J. Appl. Math. & Informatics Vol. **29**(2011), No. 1 - 2, pp. 517 - 521
Website: http://www.kcam.biz

ON m-CONVEX SETS IN PRECONVEXITY SPACES

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Abstract. In this paper, we introduce the concepts of m-convex set, mc-convex function and m^*c-convex function. We study basic properties for m-convex sets and characterization for such functions.

AMS Mathematics Subject Classification : 52A01
Key words and phrases : preconvexity, c-convex function, m-convex set, mc-convex function, mc^*-convex function

1. Introduction

In [1], Guay introduced the concept of preconvexity spaces defined by a binary relation on the power set $P(X)$ of a set X and investigated some properties. He showed that a preconvexity on a set yields a convexity space in the same manner as a proximity [3] yields a topological space. In this paper, we introduce and study the m-convex sets induced by convex sets on a preconvexity space. In fact, every m-convex set is a convex set but the collection of all m-convex sets on a preconvexity space has some special properties as studied in section 2. We also introduce the concepts of mc-convex functions and mc^*-continuous functions defined by m-convex sets and convex sets. In particular, the mc-convex function is a generalization of convex function on preconvexity spaces. In section 3, we investigate characterizations for such functions and relationships among c-convex function, mc-convex function and mc^*-continuous function.

Definition 1 ([1]). *Let X be a nonempty set. A binary relation σ on $P(X)$ is called a preconvexity on X if the relation satisfies the following properties; we write $x\sigma A$ for $\{x\}\sigma A$:*

(1) *If $A \subseteq B$, then $A\sigma B$.*
(2) *If $A\sigma B$ and $B = \emptyset$, then $A = \emptyset$.*
(3) *If $A\sigma B$ and $b\sigma C$ for all $b \in B$, then $A\sigma C$.*
(4) *If $A\sigma B$ and $x \in A$, then $x\sigma B$.*

Received September 3, 2010. Accepted October 19, 2010.

The pair (X, σ) is called a *preconvexity space*. A convexity is a reflexive and transitive relation. In a preconvexity space (X, σ), $G(A) = \{x \in X : x\sigma A\}$ is called the *convexity hull* of a subset A. A is said to be *convex* [1] if $G(A) = A$.

Theorem 1 ([1]). *For a preconvexity space (X, σ),*

(1) $G(\emptyset) = \emptyset$.
(2) $A \subseteq G(A)$ *for all* $A \subseteq X$.
(3) *If* $A \subseteq B$, *then* $G(A) \subseteq G(B)$.
(4) $G(G(A)) = G(A)$ *for* $A \subseteq X$.

Theorem 2 ([1]). *If σ is a preconvexity on X and $A \subseteq X$, then $G(A) = \cap\{C : G(C) = C$ and $A \subseteq C\}$.*

Theorem 3 ([1]). *Let σ be a preconvexity on X and $A, B \subseteq X$. Then*

(1) $A\sigma B$ *iff* $A \subseteq G(B)$.
(2) $A\sigma B$ *iff* $G(A)\sigma G(B)$.

Definition 2 ([1]). *Let σ_1, σ_2 be two preconvexities on the convexity spaces (X, σ_1) and (Y, σ_2), respectively. A function $f : X \to Y$ is said to be c-convex if $A\sigma_1 B$ implies $f(A)\sigma_2 f(B)$.*

Lemma 1 ([2]). *Let (X, σ) be a preconvexity space. Then for all $A \subseteq X$, $G(A)\sigma A$.*

2. m-convex sets on preconvexity spaces

Definition 3. *Let (X, σ) be a preconvexity space, $x \in X$ and $A \subseteq X$. Then A is called an m-convex set if $x \in A \cup F$, whenever $x\sigma(A \cup F)$ for every convex set F.*

Theorem 4. *Let (X, σ) be a preconvexity space. For an m-convex set A and a convex set F, $A \cup F$ is a convex set.*

Proof. If $x \in G(A \cup F)$, then $x\sigma(A \cup F)$. By definition of m-convex set, $x \in (A \cup F)$ and $G(A \cup F) = A \cup F$. Therefore, $A \cup F$ is convex. □

Theorem 5. *Let (X, σ) be a preconvexity space. Then*

(1) *Both $\emptyset$ and X are m-convex.*
(2) *If A and B are m-convex, then $A \cup B$ is m-convex.*
(3) *For $\alpha \in J$, if A_α is m-convex, then $\cap_{\alpha \in J} A_\alpha$ is m-convex.*

Proof. (1) For each convex set F, if $x\sigma(F \cup \emptyset)$, then $x \in G(F)$. Since $G(F) = F$, it implies $x \in F \cup \emptyset$ and hence $\emptyset$ is m-convex. From $X = G(X)$, obviously it follows that X is m-convex.

(2) Let A and B be m-convex subsets. For each convex set F, let $x\sigma((A \cup B) \cup F)$. By Theorem 4, we know that $B \cup F$ is a convex set. Since A is an m-convex set and $x\sigma(A \cup (B \cup F))$, $x \in (A \cup (B \cup F))$. It implies that $A \cup B$ is m-convex.

(3) Let A_α be an m-convex subset for $\alpha \in J$. If $x\sigma(\cap_{\alpha\in J}A_\alpha \cap F)$ for each convex set F, then since $(\cap_{\alpha\in J}A_\alpha \cap F)\sigma(A_\alpha \cap F)$ and σ is a transitive relation, $x\sigma(A_\alpha \cap F)$ for $\alpha \in J$. Since A_α is m-convex, $x \in A_\alpha \cap F$ for each α, and $x \in \cap_{\alpha\in J}A_\alpha \cap F$. Hence $\cap_{\alpha\in J}A_\alpha$ is m-convex.

□

Theorem 6. *Let (X, σ) be a preconvexity space. Then every m-convex set is convex.*

Proof. For each m-convex set F, since $\emptyset$ is a convex set, by Theorem 4, $F \cup \emptyset = F$ is convex. □

Example 1. *Let $X = \{a, b, c\}$ and a topology $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Consider a family $\mathcal{S} = \{A \subseteq X : A \subseteq cl(int(A))\}$, where cl and int denote closure and interior operators, respectively, in the topological space (X, τ_1). Set $scl(A) = \cap\{F : A \subseteq F \text{and } X - F \in \mathcal{S}\}$. Define $A\sigma B$ iff $scl(A) \subseteq scl(B)$. Then σ is a preconvexity on (X, σ). Note that:*

1. *$\{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}, X\}$ is the collection of all convex subsets on (X, σ);*
2. *$\{\emptyset, \{c\}, \{b, c\}, \{a, c\}, X\}$ is the collection of all m-convex subsets on (X, σ).*

This fact shows that in Theorem 6 the converse may not be true.

Definition 4. *Let (X, σ) be a preconvexity space and $A \subset X$.*

The set $mh(A) = \cap\{F \subseteq X : A \subseteq F, \quad F \text{ is an } m\text{-convex set } \}$ is called the m-closure of A.

Lemma 2. *Let (X, σ) be a preconvexity space and $A \subseteq X$. Then*

$$G(A) \subseteq mh(A).$$

Proof. For some $x \in G(A)$, suppose on the contrary that $x \notin mh(A)$. Then there exists an m-convex set F such that $A \subseteq F$ and $x \notin F$. Since every m-convex set is convex, $G(A) \subseteq G(F) = F$ and so $x \notin G(A)$. It contradicts that $x \in G(A)$. This completes that $G(A) \subseteq mh(A)$. □

From Theorem 5, we get the next theorem:

Theorem 7. *Let (X, σ) be a preconvexity space. Then*

1. *$mh(\emptyset) = \emptyset$.*
2. *$A \subseteq mh(A)$ for $A \in X$.*
3. *$mh(mh(A)) = mh(A)$ for $A \in X$.*
4. *$mh(A \cup B) = mh(A) \cup mh(B)$ for $A, B \in X$.*

3. mc^*-convex functions and mc-convex functions

In this section, we introduce the concepts of mc-convex functions and mc^*-continuous functions defined by m-convex sets and convex sets. We investigate characterizations for such functions and relationships among c-convex function, mc-convex function and mc^*-continuous function.

Definition 5. *Let (X, σ) and (Y, μ) be two preconvexity spaces. A function $f : X \to Y$ is said to be*

(1) *mc^*-convex if $f^{-1}(U)$ is m-convex for each m-convex set U in Y;*
(2) *mc-convex if $f^{-1}(U)$ is convex for each m-convex set U in Y.*

Theorem 8. *Let $f : X \to Y$ be a function on two preconvexity spaces (X, σ) and (Y, μ). Then the following are equivalent:*

(1) *f is mc^*-continuous.*
(2) *$f(mh(A)) \subseteq mh(f(A))$ for $A \subseteq X$.*
(3) *$mh(f^{-1}(B) \subseteq f^{-1}(mh(B))$ for $B \subseteq Y$.*

Proof. (1) $\Rightarrow$ (2) Let F be any m-convex set in Y containing $f(A)$. Then $f^{-1}(F)$ is an m-convex set containing A. Since $mh(A)$ is the smallest m-convex set containing A, $A \subseteq mh(A) \subseteq f^{-1}(F)$, and $f(A) \subseteq f(mh(A)) \subseteq F$. This implies that $f(mh(A)) \subseteq mh(f(A))$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Obvious.

□

Theorem 9. *Let $f : X \to Y$ be a function on two preconvexity spaces (X, σ) and (Y, μ). Then the following are equivalent:*

(1) *f is mc-convex.*
(2) *$f(G(A)) \subseteq mh(f(A))$ for $A \subseteq X$.*
(3) *$G(f^{-1}(B)) \subseteq f^{-1}(mh(B))$ for $B \subseteq Y$.*

Proof. (1) $\Rightarrow$ (2) Let F be any m-convex set in Y containing $f(A)$; then $f^{-1}(F)$ is a convex set containing A. So $A \subseteq G(A) \subseteq f^{-1}(F)$ and $f(A) \subseteq f(G(A)) \subseteq F$. This implies that $f(G(A)) \subseteq mh(f(A))$.

(2) $\Rightarrow$ (3) For $B \subseteq Y$, it is $f(G(f^{-1}(B))) \subseteq mh(B)$ by (2). Thus we get the result.

(3) $\Rightarrow$ (1) It is obvious.

□

Theorem 10. *Let $f : X \to Y$ be a function on two preconvexity spaces (X, σ) and (Y, μ). Then if f is c-convex, then it is mc-convex.*

Proof. Let F be any m-convex set in Y. By Theorem 6 and Lemma 1, F is convex and $G(F)\sigma F$. From f is c-convex and Lemma 2,

$$f(G(F))\sigma f(F) \subseteq G(f(F)) \subseteq mh(f(F)).$$

Hence by Theorem 9, f is mc-convex. □

Remark 1. *Finally, we have the following implications but the converses are not true in general.*

$$mc^*\text{-continuous} \Rightarrow mc\text{-convex} \Leftarrow c\text{-convex}$$

Example 2. *Let $X = \{a, b, c\}$ and a topology $\tau_2 = \{\emptyset, \{a\}, X\}$. Consider a family $\mathcal{S} = \{A \subseteq X : A \subseteq cl(int(A))\}$, where cl and int denote closure and interior operators, respectively, in the topological space (X, τ_2). Set $scl(A) = \cap\{F : A \subseteq F and\ X - F \in \mathcal{S}\}$. Define $A\mu B$ iff $scl(A) \subseteq scl(B)$. Then μ is a preconvexity on (X, μ) and*

(1) *$C = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ is the collection of all convex subsets in (X, μ);*

(2) *C is also the collection of all m-convex subsets of (X, μ).*

Consider the preconvexity σ defined in Example 1. Then we obtain the following things:

(i) The identity function $f : (X, \sigma) \to (X, \mu)$ is mc-convex. For an m-convex set $A = \{b\}$ in (X, μ), $f^{-1}(A)$ is not m-convex in (X, σ). Therefore, f is not mc*-convex.

(ii) Let us define a function $f : (X, \mu) \to (X, \sigma)$ as the following: $f(a) = f(b) = b, f(c) = c$. Then f is m-convex. For a convex set $F = \{b\}$ in (X, σ), $f^{-1}(F) = \{a, b\}$ is not convex in (X, μ). Consequently, f is not c-convex.

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