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SOME WAITING TIME ANALYSIS FOR CERTAIN QUEUEING POLICIES

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Abstract. In a M/G/I queue where the server alternates between busy and idle periods, we assume that firstly customers arrive at the system according to a Poisson process and the arrival process and customer service times are mutually independent, secondly the system has infinite waiting room, thirdly the server utilization is less than 1 and the system has reached a steady state. With these assumptions, we analyze waiting times on the systems where some vacation policies are considered.

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1. Introduction

Let us first define an M/G/I queue where the server alternates between busy and idle periods. In this system, the assumptions are made as follows: 1) Customers arrive at the system according to a poisson process with rate λ, and have a general service time with a Laplace Transform, $X^*(s)$, where its first two moments, $\overline{x}$ and $\overline{x^2}$, are finite. Further, the arrival process and customer service times are mutually independent, and are also independent of any measure of past system behavior; 2) The system has infinite waiting room; 3) The server utilization ($\rho = \lambda\overline{x}$) is less than 1 and the system has reached a steady state. After serving all customers or returning from certain vacations, the server depending on the system behavior prior to that time epoch, may decide to remain in the system or to take a new vacation. Such decision is called a vacation decision and the time is a decision epoch. If a vacation is taken, its length may also depend on past system history. The vacation lengths are characterized by a limiting distribution function in the way specified by in [7]. Despite the possible dependency of the vacation policy on past system behavior, future arrivals and service times are not affected by the policy or the vacation length. To fully

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specify a policy, we specify the server will take another vacation if the system is still empty upon returning from a vacation and server will start a vacation at a later time, given that the server first chooses to remain idle after serving all customers, and that the system is still empty at the start of the vacation. We treat a vacation in the system as a special customer with a service time equal to the vacation length. Thus, as the service discipline is the first come first service, the amount of work found by an arriving customer is equal to his waiting time.

2. Combination of individual vacation and idle time

Under this policy, the server, after serving all customers, decides whether to take a vacation or to remain idle in the system. If a vacation is taken and the system is still empty when the server returns from the vacation, the server again chooses between taking another vacation or staying idle. Once the server decides to stay idle, it remains in the system until customer arrive and are completely served. Then, the process for vacation decisions repeats.

Let α_v be the long-term fraction of decision epochs at which the server chooses to take a vacation. Thus, $1-\alpha_v$ is the fraction that the server chooses to remain idle. It is assumed that the decision of taking a vacation is independent of the vacation length. Similar to the previous discussion, we construct $Z_v(\hat{t})$ for this vacation policy. As defined previously, a cycle is the time interval in $\hat{t}$ between two decision epochs. Clearly, given that a vacation is taken, the average cycle length is the average vacation length. If the server choose to be idle, the average length for the idle time until the next sampling point is $1/\lambda$. Unconditioning these two situations with α_v and $1-\alpha_v$, the average cycle length is $\overline{c} = \alpha_v\,\overline{v}+(1-\alpha_v)/\lambda$. Based on this and the poisson-arrivals-see-time-averages property, we have

$$P_0 = \frac{(1-\alpha_v)/\lambda}{\overline{c}} = \frac{(1-\alpha_v)}{\alpha_v\overline{v}\lambda + 1 - \alpha_v}.$$

The Laplace transform for the waiting time, i.e, amount of work seen by a random arrival is

$$U_B^*(s) = \frac{1-\rho}{s-\lambda+\lambda X^*(s)}\left\{\frac{(1-\alpha_v)s + \lambda\alpha_v[1-V^*(s)]}{1+\alpha_v(\lambda\overline{v}-1)}\right\}.$$

3. Single vacation with the delay

Under the policy of delayed single vacation, the server after serving all customers remains in the system for a period of time. Let this time be referred to as changeover time. The changeover time may be used to model the delay incurred in changing the server from service mode to vacation mode. If any customer arrives during the changeover time, then the server starts to serve the customer without delay. After all customers are served, the vacation decision repeats. If no customer arrives during the changeover time, then the server takes a single vacation. At the end of the vacation, the server either stays idle if the system is empty or starts to serve the backlogged customers. After customers have been completely served, another vacation decision occurs again.

Although a changeover time denoted by $\overline{r}$ and a vacation length may depend on the past system behavior, they are assumed to be mutually independent of each other. Let the average, the stationary pdf and the Laplace transform for $\overline{r}$ be denoted by $\overline{r}, r(y)$ and $R^*(s)$, respectively. The process $Z_v(\hat{t})$ for this policy is constructed. As defined previously, a cycle is the virtual time interval between two decision epochs.

Two cases arise depending on whether or not a sampling point arrives before the end of a changeover time. If no sampling point arrives a changeover time, a single vacation is started. Hence, the average cycle length in this case is $\overline{r}_a + \overline{r} + v^*(\lambda)/\lambda$ where

$$\overline{r}_a = \frac{1}{R^*(\lambda)} \int_0^\infty y r(y) e^{-\lambda y} dy \tag{1}$$

is the conditional average changeover time given that no sampling point arrives during the time period. The probability of the occurrence of this case is $R^*(\lambda) = \int_0^\infty e^{-\lambda y} r(y) dy$. In the second case, a sampling point arrives before the changeover time expires and immediately after the sampling point is a decision epoch. Let $\overline{r}_a$ denote the average length of virtual time from the start of the changeover time until the first sampling point arrives. Then, the average cycle length for this case is $\overline{r}_a$. By conditional probability, $\overline{r}_a$ is given by

$$\overline{r}_a = \frac{1}{\lambda} - \frac{1}{1 - R^*(\lambda)} \int_0^\infty y r(y) e^{-\lambda y} dy. \tag{2}$$

The probability of occurrence of this case is $1 - R^*(\lambda)$. Hence, unconditioning both cases gives the average cycle length,

$$\overline{c} = R^*(\lambda) \left[\overline{r}_a + \overline{v} + R^*(\lambda) \frac{1}{\lambda} \right] + \left[1 - R^*(\lambda) \right] \overline{r}_a.$$

We note that the server is idle during certain portion of virtual time in a cycle. The average length of this idle time in each cycle is

$$R^*(\lambda) \left[\overline{r}_a + V^*(\lambda)/\lambda \right] + \left[1 - R^(\lambda) \right] \overline{r}_a.$$

Based on this and the poisson-arrivals-see-time-averages property, we get

$$P_0 = \frac{R^*(\lambda) \left[\overline{r}_a + V^*(\lambda) \frac{1}{\lambda} \right] + \left[1 - R^*(\lambda) \right] \overline{r}_a}{R^*(\lambda) \left[\overline{r}_a + \overline{v} + R^*(\lambda) \frac{1}{\lambda} \right] + \left[1 - R^*(\lambda) \right] \overline{r}_a}.$$

This yields the Laplace transform for waiting time for this vacation policy.

4. Alternating vacation and wait periods

In this policy, the server takes a vacation after serving all customers. Upon returning from the vacation, the server starts serving the backlogged customers, if any. In case the system is still empty at that time, the server waits in the system for the next arrival for a period of time referred to as a wait period. When the customer arrives, service is started immediately without further delay. However, if no customer arrives during the wait period, the server takes another vacation. Then, the process of vacation and wait periods repeats until customers arrive and are exhaustively served. At that point in time, the alternation of vacation and wait periods occurs again.

Assume that the vacation lengths and wait periods from an alternation renewal sequence. Further, let the r.v., the mean, the pdf and the Laplace transform for the length of a wait period be $\tilde{r}, \overline{r}, r(y)$ and $R^*(s)$, respectively. As a result, $R^*(\lambda)$ is the probability that no customer arrives during a wait period. The process $Z_v(\hat{t})$ for this policy can be constructed. We define a *cycle* as the time interval in $\hat{t}$ between the beginnings of two vacations taken immediately after the server servers all customers. It is easy to verify that the average cycle length is

$$\begin{aligned}\overline{c} &= \sum_{k=0}^{\infty}\Big[V^*(\lambda)R^*(\lambda)\Big]^k\Big\{\overline{v}+V^*(\lambda)R^*(\lambda)\overline{r}_a+V^*(\lambda)\big[1-R^*(\lambda)\big]\overline{r}_a\Big\}\\ &= \frac{\overline{v}+V^*(\lambda)R^*(\lambda)\overline{r}_a+V^*(\lambda)\Big[1-R^*(\lambda)\Big]\overline{r}_a}{1-V^*(\lambda)R^*(\lambda)} \qquad (3)\end{aligned}$$

where $\overline{r}_a$ is the average length of a wait period conditioned that no customer arrives, while $\overline{r}_a$ is the average time from the start of a wait period until the arrival of the next customer, given that at least one customer arrival during the wait period. These quantities are given by (1) and (2), respectively.

The last two terms in the numerator of (3) represent the average time interval during an average cycle at which the server is idle in the system. Using the poisson-arrivals-see-time-averages property, the probability that a sampling point finds the server being idle upon arrival is

$$P_0 = \frac{V^*(\lambda)R^*(\lambda)\overline{r}_a+V^*(\lambda)\big[1-R^*(\lambda)\big]\overline{r}_a}{\overline{v}+V^*(\lambda)R^*(\lambda)\overline{r}_a+V^*(\lambda)\big[1-R^*(\lambda)\big]\overline{r}_a}.$$

This yields the Laplace transform for waiting time for this policy.

5. Combination types of vacation with idle times

This policy considers (a) the system has a finite number, N, types of vacation, and (b) if the system is still empty after all types of vacation have been taken sequentially, the server stays idle in the system to wait for the next arrival.

Let a *cycle* for $Z_v(\hat{t})$ be the virtual time period between the start of two adjacent type-1 vacation. The average cycle length for the policy is

$$\overline{c} = \sum_{k=1}^{N}\prod_{j=1}^{k-1}V_j^*(\lambda)\overline{v}_k+\prod_{j=1}^{N}V_j^*(\lambda)\frac{1}{\lambda}.$$

Again, we note that the last term is the average duration of a cycle for which the server is idle. Using the poisson-arrivals-see-time-averages property, we have

$$P_0 = \frac{\prod_{j=1}^{N} V_j^*(\lambda)}{\lambda \sum_{k=1}^{N} \prod_{j=1}^{k-1} V_j^*(\lambda)\overline{v}_k + \prod_{j=1}^{N} V_j^*(\lambda)}.$$

Similarly, the average duration of $\hat{t}$ for which $Z_v(\hat{t})$ corresponds to a type-i vacation is $\prod_{j=1}^{i-1} V_j^*(\lambda)\overline{v}_i$. Thus, by the poisson-arrivals-see-time-averages property, the probability that a sampling that a sampling point finds $Z_v(\hat{t})$ having type-i vacation work is

$$P_i = \frac{\lambda \prod_{j=1}^{N} V_j^*(\lambda)\overline{v}_i}{\lambda \sum_{k=1}^{N} \prod_{j=1}^{k-1} V_j^*(\lambda)\overline{v}_k + \prod_{j=1}^{N} V_j^*(\lambda)}.$$

This gives the Laplace transform for the customer waiting time.

6. Conclusions

We analyzed the waiting times with some vacation policies. Some equations of the combination of individual vacation and idle time, single vacation with the delay, alternating vacation and wait periods, and combination types of vacation with idle times, based on the poisson-arrivals-see-time-averages property, have been obtained.

The analysis approach in this paper may be applicable to other related queueing models like variants of priority queues. These results can serve as a basis for the formulation and solution of certain optimization problems involved with vacation models. Using the analysis on the vacation system.

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