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ON THE SYMMETRY PROPERTIES OF THE GENERALIZED HIGHER-ORDER EULER POLYNOMIALS†

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Abstract. In this paper we prove a generalized symmetry relation between the generalized Euler polynomials and the generalized higher-order (attached to Dirichlet character) Euler polynomials. Indeed, we prove a relation between the power sum polynomials and the generalized higher-order Euler polynomials..

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1. Introduction

Let d be a fixed positive integer and let χ be the Dirichlet's character with conductor d. Then the generalized Euler numbers and polynomials attached to χ are defined as

$$\begin{aligned} \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{at}}{e^{dt}+1} &= \sum_{n=0}^{\infty} E_{n,\chi}\frac{t^n}{n!}, \\ \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{at}}{e^{dt}+1}e^{xt} &= \sum_{n=0}^{\infty} E_{n,\chi}(x)\frac{t^n}{n!}, \text{ for } |t| < \frac{\pi}{d}, \text{ (see [1-4])}. \end{aligned} \quad (1)$$

For a real or complex parameter α, we define the generalized higher-order Euler numbers and polynomials, of order α, attached to χ as follows:

$$\begin{aligned} \left(\frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{at}}{e^{dt}+1}\right)^{\alpha} &= \sum_{n=0}^{\infty} E_{n,\chi}^{(\alpha)}\frac{t^n}{n!}, \text{ and} \\ \left(\frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{at}}{e^{dt}+1}\right)^{\alpha} e^{xt} &= \sum_{n=0}^{\infty} E_{n,\chi}^{(\alpha)}(x)\frac{t^n}{n!}, \text{ where } |t| < \frac{\pi}{d}. \end{aligned} \quad (2)$$

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The main purpose of this paper is to prove an identity of symmetry for the generalized higher-order Euler polynomials. It turn out that the recurrence relation and multiplication theorem for the generalized Euler polynomials attached to χ.

For the basic definitions and properties of the Euler polynomials, see [1-6].

2. Symmetry Identities related to the generalized higher-order Euler polynomials

Theorem 1. *Let χ be the Dirichlet's character with conductor $d \in \mathbb{N}^*$, and a, b be the natural numbers with the same parity. For $n, m \in \mathbb{N}$, we have*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}E_{n-k,\chi}^{(m-1)}(aY)\sum_{j=0}^{d-1}\sum_{i=0}^{a-1}\chi(j)(-1)^{i+j}E_{k,\chi}^{(m)}\left(bX+\frac{bd}{a}i+\frac{b}{a}j\right)\\
=&\sum_{k=0}^{n}\binom{n}{k}b^k a^{n-k}E_{n-k,\chi}^{(m-1)}(bY)\sum_{j=0}^{d-1}\sum_{i=0}^{b-1}\chi(j)(-1)^{i+j}E_{k,\chi}^{(m)}\left(aX+\frac{ad}{b}i+\frac{a}{b}j\right).
\end{aligned}\tag{3}$$

Proof. The generating function of the left hand side of (3) is

$$\begin{aligned}
&\sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}E_{n-k,\chi}^{(m-1)}(aY)\sum_{j=0}^{d-1}\sum_{i=0}^{a-1}\chi(j)(-1)^{i+j}E_{k,\chi}^{(m)}\left(bX+\frac{bd}{a}i+\frac{b}{a}j\right)\frac{t^n}{n!}\\
=&\sum_{j=0}^{d-1}\sum_{i=0}^{a-1}\chi(j)(-1)^{i+j}\sum_{n\geq 0}\sum_{k=0}^{n}E_{n-k,\chi}^{(m-1)}(aY)\frac{(bt)^{n-k}}{(n-k)!}E_{k,\chi}^{(m)}\left(bX+\frac{bd}{a}i+\frac{b}{a}j\right)\frac{(at)^k}{k!}\\
=&\sum_{j=0}^{d-1}\sum_{i=0}^{a-1}\chi(j)(-1)^{i+j}\left(\sum_{k\geq 0}E_{k,\chi}^{(m-1)}(aY)\frac{(bt)^k}{(k!}\right)\\
&\times\left(\sum_{k\geq 0}E_{k,\chi}^{(m)}\left(bX+\frac{bd}{a}i+\frac{b}{a}j\right)\frac{(at)^k}{(k!}\right)\\
=&\sum_{j=0}^{d-1}\sum_{i=0}^{a-1}\chi(j)(-1)^{i+j}\left(\frac{2\sum_{j=0}^{d-1}(-1)^j\chi(j)e^{jbt}}{e^{dbt}+1}\right)^{m-1}e^{abYt}\\
&\times\left(\frac{2\sum_{j=0}^{d-1}(-1)^j\chi(j)e^{jat}}{e^{dbt}+1}\right)^{m}e^{\left(bX+\frac{bd}{a}i+\frac{b}{a}j\right)at}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
&\sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}E_{n-k,\chi}^{(m-1)}(aY)\sum_{j=0}^{d-1}\sum_{i=0}^{a-1}\chi(j)(-1)^{i+j}E_{k,\chi}^{(m)}\left(bX+\frac{bd}{a}i+\frac{b}{a}j\right)\frac{t^n}{n!}\\
=&\frac{1}{2}\left(\frac{2\sum_{j=0}^{d-1}(-1)^j\chi(j)e^{jat}}{e^{dat}+1}\right)^{m}e^{abXt}\left(1-(-1)^a e^{dabt}\right)
\end{aligned}$$

$$\times \left(\frac{2\sum_{j=0}^{d-1}(-1)^j \chi(j) e^{jbt}}{e^{dbt}+1} \right)^m e^{abYt},$$

and in the same way we obtain the generating function of the right hand side of (3) is

$$\begin{aligned} &\sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k} b^k a^{n-k} E_{n-k,\chi}^{(m-1)}(bY)\sum_{j=0}^{d-1}\sum_{i=0}^{b-1}\chi(j)(-1)^{i+j}E_{k,\chi}^{(m)}\left(aX+\frac{ad}{b}i+\frac{a}{b}j\right)\frac{t^n}{n!} \\ = \quad & \frac{1}{2}\left(\frac{2\sum_{j=0}^{d-1}(-1)^j \chi(j) e^{jat}}{e^{dat}+1} \right)^m e^{abXt}\left(1-(-1)^b e^{dabt}\right) \\ & \times \left(\frac{2\sum_{j=0}^{d-1}(-1)^j \chi(j) e^{jbt}}{e^{dbt}+1} \right)^m e^{abYt}. \end{aligned}$$

Since $a \equiv b \pmod 2$ then we have the same generating functions. Thus, the coefficients of these generating functions are the same. Hence, we obtain our desired Theorem 1. □

Remark

1) Our theorem 1 is valid for arbitrary conductor d of the Dirichlet character χ. a and b must only have the same parity.

2) Let $m = 1$ in (3). Then we have

$$\begin{aligned} & a^n \sum_{j=0}^{d-1}\sum_{i=0}^{a-1}\chi(j)(-1)^{i+j}E_{n,\chi}\left(bX+\frac{bd}{a}i+\frac{b}{a}j\right) \\ = \quad & b^n \sum_{j=0}^{d-1}\sum_{i=0}^{b-1}\chi(j)(-1)^{i+j}E_{n,\chi}\left(aX+\frac{ad}{b}i+\frac{a}{b}j\right). \end{aligned}$$

3) Let $m = 1$ in (3) and $\chi = 1$. Then we have

$$a^n \sum_{i=0}^{a-1}(-1)^{i+j}E_n\left(bX+\frac{b}{a}i\right) = b^n \sum_{i=0}^{b-1}(-1)^{i}E_n\left(aX+\frac{a}{b}i\right),$$

for all a and b with same parity. For instance,

$$\sum_{i=0}^{a-1}(-1)^i E_n\left(X+\frac{i}{a}\right) = a^{-n}E_n(aX), \text{ if } a \text{ is odd}, \quad \text{(here we take } b=1).$$

And

$$\sum_{i=0}^{a-1}(-1)^i E_n\left(X+\frac{i}{a}\right) = a^{-n}2^n\left(E_n(aX)-E_n\left(aX+\frac{a}{2}\right)\right), \text{ if } a \text{ is even},$$

(here we take $b = 2$).

Now, for the rest of this paper, we set

$$\tilde{T}_{k,\chi}(n) = \sum_{i=0}^{d-1}\sum_{j=0}^{n-1}(-1)^{i+j}\chi(j)(i+jd)^k.$$

By (1), we can also easily see that

$$(-1)^{n-1}E_{k,\chi}(nd) + E_{k,\chi} = 2\tilde{T}_{k,\chi}(n).$$

Note that if χ_0 is the trivial Dirichlet's character, then

$$\tilde{T}_k(n) := \tilde{T}_{k,\chi_0}(n) = \sum_{j=1}^{n}(-1)^{i-1}i^k = \frac{1}{2}\Big((-1)^{n-1}E_k(n) + E_k\Big).$$

Theorem 2. *Let χ be the Dirichlet's character with an arbitrary conductor $d \in \mathbb{N}^*$. For $a, b \in \mathbb{N}$ with same parity we have*

$$\begin{aligned} &\sum_{j=0}^{n}\binom{n}{j}b^j a^{n-j}E^{(m)}_{n-j,\chi}(bX)\sum_{k=0}^{j}\binom{j}{k}\tilde{T}_{k,\chi}(a)E^{(m-1)}_{j-k,\chi}(aY) \\ = &\sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j}E^{(m)}_{n-j,\chi}(aX)\sum_{k=0}^{j}\binom{j}{k}\tilde{T}_{k,\chi}(b)E^{(m-1)}_{j-k,\chi}(bY). \qquad (4) \end{aligned}$$

Proof. For $a, b \in \mathbb{N}$ with $a \equiv b \pmod 2$, let us consider the following functional equation:

$$\begin{aligned} F_{d,m}(a,b;X,Y) &= \frac{1}{2}\left(\frac{2\sum_{j=0}^{d-1}(-1)^j\chi(j)e^{jat}}{e^{dat}+1}\right)^m e^{abXt}\Big(1-(-1)^a e^{dabt}\Big) \\ &\quad \times \left(\frac{2\sum_{j=0}^{d-1}(-1)^j\chi(j)e^{jbt}}{e^{dbt}+1}\right)^m e^{abYt} \qquad (5) \\ &= \left(\frac{2\sum_{j=0}^{d-1}(-1)^j\chi(j)e^{jat}}{e^{dat}+1}\right)^m e^{abXt}\left(\frac{1-(-1)^a e^{dabt}}{e^{bdt}+1}\right) \\ &\quad \times \left(\sum_{j=0}^{d-1}\chi(j)(-1)^j e^{jbt}\right) \times \left(\frac{2\sum_{j=0}^{d-1}(-1)^j\chi(j)e^{jbt}}{e^{dbt}+1}\right)^{m-1} e^{abYt}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} &\left(\frac{1-(-1)^a e^{abdt}}{e^{bdt}+1}\right)\left(\sum_{i=0}^{d-1}\chi(i)e^{ibt}(-1)^i\right) \\ = &\left(\sum_{l=0}^{a-1}e^{ldbt}(-1)^l\right)\left(\sum_{i=0}^{d-1}\chi(i)(-1)^i e^{ibt}\right) \\ = &\sum_{j=0}^{d-1}\sum_{i=0}^{a-1}\chi(j)(-1)^{i+j}e^{(id+j)bt} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{d-1}\sum_{i=0}^{a-1}\chi(j)(-1)^{i+j}(id+j)^k\right)\frac{b^k t^k}{k!}\\
&= \sum_{k=0}^{\infty}\tilde{T}_{k,\chi}(a)\frac{b^k t^k}{k!}.
\end{aligned} \tag{6}
$$

By (5) and (6), we have

$$
\begin{aligned}
&F_{d,m}(a,b;X,Y)\\
&= \left(\sum_{i=0}^{\infty}E_{i,\chi}^{(m)}(bX)\frac{a^i t^i}{i!}\right)\left(\sum_{l=0}^{\infty}\tilde{T}_{l,\chi}(a)\frac{b^l t^l}{l!}\right)\left(\sum_{k=0}^{\infty}E_{k,\chi}^{(m-1)}(aY)\frac{b^k t^k}{k!}\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}b^j a^{n-j}E_{n-j,\chi}^{(m)}(bX)\sum_{k=0}^{j}\tilde{T}_{k,\chi}(a)\binom{j}{k}E_{j-k,\chi}^{(m-1)}(aY)\right)\frac{t^n}{n!}.
\end{aligned} \tag{7}
$$

By the symmetry of $F_{d,m}(a,b;X,Y)$ in a and b (because $a \equiv b \pmod 2$), we also see that

$$
F_{d,m}(a,b;X,Y) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j}E_{n-j,\chi}^{(m)}(aX)\sum_{k=0}^{j}\binom{j}{k}\tilde{T}_{k,\chi}(b)E_{j-k,\chi}^{(m-1)}(bY)\right)\frac{t^n}{n!}. \tag{8}
$$

By comparing the coefficients on the both sides of (7) and (8), we obtain the Theorem 2. □

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