the complex plane into $2n$ open sectors

$$S_j := \{z : \tau_j < \arg z < \tau_{j+1}\},\ j = 0, 1, 2, \cdots, 2n-1$$

with the same vertex at the origin.

The following theorem is a well-known result (see Markushevich [6] for details). Here we regard 0 as an even number for our convenience.

Theorem A. *The function $f(z) = e^{p(z)}$ has radial limits on each sector S_j:*

$$\lim_{\substack{|z|=r\to\infty \\ z\in S_j}} |f(z)| = \begin{cases} 0, & \text{if } j \text{ is odd,} \\ \infty, & \text{if } j \text{ is even.} \end{cases}$$

Furthermore the limits are uniform on any closed subsector of S_j.

For a transcendental entire function $f(z)$, it is known that if there is a nonzero finite point z_0 such that the family $\{f(2^n z)\}$ fails to be normal in every neighborhood of z_0, then the ray emanating from the origin and passing through the point z_0 is a Julia line of $f(z)$ and the converse is also true [5, 6]. It is also known that every transcendental entire function has at least one Julia line [2].

From the above theorem we can show that the function $f(z) = e^{p(z)}$ has $2n$ Julia lines and the locations of the Julia lines are determined by the argument of the leading coefficient of the polynomial $p(z)$.

Theorem 1. *Suppose that $p(z)$ is the polynomial of degree n described as in (1) and (2) above. Then the function $f(z) = e^{p(z)}$ has $2n$ Julia lines.*

Proof. We will show that every ray $L_j (j = 0, 1, \cdots, 2n-1)$ is a Julia line of $f(z)$, where L_j is defined as in (4).

For any point z_0 on L_j and $\epsilon > 0$ we can choose two points z_1 and z_2 from the neighborhood of z_0, $N(z_0, \epsilon)$, such that

$$\lim_{k\to\infty} f(2^k z_1) = 0, \text{ and } \lim_{k\to\infty} f(2^k z_2) = \infty$$

by Theorem A. So the family $\{f_k(z) := f(2^k z)\}$ is not normal at the point z_0.

On the other hand, for each point w_0 ($w_0 \neq 0$) which is not on any ray L_j ($j = 0, 1, \cdots, 2n-1$), we can choose a positive number ϵ such that

$$\left(\bigcup_{j=0}^{2n-1} L_j\right) \bigcap N(w_0, \epsilon) = \emptyset.$$

Then the family $\{f_k(z)\}$ converges identically either to 0 or ∞ on the set $N(w_0, \epsilon)$. Hence the family is normal at w_0. Therefore all the rays L_j are the only Julia lines of $f(z)$. This completes the proof. □

3. Maximum curves

For a transcendental entire function $f(z)$, the maximum function $M(r, f)$ is strictly increasing and continuous on $[0, \infty)$. In this section we discuss some properties of the maximum curve $M_c(f)$.

Theorem 2. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function. Suppose that there exists a point $z_0 \in \mathbb{C} \setminus \{0\}$ such that the principal argument of $a_n z_0^n$ are the same for all $a_n \neq 0$. Then

$$|f(z_0)| = \max_{|z|=|z_0|} |f(z)| = M(|z_0|, f).$$

Proof. Let $\theta_0 = \operatorname{Arg}(a_n z_0^n)$ for all $a_n \neq 0$. We have

$$\operatorname{Arg}(e^{-i\theta_0} a_n z_0^n) = 0 \text{ and } e^{-i\theta_0} a_n z_0^n > 0.$$

So for all z with $|z| = |z_0|$, the inequality

$$|f(z_0)| = |e^{-i\theta_0} f(z_0)| = \left|\sum_{n=0}^{\infty} e^{-i\theta_0} a_n z_0^n\right| = \sum_{n=0}^{\infty} |a_n z_0^n| = \sum_{n=0}^{\infty} |a_n z^n|$$

$$\geq \left|\sum_{n=0}^{\infty} a_n z^n\right| = |f(z)|$$

holds. Hence $M(|z_0|, f) = |f(z_0)|$. This completes the proof. □

In the above theorem if all the principal argument of $a_n z_0^n$ are the same, then $\mathrm{Arg}(a_n z_0^n) = \mathrm{Arg}(a_n (rz_0)^n)$ for all $a_n \neq 0$ and $r > 0$. Therefore the ray $\theta_0 = \mathrm{Arg}(a_n z_0^n)$ is a maximum curve of $f(z)$, and its maximum function is given by

$$M(r, f) = |f(z)|_{z=re^{i\theta_0}}.$$

To illustrate, we consider the function

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

By choosing $z_0 = i$, we have

$$\mathrm{Arg}\left((-1)^n \frac{i^{2n+1}}{(2n+1)!}\right) = \mathrm{Arg}\left(\frac{i}{(2n+1)!}\right) = \frac{\pi}{2},$$

for all $n = 0, 1, 2, \cdots$. Thus $\theta_0 := \mathrm{Arg}(z_0) = \frac{\pi}{2}$ is a maximum curve of sin$z$ and its maximum function is given by

$$M(r, \sin z) = \left|\frac{e^{iz} - e^{-iz}}{2i}\right|_{z=re^{\frac{\pi}{2}i}} = \frac{e^r - e^{-r}}{2}.$$

Theorem 3. *Let $f(z)$ be a nonconstant, zero free entire function. Then the maximum curves of $f(z)$ intersect the circle $|z| = r$ at finitely many points for each $r > 0$.*

Proof. Since $f(z)$ is a zero free entire function, there is an entire function $g(z)$ such that $f(z) = e^{g(z)}$. Suppose that the maximum curves of $f(z)$ pass infinitely many points $\{r_0 e^{i\theta_\alpha}\}$ on the circle $|z| = r_0$. Then the harmonic function Re $g(z)$ takes the same value $\log M(r_0, f)$ at points $r_0 e^{i\theta_\alpha}$ for all α, yet the set $\{r_0 e^{i\theta_\alpha}\}$ has a limit point on the circle $|z| = r_0$. Hence the harmonic function Re $g(z)$ is a constant function with the value $\log M(r_0, f)$ by identity theorem for harmonic functions which means that the functions $g(z)$ and $f(z)$ are constants on the complex plane. And it is evident that the maximum curves of $f(z)$ touch at least one point of the circle $|z| = r$ for all $r > 0$. This completes proof. □

We have the following

Corollary. *If $p(z)$ is a nonconstant polynomial, then the maximum curves of $e^{p(z)}$ intersect the circle of radius $r(> 0)$ centered at the origin at fiitely many points.*

Theorem 4. *Let $f_1(z)$ and $f_2(z)$ be noncostants, zero free entire functions. Suppose that $f_1(z)$, $f_2(z)$ share the maximum curves and that $M(r, f_1) = M(r, f_2)$. Then $f_1(z)$ and $f_2(z)$ have the same modulus for all $z \in \mathbb{C}$.*

Proof. Let $g_1(z)$ and $g_2(z)$ be entire functions such that $f_1(z) = e^{g_1(z)}$ and $f_2(z) = e^{g_2(z)}$. Let $u_j(z) := \text{Re } g_j(z)$ and let $v_j(z)$ be a harmonic conjugate of $u_j(z),\ j = 1, 2$. Since $M(r, f_1) = M(r, f_2)$, we have $u_1(z) = u_2(z)$, and there is a real number α such that

$$v_1(z) = v_2(z) + \alpha$$

for all z on the maximum curves. So $g_1(z)$ and $g_2(z)$ can be written as

$$g_1(z) = u_1(z) + iv_1(z) \tag{5}$$

and

$$g_2(z) = u_2(z) + iv_2(z) = g_1(z) + i\alpha \tag{6}$$

for all $z \in \mathbb{C}$ by the identity theorem.

The equations (5) and (6) lead to

$$\left|\frac{f(z)}{g(z)}\right| = \left|\frac{e^{g_1(z)}}{e^{g_2(z)}}\right| = \left|\frac{1}{e^{i\alpha}}\right| = 1,$$

the described result. □

Corollary. *Let $p_1(z)$ and $p_2(z)$ be nonconstant polynomials. Suppose that $e^{p_1(z)}$ and $e^{p_2(z)}$ share the maximum curves and that $M(r, e^{p_1(z)}) = M(r, e^{p_2(z)})$. Then $e^{p_1(z)}$ and $e^{p_2(z)}$ have the same modulus at each point $z \in \mathbb{C}$.*

4. Examples

J. Clunie [1] posed the question as to whether the maximum curve $M_c(f)$ can have an isolated point. And the answer was given by T.Tyler. The following theorem is due to Tyler [7].

Theorem B. *If for an entire function $f(z)$, two different beta curves meet at a point $z_0 = r_0e^{i\theta_0},\ r_0 > 0$, then*

$$\frac{d}{dz}\left\{\frac{f'(z)}{f(z)}\right\}\Big|_{z=r_0e^{i\theta_0}} = 0.$$

Tyler gave an example of an entire function whose maximum point is isolated: polynomial $p(z) = C(z^2 + 1)^2 + z(z^2 - 1)^2,\ C > 1$ has an isolated maximum point at $z_0 = -1$.

Example 1. The function $e^{p(z)}$, where $p(z) = C(z^2 + 1)^2 + z(z^2 - 1)^2,\ C > 1$ has the same property *i.e.*, $e^{p(z)}$ has an isolated maximum point at $z = -1$.

This can be seen as follows: on the unit circle $|z| = 1$,

$$\begin{aligned}\text{Re } p(z) &\le |p(z)| \le C\left|z^2\left(z+\frac{1}{z}\right)^2\right| + \left|z^2\left(z-\frac{1}{z}\right)^2\right| \\ &\le 4C\cos^2\theta + 4\sin^2\theta = 4C\left[1-\left(1-\frac{1}{C}\right)\sin^2\theta\right] \\ &\le 4C = p(1) = p(-1).\end{aligned}$$

Hence the function $\left|e^{p(e^{i\theta})}\right|$ takes a maximum at $z = -1$, *i.e.*, $M(1, e^{p(z)}) = e^{p(-1)}$.

And $p(r) > p(-r)$ in some deleted neighborhood of $r = -1$ of $\mathbb{R}$. Since

$$\frac{d}{dz}\left\{\frac{(e^{p(z)})'}{e^{p(z)}}\right\}\Big|_{z=-1} = 16C - 8 \neq 0$$

the negative real line is the only beta curve passing the point $z = -1$ by Theorem B. Hence $z = -1$ is an isolated maximum point of $e^{p(z)}$.

Example 2. A component of maximum curves of a transcendental entire function may not start from the origin. It may have finite length and split into different directions.

As an example consider a polynomial $p(z) = z^3 + 3z^2 - 4z$ whose real part is

$$\begin{aligned} s(r,\theta) := \text{Re } p(z) &= r^3\cos 3\theta + 3r^2\cos 2\theta - 4r\cos\theta \\ &= 4r^3\cos^3\theta + 6r^2\cos^2\theta - (3r^3+4r)\cos\theta - 3r^2. \end{aligned} \tag{7}$$

We substitute t for $\cos\theta$ in (7) and let

$$h(r,t) := 4r^3t^3 + 6r^2t^2 - (3r^3+4r)t - 3r^2.$$

The function $h(r,t)$ has a local maximum at $t_r = \frac{-3-\sqrt{21+9r^2}}{6r}$ when $r \ge \frac{2}{3}(1+\sqrt{2})$ and has no local maximum when $r < \frac{2}{3}(1+\sqrt{2})$ on the interval $-1 \le t \le 1$.

We compare the values of $h(r,t)$ at two end points of the interval and at a local maximum point:

$$h(r,t_r) - h(r,1) \quad \begin{cases} > 0 & \text{if} \quad \frac{2}{3}(1+\sqrt{2}) < r < \frac{25}{12}, \\ = 0 & \text{if} \quad r = \frac{25}{12}, \\ < 0 & \text{if} \quad r > \frac{25}{12}, \end{cases} \tag{8}$$

$$h(r,t_r) - h(r,-1) \quad \begin{cases} > 0 & \text{if } r > \frac{2}{3}(1+\sqrt{2}), \\ = 0 & \text{if } r = \frac{2}{3}(1+\sqrt{2}), \\ < 0 & \text{if } 0 < r < \frac{2}{3}(1+\sqrt{2}), \end{cases} \tag{9}$$

$$h(r,1) - h(r,-1) \quad \begin{cases} < 0 & \text{if} \quad r < 2, \\ = 0 & \text{if} \quad r = 0, \\ > 0 & \text{if} \quad r > 2. \end{cases} \tag{10}$$

The above relations (8-10) lead to the following results:

$$\max h(r,t) = \begin{cases} h(r,-1) & \text{if} \quad 0 < r \leq \frac{2}{3}(1+\sqrt{2}), \\ h(r,t_r) & \text{if} \quad \frac{2}{3}(1+\sqrt{2}) \leq r \leq \frac{25}{12}, \\ h(r,1) & \text{if} \quad r \geq \frac{25}{12} \end{cases}$$

and

$$M(r, e^{p(z)}) = \begin{cases} e^{h(r,-1)} & \text{if} \quad 0 < r \leq \frac{2}{3}(1+\sqrt{2}), \\ e^{h(r,t_r)} & \text{if} \quad \frac{2}{3}(1+\sqrt{2}) \leq r \leq \frac{25}{12}, \\ e^{h(r,1)} & \text{if} \quad r \geq \frac{25}{12}. \end{cases}$$

We note that

$$t_r|_{r=\frac{2}{3}(1+\sqrt{2})} = t_r = \frac{-3-\sqrt{21+9r^2}}{6r}\Big|_{r=\frac{2}{3}(1+\sqrt{2})} = -1,$$

and

$$t_r|_{r=\frac{25}{12}} = t_r = \frac{-3-\sqrt{21+9r^2}}{6r}\Big|_{r=\frac{25}{12}} = -\frac{43}{50} \neq 1.$$

So the maximum curve of $e^{z^3+3z^2-4z}$ goes along the negative real line when $0 < r \leq \frac{2}{3}(1+\sqrt{2})$, and it splits into two curves at $r = \frac{2}{3}(1+\sqrt{2})$. On the interval $\frac{2}{3}(1+\sqrt{2}) \leq r \leq \frac{25}{12}$ maximum curves are determined by $\cos\theta = \frac{-3-\sqrt{21+9r^2}}{6r}$. These two curves do not touch the positive real line. Finally, the curve goes along the positive real line when $r \geq \frac{25}{12}$.

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