HOPF HYPERSURFACES IN COMPLEX TWO-PLANE
GRASSMANNIANS WITH LIE PARALLEL
NORMAL JACOBI OPERATOR

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Abstract. In this paper we give some non-existence theorems for Hopf
hypersurfaces in the complex two-plane Grassmannian $G_2(C^{m+2})$ with
Lie parallel normal Jacobi operator $R_N$ and totally geodesic $\mathcal{D}$ and $\mathcal{D}^\perp$
components of the Reeb flow.

0. Introduction

The Jacobi fields along geodesics of a given Riemannian manifold $(M, g)$
play an important role in the study of differential geometry. It satisfies a very
well-known differential equation. This classical differential equation naturally
inspires the so-called Jacobi operators. That is, if $R$ is the curvature operator
of $M$ and $X$ is any vector field tangent to $M$, the Jacobi operator with respect
to $X$ at $x \in M$, $R_X \in \text{End}(T_x M)$, is defined as $R_X(Y)(x) = (\langle R(Y, X)X \rangle)(x)$ for
all $Y \in T_x M$, being a self-adjoint endomorphism of the tangent bundle $TM$ of
$M$. Clearly, each vector field $X$ tangent to $M$ provides a Jacobi operator with
respect to $X$ (See [7] and [9]).

If the structure vector field $\xi = -JN$ of a real hypersurface $M$ in complex
projective space $P_n(\mathbb{C})$ is invariant under the shape operator, $\xi$ is said to be
Hopf, where $J$ denotes a Kähler structure of $P_n(\mathbb{C})$, and $N$ is a unit normal
vector field of $M$ in $P_n(\mathbb{C})$.

In the quaternionic projective space $\mathbb{H}P^m$ Pérez and Suh [10] classified the
real hypersurfaces in $\mathbb{H}P^m$ with $\mathcal{D}^\perp$-parallel curvature tensor $\nabla_{\xi} R = 0$ for $\nu = 1, 2, 3$, where $R$ denotes the curvature tensor of $M$ in $\mathbb{H}P^m$ and $\mathcal{D}^\perp$ is a
distribution defined by $\mathcal{D}^\perp = \text{Span} \{\xi_1, \xi_2, \xi_3\}$. In this case they are congruent
to a tube of radius $\frac{1}{2}$ over a totally geodesic quaternionic submanifold $\mathbb{H}P^{k}$ in
$\mathbb{H}P^m$, $2 \leq k \leq m - 2$.

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Jacobi operator is invariant under the shape operator $A$.

Berndt and Suh [3] proved the following:

In quaternionic space forms, Berndt [1] introduced the notion of normal Jacobi operator $R_N X = R(X, N) N = \text{End} (T_x M)$, $x \in M$ for real hypersurfaces $M$ in a quaternionic projective space $\mathbb{H}P^n$ or in a quaternionic hyperbolic space $\mathbb{H}H^n$, where $R$ denotes the curvature tensor of $\mathbb{H}P^n$ and $\mathbb{H}H^n$ respectively. Berndt [1] also showed that “the curvature adaptedness”, when the normal Jacobi operator $R_N$ commutes with the shape operator $A$, is equivalent to the fact that the distributions $\mathcal{D}$ and $\mathcal{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator $A$ of $M$, where $T_x M = \mathcal{D} \oplus \mathcal{D}^\perp$, $x \in M$.

Now let us consider a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ which consists of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The situation for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+1})$ with parallel normal Jacobi operator $R_N$ is not so simple and will be quite different from the cases in $\mathbb{H}P^n$.

In this paper the present authors consider a real hypersurface $M$ in the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator, that is, $\mathcal{L}_X R_N = 0$ for any $X \in T_x M$, $x \in M$, where $R$ and $N$ respectively denote the curvature tensor of the ambient space $G_2(\mathbb{C}^{m+2})$ and a unit normal vector field of $M$ in $G_2(\mathbb{C}^{m+2})$. The curvature tensor $\hat{R}(X, Y) Z$ for any vector fields $X, Y$ and $Z$ on $G_2(\mathbb{C}^{m+2})$ is explicitly defined in Section 1. Then the normal Jacobi operator $R_N$ for the unit normal vector field $N$ can be defined from the curvature tensor $\hat{R}(X, Y) N$ by putting $Y = Z = N$.

The ambient space $G_2(\mathbb{C}^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure $J$ and a quaternionic Kähler structure $\mathcal{J}$ not containing $J$ (See Berndt [2]). From these two structures $J$ and $\mathcal{J}$, we have geometric conditions naturally induced on a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ such that $[\xi] = \text{Span} \{\xi\}$ or $\mathcal{D}^\perp = \text{Span} \{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator. By these two conditions, Berndt and Suh [3] proved the following:

**Theorem A.** Let $M$ be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathcal{D}^\perp$ are invariant under the shape operator of $M$ if and only if

(A) $M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or

(B) $m$ is even, say $m = 2n$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{HP}^n$ in $G_2(\mathbb{C}^{m+2})$.

The structure vector field $\xi$ of a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ is said to be a Reeb vector field. Moreover, the Reeb vector field $\xi$ is said to be Hopf if it is invariant under the shape operator $A$. The 1-dimensional foliation of $M$ by
the integral manifolds of the Reeb vector field $\xi$ is said to be a *Hopf foliation* of $M$. We say that $M$ is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of $M$ is totally geodesic. By the formulas in section 2 it can be easily checked that $M$ is Hopf if and only if the Reeb vector field $\xi$ is Hopf.

The flow generated by the integral curves of the Reeb vector field is said to be a *geodesic Reeb flow* if $M$ becomes a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. We say that $M$ is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of $M$ is totally geodesic. By the formulas in section 2 it can be easily checked that $M$ is Hopf if and only if the Reeb vector field $\xi$ is Hopf.

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any parallel displacement $\phi_t$ generated from the flow $\phi_t$ such that $\phi_t(x) = \gamma(t)$ and $\gamma(0) = x$ for the integral curve $\gamma$ of $X$ in $T_x M$, $x \in M$.

Then the authors prove the following for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operators:

**Theorem 1.** Let $M$ be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator. If the integral curves of $D$ and $D^\perp$ components of the Reeb vector field $\xi$ are totally geodesic, then $\xi$ belongs to either the distribution $\mathcal{D}$ or the distribution $\mathcal{D}^\perp$.

On the other hand, in the paper [6] of Jeong and Suh, we gave non-existence theorems for real hypersurfaces $M$ in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator, that is, $\mathcal{L}_\xi R_N = 0$ as follows:

**Theorem B.** There does not exist any real hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathcal{L}_\xi R_N = 0$ if the Reeb vector field $\xi \in \mathcal{D}^\perp$.

**Theorem C.** There does not exist any real hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathcal{L}_\xi R_N = 0$ if the Reeb vector field $\xi \in \mathcal{D}$.

Then as an application of Theorem 1 to Theorems B and C the authors can assert the following:

**Theorem 2.** There does not exist any Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator if the integral curves of $D$ and $D^\perp$ components of the Reeb vector field are totally geodesic.

1. **Riemannian geometry of $G_2(\mathbb{C}^{m+2})$**

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details refer to [2], [3], and [4]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = SU(2) \times SU(m) \subset G$. The space $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space $G/K$, which we equip with the unique analytic structure for which the natural action of $G$ on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of $G$ and $K$, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form $B$ of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $\text{Ad}(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $\mathfrak{o} = \mathfrak{k} = \mathfrak{k}$ and identify $T_o G_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since $B$ is negative definite on $\mathfrak{g}$, negative $B$ restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $\text{Ad}(K)$-invariance of $B$ this inner product can be extended to a $G$-invariant Riemannian metric $g$ on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize $g$ such that the maximum sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight. When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{CP}^2$ with constant holomorphic sectional curvature eight. When $m = 2$,
we note that the isomorphism Spin(6) \cong SU(4) yields an isometry between $G_2(\mathbb{C}^2)$ and the real Grassmann manifold $G_m^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}^6$. From now on, in this paper we will assume $m \geq 3$.

The Lie algebra $\mathfrak{f}$ has the direct sum decomposition

$$\mathfrak{f} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R},$$

where $\mathfrak{R}$ is the center of $\mathfrak{f}$. Viewing $\mathfrak{f}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure $J$ and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If $J_\nu$, $\nu = 1, 2, 3$, is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_\nu = J_\nu J$, and $JJ_\nu$ is a symmetric endomorphism with $(JJ_\nu)^2 = I$ and $\text{tr}(JJ_\nu) = 0$.

A canonical local basis $J_1, J_2, J_3$ of $\mathfrak{J}$ consists of three local almost Hermitian structures $J_\nu$ in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$, where the index $\nu$ is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\nabla$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis $J_1, J_2, J_3$ of $\mathfrak{J}$ three local one-forms $q_1, q_2, q_3$ such that

$$\nabla_X J_\nu = q_{\nu+2}(X) J_{\nu+1} - q_{\nu+1}(X) J_{\nu+2}$$

for all vector fields $X$ on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\hat{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\hat{R}(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ$$

$$+ \sum_{\nu=1}^{3} \{g(J_\nu Y, Z)J_\nu X - g(J_\nu X, Z)J_\nu Y - 2g(J_\nu X, Y)J_\nu Z\}$$

$$+ \sum_{\nu=1}^{3} \{g(J_\nu JY, Z)J_\nu JX - g(J_\nu JX, Z)J_\nu JY\},$$

where $J_1, J_2, J_3$ is any canonical local basis of $\mathfrak{J}$.

2. Some fundamental formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

Now in this section we want to derive some fundamental formulas which will be used in the proof of our theorems and the equation of Codazzi for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ (See [3], [4], [12], [13], and [14]).

Let $M$ be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a submanifold of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on $M$ will also be denoted by $g$, and $\nabla$ denotes the Riemannian connection of $(M, g)$. Let $N$ be a local unit normal field of $M$ and $A$ the shape operator of $M$ with respect to $N$. The Kähler structure $J$ of $G_2(\mathbb{C}^{m+2})$ induces on $M$ an almost contact metric structure $(\phi, \xi, \eta, g)$. More explicitly, we can define a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$ and its dual 1-form $\eta$ on $M$ by $g(\phi X, Y) = g(JX, Y)$.
and \( \eta(X) = g(X, \xi) \) for any tangent vector fields \( X \) and \( Y \) on \( M \). Then they satisfy the following
\[
\phi^2 X = -X + \eta(X)\xi, \quad \phi \xi = 0, \quad \eta(\phi X) = 0 \quad \text{and} \quad \eta(\xi) = 1
\]
for any tangent vector field \( X \).

Furthermore, let \( J_1, J_2, J_3 \) be a canonical local basis of \( \mathfrak{F} \). Then each \( J_\nu \)
defines an almost contact metric structure \( (\phi_\nu, \xi_\nu, \eta_\nu, g) \) on \( M \) in such a way that a tensor field \( \phi_\nu \) of type \((1, 1)\), a vector field \( \xi_\nu \) and its dual 1-form \( \eta_\nu \) on \( M \) defined by \( g(\phi_\nu X, Y) = g(J_\nu X, Y) \) and \( \eta_\nu(X) = g(\xi_\nu, X) \) for any tangent vector fields \( X \) and \( Y \) on \( M \). Then they also satisfy the following
\[
\phi_\nu^2 X = -X + \eta_\nu(X)\xi, \quad \phi_\nu \xi = 0, \quad \eta_\nu(\phi_\nu X) = 0 \quad \text{and} \quad \eta_\nu(\xi_\nu) = 1
\]
for any vector field \( X \) tangent to \( M \) and \( \nu = 1, 2, 3 \).

Using the above expression (1.2) for the curvature tensor \( R \) of the ambient space \( G_2(\mathbb{C}^{m+2}) \), the equation of Codazzi becomes
\[
(\nabla_X Y - \nabla_Y X) = \eta(Y)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi
\]
\[+ \sum_{\nu=1}^{3} \left\{ \eta_\nu(\phi X)\phi_\nu Y - \eta_\nu(\phi Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu \right\} \]
\[+ \sum_{\nu=1}^{3} \left\{ \eta(X)\phi_\nu Y - \eta(Y)\phi_\nu X \right\} \xi_\nu .
\]

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:
\[
\phi_{\nu+1} \xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu \xi_{\nu+1} = \xi_{\nu+2},
\]
\[
\phi \xi_\nu = \phi_\nu \xi, \quad \eta_\nu(\phi X) = \eta(\phi_\nu X),
\]
\[
\phi_\nu \phi_{\nu+1} X = \phi_{\nu+2} X + \eta_{\nu+1}(X)\xi_\nu,
\]
\[
\phi_{\nu+1} \phi_\nu X = -\phi_{\nu+2} X + \eta_\nu(X)\xi_{\nu+1}.
\]

Now let us note that
\[
(\nabla_X \phi) Y = \eta(Y) AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX,
\]
\[
\nabla_X \xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX,
\]

for any vector field \( X \) tangent to \( M \) in \( G_2(\mathbb{C}^{m+2}) \), where \( N \) denotes a unit normal vector field of \( M \) in \( G_2(\mathbb{C}^{m+2}) \). Then from this and the formulas (1.1) and (2.1) we have that
where the terms in the right side can be given respectively as follows:

\begin{equation}
(\nabla_X \phi_\nu) Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX, Y)\xi_\nu.
\end{equation}

Summing up these formulas, we find the following

\begin{equation}
\nabla_X (\phi_\nu \xi) = \nabla_X (\phi \xi) = (\nabla_X \phi) \xi_\nu + \phi(\nabla_X \xi_\nu)
\end{equation}

Moreover, from \(JJ_\nu = J_\nu J\), \(\nu = 1, 2, 3\), it follows that

\begin{equation}
\phi_\nu X = \phi_\nu X + \eta_\nu(X)\xi - \eta(X)\xi_\nu.
\end{equation}

### 3. Lie parallel normal Jacobi operator

Let \(M\) be a real hypersurface in \(G_2(\mathbb{C}^{m+2})\) with Lie parallel normal Jacobi operator, that is, \(\mathcal{L}_X R_N = 0\) for any vector field \(X\) tangent to \(M\). Then first of all, we write the normal Jacobi operator \(R_N\), which is given by

\begin{equation}
R_N(X) = R(X, N)N = X + 3\eta(X)\xi + 3\sum_{\nu=1}^3 \eta_\nu(X)\xi_\nu
\end{equation}

where we have used the following

\begin{align*}
g(J_\nu J_N, N) &= -g(J_N, J_\nu N) = -g(\xi, \xi_\nu) = -\eta_\nu(\xi), \\
g(J_\nu J_X, N) &= g(X, J_\nu N) = -g(X, J_\nu \xi) \\
&= -g(X, \phi_\nu \xi + \eta(\xi_\nu)N) = -g(X, \phi_\nu \xi),
\end{align*}

and

\(J_\nu J_N = -J_\nu \xi = -\phi_\nu \xi - \eta_\nu(\xi)N\).

Of course, by (2.7) we know that the normal Jacobi operator \(R_N\) is a symmetric endomorphism of \(T_x M\), \(x \in M\).

Now let us consider the Lie derivative of the normal Jacobi operator along any direction. Then for any vector fields \(X\) and \(Y\) tangent to \(M\) it is given by

\begin{equation}
(\mathcal{L}_X R_N) Y = \mathcal{L}_X (R_N Y) - R_N (\mathcal{L}_X Y)
\end{equation}

where the terms in the right side can be given respectively as follows:

\begin{align*}
(\nabla_X R_N) Y &= 3(\nabla_X \eta)(Y)\xi + 3\eta(Y)\nabla_X \xi + 3\sum_{\nu=1}^3 (\nabla_X \eta_\nu)(Y)\xi_\nu.
\end{align*}
\[ \nabla_{RX} Y = \nabla_Y X + 3\eta(Y)\nabla_{\xi} X + 3\sum_{\nu=1}^{3} \eta_{\nu}(Y)\nabla_{\xi_{\nu}} X - \sum_{\nu=1}^{3} \eta_{\nu}(\nabla_{\phi_{\nu}} Y)X + \sum_{\nu=1}^{3} \eta_{\nu}(\nabla_{\phi_{\nu}} Y)\nabla_{\phi_{\nu}} X \]

and

\[ \bar{R}_{X}(\nabla_Y X) = \nabla_Y X + 3\eta(\nabla_Y X)\xi + 3\sum_{\nu=1}^{3} \eta_{\nu}(\nabla_{Y}X)\xi_{\nu} - \sum_{\nu=1}^{3} \{ \eta_{\nu}(\xi)(\phi_{\nu}\phi_{Y} - \eta(Y)\xi_{\nu}) - \eta_{\nu}(\phi\nabla_{Y}X)\phi_{\nu}\xi \}. \]

Then by the formulas given in section 2, (3.2) gives the following for a real hypersurface \( M \) in \( G_2(\mathbb{C}^{m+2}) \) with Lie parallel normal Jacobi operator \( \bar{R}_{N} \):

\[ (\mathcal{L}_{X}\bar{R}_{N})Y = 3g(\phi_{AX}, Y)\xi + 3\eta(Y)\phi_{AX} + 3\sum_{\nu=1}^{3} g(\phi_{\nu} AX, Y)\xi_{\nu} \]

\[ + 3\sum_{\nu=1}^{3} \eta_{\nu}(Y)\phi_{\nu} AX \]

\[ - \sum_{\nu=1}^{3} X(\eta_{\nu}(\xi))(\phi_{\nu} Y - \eta(Y)\xi_{\nu}) \]

\[ + \eta_{\nu}(\xi)\left\{ -q_{\nu+1}(X)\phi_{\nu+2} Y + q_{\nu+2}(X)\phi_{\nu+1} Y + \eta(Y)\phi_{\nu} AX - g(AX, Y)\xi_{\nu} + \eta(Y)\phi_{\nu} AX - g(AX, Y)\phi_{\nu} \xi - g(\phi AX, Y)\xi_{\nu} - \eta(Y)(q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_{\nu} AX) \right\} \]

\[ - g(\phi_{\nu} AX, Y)\phi_{\nu} \xi - \eta(Y)\eta_{\nu}(AX)\phi_{\nu} \xi + g(AX, Y)\eta_{\nu}(\xi)\phi_{\nu} \xi \]

\[ - \eta_{\nu}(\phi Y)\left\{ \eta_{\nu}(\xi)AX - g(AX, \xi)\xi_{\nu} + \phi_{\nu} \phi AX \right\} \]

\[ - 3\eta(Y)\nabla_{\xi} X - 3\sum_{\nu=1}^{3} \eta_{\nu}(Y)\nabla_{\xi_{\nu}} X \]

\[ + \sum_{\nu=1}^{3} \eta_{\nu}(\xi)(\nabla_{\phi_{\nu}} Y - \eta(Y)\nabla_{\xi_{\nu}} Y) - \eta_{\nu}(\phi Y)\nabla_{\phi_{\nu}} X \]

\[ + 3\eta(\nabla_{Y} X)\xi + 3\sum_{\nu=1}^{3} \eta_{\nu}(\nabla_{Y} X)\xi_{\nu} \]

\[ - \sum_{\nu=1}^{3} \eta_{\nu}(\xi)(\phi_{\nu} \phi_{Y} X - \eta(Y)\nabla_{Y} X)\xi_{\nu} - \eta_{\nu}(\phi_{Y})\nabla_{\phi_{Y} X})\phi_{\nu} \xi \]

\[ = 0, \]
where in the first equality we have used the following formulas
\[
3 \sum_{\nu=1}^{3} g(q_{\nu+1}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2}, Y)\xi_{\nu} \\
+ 3 \sum_{\nu=1}^{3} \eta_{\nu}(Y) \{ q_{\nu+1}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} \} = 0
\]
and
\[
\sum_{\nu=1}^{3} \{ \eta_{\nu+1}(\phi Y)q_{\nu+2}(X)\phi_{\nu}\xi - \eta_{\nu+2}(\phi Y)q_{\nu+1}(X)\phi_{\nu}\xi \\
- \eta_{\nu}(\phi Y)q_{\nu+1}(X)\phi_{\nu+2}\xi + \eta_{\nu}(\phi Y)q_{\nu+2}(X)\phi_{\nu+1}\xi \} = 0.
\]
In particular by putting \( X = \xi \) in (3.3) we have the following
\[
(\mathcal{L}_\xi \mathcal{R}_N)Y = 3g(\phi A\xi, Y)\xi + 3 \sum_{\nu=1}^{3} g(\phi_{\nu} A\xi, Y)\xi_{\nu} \\
+ 3 \sum_{\nu=1}^{3} \eta_{\nu}(Y)\phi_{\nu} A\xi \\
- \sum_{\nu=1}^{3} \left[ \xi(\eta_{\nu}(\xi))(\phi_{\nu}\phi Y - \eta(Y)\xi_{\nu}) \\
+ \eta_{\nu}(\xi) \left\{ - q_{\nu+1}(\xi)\phi_{\nu+2}\phi Y + q_{\nu+2}(\xi)\phi_{\nu+1}\phi Y \\
+ \eta_{\nu}(\phi Y) A\xi - g(A\xi, \phi Y)\xi_{\nu} \\
+ \eta(Y) \phi_{\nu} A\xi - g(A\xi, Y)\phi_{\nu}\xi - g(\phi A\xi, Y)\xi_{\nu} \\
- \eta(Y)(q_{\nu+2}(\xi)\xi_{\nu+1} - q_{\nu+1}(\xi)\xi_{\nu+2} + \phi_{\nu} A\xi) \right\} \\
- g(\phi_{\nu} A\xi, \phi Y)\phi_{\nu}\xi - \eta(Y)\eta_{\nu}(A\xi)\phi_{\nu}\xi + g(\phi Y, A\xi, Y)\eta_{\nu}(\xi)\phi_{\nu}\xi \\
- \eta_{\nu}(\phi Y) \left\{ \eta_{\nu}(\xi) A\xi - g(\xi, \phi_{\nu} A\xi) + \phi_{\nu} \phi A\xi \right\} \\
- 3 \sum_{\nu=1}^{3} \eta_{\nu}(Y)\phi A\xi_{\nu} + 3 \sum_{\nu=1}^{3} \eta_{\nu}(\phi AY)\xi_{\nu} \\
+ \sum_{\nu=1}^{3} \left[ \eta_{\nu}(\xi) \left\{ \phi A\phi_{\nu}\phi Y - \eta(Y)\phi_{\nu} A\xi \right\} - \eta_{\nu}(\phi Y)\phi A\phi_{\nu}\xi \right] \\
+ \sum_{\nu=1}^{3} \left[ \eta_{\nu}(\xi) \left\{ \phi A Y - \eta(AY)\phi_{\nu}\xi \right\} - \eta_{\nu}(AY)\phi_{\nu}\xi \right] \\
+ \eta(AY)\eta_{\nu}(\xi)\phi_{\nu}\xi \right \} = 0,
\]
where in the first equality we have used the second formula of (2.3). From this, by putting \( Y = \xi \) in (3.4) we have the following
\[
(\mathcal{L}_\xi \mathcal{R}_N)\xi = 6 \sum_{\nu=1}^{3} g(\phi_{\nu} A\xi, \xi)\xi_{\nu} + 4 \sum_{\nu=1}^{3} \eta_{\nu}(\xi)\phi_{\nu} A\xi
\]
\[\sum_{\nu=1}^{3} \left[ \xi(\eta_{\nu}(\xi))\xi_{\nu} + \eta_{\nu}(\xi)\{q_{\nu+2}(\xi)\xi_{\nu+1} - q_{\nu+1}(\xi)\xi_{\nu+2}\} \right] - 4\sum_{\nu=1}^{3} \eta_{\nu}(\xi)\phi \xi_{\nu} = 0.\]

### 4. Lie parallel normal Jacobi operator

In this section we want to prove the following:

**Proposition 4.1.** Let \( M \) be a Hopf real hypersurface in \( G_{2}(\mathbb{C}^{n+2}) \) with Lie parallel normal Jacobi operator. If the integral curves of \( D \) and \( D^{\perp} \) components of the Reeb vector field \( \xi \) are totally geodesic, then \( \xi \) belongs to either the distribution \( D \) or the distribution \( D^{\perp} \).

**Proof.** When the function \( \alpha = g(A\xi, \xi) \) identically vanishes, the proposition was proved directly by Pérez and Suh [11]. Thus we consider only the case that the function \( \alpha \) is non-vanishing in this proof.

By putting \( A\xi = \alpha \xi \) into (3.5) we have

\[\sum_{\nu=1}^{3} \eta_{\nu}(\xi)(\alpha \phi_{\xi} - \phi A\xi_{\nu}) = 0,\]

where we have used the following formula

\[\sum_{\nu=1}^{3} \left[ \xi(\eta_{\nu}(\xi))\xi_{\nu} + \eta_{\nu}(\xi)\{q_{\nu+2}(\xi)\xi_{\nu+1} - q_{\nu+1}(\xi)\xi_{\nu+2}\} \right] = 0.\]

Now let us put \( \xi = \eta(X_{0})X_{0} + \eta(\xi_{1})\xi_{1} \) for some unit \( X_{0} \in D \) and \( \xi_{1} \in D^{\perp} \). Then naturally we know that \( \eta(\xi_{2}) = \eta(\xi_{3}) = 0 \). Hereafter, unless otherwise stated, let us assume \( \eta(X_{0})\eta(\xi_{1}) \neq 0 \).

Then (4.1) reduces to

\[\alpha \phi_{\xi_{1}} - \phi A\xi_{1} = 0.\]

From this, by taking the structure tensor \( \phi \) and also using that \( \xi \) is principal, we have

\[A\xi_{1} = \alpha \xi_{1} \quad \text{and} \quad AX_{0} = \alpha X_{0}.\]

Then putting \( X = X_{0} \) and \( Y = \xi \) into (3.3) and using (4.2) gives

\[0 = (L_{X_{0}}R_{N})\xi + 3\alpha \phi X_{0} + 3\alpha \sum_{\nu=1}^{3} g(\phi_{\nu} X_{0}, \xi)\xi_{\nu} + 3\alpha \eta_{1}(\xi)\phi_{1} X_{0} + \eta_{1}(\xi)\{q_{3}(X_{0})\xi_{2} - q_{2}(X_{0})\xi_{3}\} - 3\nabla_{\xi} X_{0} - 4\eta_{1}(\xi)\nabla_{\xi} X_{0} + 3\eta(\nabla_{\xi} X_{0})\xi + 3\sum_{\nu=1}^{3} \eta_{\nu}(\nabla_{\xi} X_{0})\xi_{\nu} - \eta_{1}(\xi)\phi_{1} \phi_{\xi} X_{0} + \eta_{1}(\xi)\eta(\nabla_{\xi} X_{0})\xi_{1} + \sum_{\nu=1}^{3} \eta_{\nu}(\phi_{\xi} X_{0})\phi_{\xi} \xi_{\nu},\]

where we have used

\[X_{0}(\eta_{1}(\xi))\xi_{1} = g(\nabla_{X_{0}}\xi_{1}, \xi)\xi_{1} + g(\xi_{1}, \nabla_{X_{0}}\xi)\xi_{1}.\]
\[ \begin{align*}
&= g(\phi_1 AX_0, \xi_1) + g(\xi_1, \phi AX_0) \\
&= -\alpha g(X_0, \phi_1 \xi) - \alpha g(\phi_1, X_0) \xi_1 \\
&= -2\alpha g(X_0, \phi_1 (\eta(X_0) X_0 + \eta(\xi_1) \xi_1)) \\
&= -2\alpha \eta(X_0) g(X_0, \phi_1 X_0) \\
&= 0.
\end{align*} \]

From this, together with (2.3) and (2.4), and using \( \phi X_0 \in \mathfrak{D}, \nabla_\xi X_0 \in \mathfrak{D} \) and \( \eta(\nabla_\xi X_0) = 0 \), we have

\[ 0 = (L_{X_0} R_N) \xi \]
\[ = 3\alpha (\phi X_0 + \eta_1 (\xi) \phi_1 X_0) + \eta_1 (\xi) (q_3(X_0) \xi_2 - q_2(X_0) \xi_3) \\
- 3\nabla_\xi X_0 - 4 \eta_1 (\xi) \nabla_\xi X_0 - \eta_1 (\xi) \phi_1 \nabla_\xi X_0 \\
+ \sum_{\nu=1}^{3} \eta_\nu (\phi \nabla_\xi X_0) \phi_\nu \xi,
\]

because we know the following

\[ g(\phi X_0, \xi_\nu) = -g(X_0, \phi_\nu \xi) = -g(X_0, \phi_\nu \xi) = 0, \]
\[ \eta(\nabla_\xi X_0) = g(\nabla_\xi X_0, \xi) = g(\nabla_\xi X_0, \eta(X_0) X_0 + \eta(\xi_1) \xi_1) = 0 \]

and

\[ g(\nabla_\xi X_0, \xi_\nu) = -g(X_0, \nabla_\xi \xi_\nu) \\
= -\alpha g(X_0, \phi_\nu \xi) \\
= -\alpha g(X_0, \phi_\nu \xi) \\
= \alpha g(\phi X_0, \xi_\nu) \\
= 0
\]

for any \( \nu = 1, 2, 3 \).

On the other hand, we know that

\[ \nabla_\xi X_0 \in \mathfrak{D}, \]

because

\[ g(\nabla_\xi X_0, \xi_\nu) = -g(X_0, \nabla_\xi \xi_\nu) \\
= -g(X_0, q_{\nu+2}(\xi_1) \xi_{\nu+1} - q_{\nu+1}(\xi_1) \xi_{\nu+2} + \phi_\nu A \xi_1) \\
= -\alpha g(X_0, \phi_\nu \xi_1) \\
= 0.
\]

Moreover, the following formulas hold

\[ g(\phi \nabla_\xi X_0, \xi_2) = 0 \quad \text{and} \quad g(\phi \nabla_\xi X_0, \xi_3) = 0. \]

In fact, differentiating \( g(\phi X_0, \xi_2) = 0 \) gives

\[ 0 = g(\nabla_\xi \phi) X_0, \xi_2) + g(\phi \nabla_\xi X_0, \xi_2) + g(\phi X_0, \nabla_\xi \xi_2) \\
= g(\phi \nabla_\xi X_0, \xi_2) + \alpha g(\phi X_0, \phi \xi_2) \]
\[ = g(\phi \nabla_X X_0, \xi_2) \]

and similarly the latter term comes from \( g(\phi X_0, \xi_3) = 0 \).

By taking the inner product (4.3) with \( \xi_3 \), and using the facts that \( \phi X_0 \), \( \phi_1 X_0 \), \( \nabla_X X_0 \) and \( \nabla_{\xi}, X_0 \) belong to the distribution \( \mathcal{D} \), we have

\[
0 = -\eta_1(\xi)q_2(X_0) - \eta_1(\xi)g(\phi_1 \phi \nabla_X X_0, \xi_3) + \eta_1(\phi \nabla_X X_0)g(\phi_1 \xi, \xi_3) \\
= -\eta_1(\xi)q_2(X_0).
\]

Similarly, by taking the inner product with \( \xi_2 \) to (4.3), we have the following relations

\[
q_2(X_0) = 0 \quad \text{and} \quad q_3(X_0) = 0
\]

under the assumption of \( \eta_1(\xi) \neq 0 \). Then (4.4), (4.5) and (4.6) give

\[
0 = (L_{X_0} \hat{R}_N) \xi
\]

\[
= 3\alpha(\phi X_0 + \eta(\xi)\phi_1 X_0) - 3\nabla_X X_0 - 4\eta(\xi)\nabla_{\xi} X_0 - \eta_1(\xi)\phi_1 \phi \nabla_X X_0 + \eta_1(\phi \nabla_X X_0)\phi_1 \xi.
\]

On the other hand, by the assumption of \( M \) being Hopf and using (4.2), we have

\[
\nabla_{\xi} \xi = \phi A \xi \\
= \phi A(\eta(X_0)X_0 + \eta(\xi_1)\xi_1) \\
= \alpha(\eta(X_0)\phi X_0 + \eta(\xi_1)\eta(X_0)\phi_1 X_0) \\
= \alpha\eta(X_0)(\phi X_0 + \eta(\xi_1)\phi_1 X_0) \\
= 0.
\]

Consequently, we see

\[
(4.8) \quad \phi X_0 + \eta(\xi_1)\phi_1 X_0 = 0.
\]

from the assumption of \( \alpha \neq 0 \) and \( \eta(X_0) \neq 0 \).

Substituting (4.8) into (4.7), we have

\[
0 = (L_{X_0} \hat{R}_N) \xi
\]

\[
= -3\nabla_X X_0 - 4\eta(\xi)\nabla_{\xi} X_0 - \eta_1(\xi)\phi_1 \phi \nabla_X X_0 + \eta_1(\phi \nabla_X X_0)\phi_1 \xi.
\]

Now, by applying the operator \( \phi_1 \) to (4.8) we have

\[
(4.9) \quad \phi_1 \phi X_0 = \eta(\xi_1)X_0.
\]

Then by differentiating (4.9) along the direction of the Reeb vector field \( \xi \) and using (2.1), (2.3), (2.4), (2.5) and (4.8), we have

\[
(4.10) \quad q_2(\xi)\eta(\xi_1)\phi_2 X_0 + q_3(\xi)\eta(\xi_1)\phi_3 X_0 + \phi_1 \phi \nabla_X X_0 = \eta(\xi_1)\nabla_X X_0.
\]

By taking the inner product (4.10) with \( \xi_2 \) and \( \xi_3 \) respectively and using the fact that \( \nabla_{\xi} X_0, \phi_\nu X_0 \in \mathcal{D}, \nu = 1, 2, 3 \), we have the following respectively

\[
(4.11) \quad g(\nabla_X X_0, \phi_3 X_0) = 0 \quad \text{and} \quad g(\nabla_X X_0, \phi_2 X_0) = 0.
\]
On the other hand, the assumption that $\mathfrak{D}$-component of $\xi$ is totally geodesic and (4.2) give

$$q_2(\xi_1) = q_3(\xi_1) = 0.$$  

Let us differentiate the formula (4.9) along the direction of $\xi_1$. Then by virtue of the formulas (2.3), (2.4), (2.5) and (4.12), we have

$$\phi_1 \phi \nabla_{\xi_1} X_0 = \eta(\xi_1) \nabla_{\xi_1} X_0.$$  

On the other hand, by taking the inner product (4.10) with $\phi_2 X_0$, $\phi_3 X_0$ respectively and using (2.1), (2.7) and (4.11) respectively we have

$$q_2(\xi) = 0 \quad \text{and} \quad q_3(\xi) = 0.$$  

Then (4.10) implies that

$$\phi_1 \phi \nabla_{\xi} X_0 = \eta(\xi_1) \nabla_{\xi} X_0.$$  

Moreover, by differentiating (4.8) along the direction of $\xi$ and using (2.3), (2.4), (2.5) and (4.14), we have

$$\phi \nabla_{\xi} X_0 = \alpha \eta(\xi) \eta(X_0) \xi_1 - \eta(\xi) \phi_1 \nabla_{\xi} X_0.$$  

From this, by applying $\phi$ and using (4.15) we have

$$\nabla_{\xi} X_0 = -\alpha \eta(\xi_1) \phi_1 X_0.$$  

Now differentiating (4.8) along the direction $\xi_1$ and using (2.3) and (2.5), we have

$$\alpha \eta(X_0) \xi_1 + \phi \nabla_{\xi_1} X_0 = -\eta(\xi) \phi_1 \nabla_{\xi_1} X_0.$$  

Similarly, by applying $\phi$ to above equation and using (4.13) we have

$$\nabla_{\xi_1} X_0 = \alpha \phi_1 X_0.$$  

Then (4.16) and (4.17) give

$$\nabla_{\xi} X_0 = -\eta(\xi_1) \nabla_{\xi_1} X_0.$$  

On the other hand, we know that

$$\nabla_{\xi} X_0 = \nabla_{\eta(X_0) X_0 + \eta(\xi_1) \xi_1} X_0$$

$$= \eta(X_0) \nabla_{X_0} X_0 + \eta(\xi_1) \nabla_{\xi_1} X_0$$

$$= \eta(\xi_1) \nabla_{\xi_1} X_0,$$

because the $\mathfrak{D}$-component of the Reeb vector field $\xi$ is totally geodesic. From (4.18) and (4.19) we see that $\eta(\xi_1) \nabla_{\xi_1} X_0 = 0$. This means that $\nabla_{\xi_1} X_0 = 0$. From this together with (4.17), it follows that $\phi_1 X_0 = 0$. This gives a contradiction. So we only have $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$. \qed
5. Lie parallel normal Jacobi operator for $\xi \in \mathcal{D}^\perp$

In order to give a complete proof of Theorem 2, first we consider the case that the Reeb vector field $\xi$ belongs to the distribution $\mathcal{D}^\perp$. Now in this direction we introduce some lemmas given in Jeong and Suh [6] as follows:

**Lemma 5.A.** Let $M$ be a real hypersurface in $G_2(C^{m+2})$ satisfying Lie $\xi$-parallel normal Jacobi operator and $\xi \in \mathcal{D}^\perp$. Then $A\xi = \alpha \xi + \beta U$, where $U$ is a unit vector field orthogonal to $\xi$ and belongs to $\mathcal{D}$.

Moreover, from Lemma 5.A, they proved the following lemmas:

**Lemma 5.B.** Let $M$ be a real hypersurface in $G_2(C^{m+2})$ satisfying Lie $\xi$-parallel normal Jacobi operator and $\xi \in \mathcal{D}^\perp$. Then $\beta$ identically vanishes, that is, the Reeb vector field $\xi$ is principal.

**Lemma 5.C.** Let $M$ be a real hypersurface in $G_2(C^{m+2})$ satisfying Lie $\xi$-parallel normal Jacobi operator and $\xi \in \mathcal{D}^\perp$. Then $g(AD, \mathcal{D}^\perp) = 0$.

From these lemmas we assert:

**Lemma 5.1.** Let $M$ be a real hypersurface in $G_2(C^{m+2})$ satisfying Lie parallel normal Jacobi operator and $\xi \in \mathcal{D}^\perp$. Then the Reeb vector $\xi$ is principal and $g(AD, \mathcal{D}^\perp) = 0$.

Before going to give our proof of Theorem 2 in the introduction, let us check "What kind of model hypersurfaces given in Theorem A satisfy Lie parallel normal Jacobi operator." In other words, it will be an interesting problem to know whether there exist real hypersurfaces in $G_2(C^{m+2})$ satisfying the condition $L_X R_N = 0$ for $\xi \in \mathcal{D}^\perp$.

Then by virtue of Lemmas 5.1, we are able to recall the proposition given by Berndt and Suh [3] as follows:

**Proposition A.** Let $M$ be a connected real hypersurface of $G_2(C^{m+2})$. Suppose that $AD \subset \mathcal{D}$, $A\xi = \alpha \xi$, and $\xi$ is tangent to $\mathcal{D}^\perp$. Let $J_1$ be the almost Hermitian structure such that $JN = J_1 N$. Then $M$ has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures

$$\alpha = \sqrt{8} \cot(\sqrt{8}r), \quad \beta = \sqrt{2} \cot(\sqrt{2}r), \quad \lambda = -\sqrt{2} \tan(\sqrt{2}r), \quad \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces we have

$$T_\alpha = \mathbb{R} \xi = \mathbb{R} JN = \mathbb{R} \xi_1,$$

$$T_\beta = \mathbb{C} \xi = \mathbb{C} JN = \mathbb{R} \xi_2 \oplus \mathbb{R} \xi_3,$$

$$T_\lambda = \{X|X \perp \xi, JX = J_1 X\},$$

$$T_\mu = \{X|X \perp \xi, JX = -J_1 X\},$$
where \( R \xi, C \xi \) and \( H \xi \) respectively denotes real, complex and quaternionic span of the structure vector \( \xi \) and \( C^+ \xi \) denotes the orthogonal complement of \( C \xi \) in \( H \xi \).

In the proof of Lemma 5.C (See Section 4 in [6]) we have asserted that \( A \xi_2 = 0 \) and \( A \xi_3 = 0 \). But the principal curvature \( \beta = \sqrt{2} \cot(\sqrt{2}r) \) given in Proposition 4 is never vanishing for any \( r \in (0, \frac{\pi}{4}) \). So this gives a contradiction. Accordingly, we completed the proof of our Theorem 2 for the case \( \xi \in D^\perp \).

### 6. Lie parallel normal Jacobi operator for \( \xi \in D \)

Next we consider the case that the Reeb vector field \( \xi \) belongs to the distribution \( D \). Then in this section we introduce the following lemmas due to Jeong and Suh [6] for hypersurfaces in \( G_2(\mathbb{C}^{m+2}) \) with Lie \( \xi \)-parallel normal Jacobi operator.

**Lemma 6.A.** Let \( M \) be a real hypersurface in \( G_2(\mathbb{C}^{m+2}) \) satisfying Lie \( \xi \)-parallel normal Jacobi operator and \( \xi \in D \). Then the Reeb vector \( \xi \) is principal.

Then by using Lemma 6.A, Jeong and Suh [6] also verified the following:

**Lemma 6.B.** Let \( M \) be a real hypersurface in \( G_2(\mathbb{C}^{m+2}) \) satisfying Lie \( \xi \)-parallel normal Jacobi operator and \( \xi \in D \). Then \( g(A \xi, D^\perp) = 0 \).

By virtue of these Lemmas 6.A and 6.B we have

**Lemma 6.C.** Let \( M \) be a real hypersurface in \( G_2(\mathbb{C}^{m+2}) \) satisfying Lie parallel normal Jacobi operator and \( \xi \in D \). Then the Reeb vector field \( \xi \) is principal and \( g(A \xi, D^\perp) = 0 \).

From this Lemma 6.1, together with Theorem A due to Berndt and Suh [3], we have that \( M \) is locally a tube over a totally geodesic and totally real quaternionic projective space \( \mathbb{HP}^{m}, m = 2n \). So for the geometrical structure of such a tube we recall the following proposition.

**Proposition B.** Let \( M \) be a connected real hypersurface of \( G_2(\mathbb{C}^{m+2}) \). Suppose that \( A \xi \subset D \), \( A \xi = \alpha \xi \), and \( \xi \) is tangent to \( D \). Then the quaternionic dimension \( m \) of \( G_2(\mathbb{C}^{m+2}) \) is even, say \( m = 2n \), and \( M \) has five distinct constant principal curvatures

\[
\alpha = -2 \tan(2r), \quad \beta = 2 \cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)
\]

with some \( r \in (0, \pi/4) \). The corresponding multiplicities are

\[
m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu)
\]

and the corresponding eigenspaces are

\[
T_\alpha = R \xi, \quad T_\beta = J \lambda \xi, \quad T_\gamma = J \xi, \quad T_\lambda, \quad T_\mu,
\]

where

\[
T_\lambda \oplus T_\mu = (HC \xi)^\perp, \quad J T_\lambda = T_\lambda, \quad J T_\mu = T_\mu, \quad JT_\lambda = T_\mu.
\]
Now, using the assumption that $M$ is Hopf in (3.4), we have the following

$$\left( L_{\xi} \tilde{R}_N \right) Y$$

\[ = 4\alpha \sum_{\nu=1}^{3} g(\phi_{\nu} \xi, Y) \xi_{\nu} + 4\alpha \sum_{\nu=1}^{3} \eta_{\nu}(Y) \phi_{\nu} \xi \]

\[ - 3 \sum_{\nu=1}^{3} \eta_{\nu}(Y) \phi A \xi_{\nu} + 3 \sum_{\nu=1}^{3} \eta_{\nu}(\phi A Y) \xi_{\nu} \]

\[ + \sum_{\nu=1}^{3} \left\{ \eta_{\nu}(\xi) \left( \phi A \phi_{\nu} Y - \eta(Y) \phi A \xi_{\nu} \right) - \eta_{\nu}(\phi Y) \phi A \phi_{\nu} \xi \right\} \]

\[ + \sum_{\nu=1}^{3} \left\{ \eta_{\nu}(\xi) \left( \phi_{\nu} A Y - \alpha \eta(Y) \phi_{\nu} \xi \right) - \eta_{\nu}(A Y) \phi_{\nu} \xi + \eta(Y) \eta_{\nu}(\xi) \phi_{\nu} \xi \right\} \]

\[ = 0 . \]

Moreover, using the fact that the Reeb vector field $\xi$ belongs to the distribution $\mathfrak{D}$, we have

$$\left( L_{\xi} \tilde{R}_N \right) Y = 4\alpha \sum_{\nu=1}^{3} g(\phi_{\nu} \xi, Y) \xi_{\nu} + 4\alpha \sum_{\nu=1}^{3} \eta_{\nu}(Y) \phi_{\nu} \xi$$

$$- 3 \sum_{\nu=1}^{3} \eta_{\nu}(Y) \phi A \xi_{\nu} + 3 \sum_{\nu=1}^{3} \eta_{\nu}(\phi A Y) \xi_{\nu}$$

$$- \sum_{\nu=1}^{3} \eta_{\nu}(\phi Y) \phi A \phi_{\nu} \xi - \sum_{\nu=1}^{3} \eta_{\nu}(A Y) \phi_{\nu} \xi$$

$$= 0$$

for any $Y \in T_x M$, $x \in M$.

Let us construct a sub-distribution $\mathfrak{D}_0$ of the distribution $\mathfrak{D}$ in such a way that

$$[\xi] \oplus \mathfrak{D}_0 = \mathfrak{D},$$

where $[\xi]$ denotes an one-dimensional vector subspace spanned by the Reeb vector field $\xi$. Then $\mathfrak{D}_0$ becomes $\mathfrak{D}_0 = \{ Y \in \mathfrak{D} \mid Y \perp \xi \}$. Here, if we substitute any $Y \in \mathfrak{D}_0$ in (6.1) and use $\xi \in \mathfrak{D}$, the left side of (6.1) becomes

$$\left( L_{\xi} \tilde{R}_N \right) Y = 4\alpha \sum_{\nu=1}^{3} g(\phi_{\nu} \xi, Y) \xi_{\nu} + 3 \sum_{\nu=1}^{3} \eta_{\nu}(\phi A Y) \xi_{\nu}$$

$$- \sum_{\nu=1}^{3} \eta_{\nu}(\phi Y) \phi A \phi_{\nu} \xi - \sum_{\nu=1}^{3} \eta_{\nu}(A Y) \phi_{\nu} \xi.$$

From this, putting $Y = \phi_{\mu} \xi \in T_{\xi}$, and using $A \phi_{\mu} \xi = 0$, $\mu = 1, 2, 3$, given in Proposition $B$, it becomes

$$\left( L_{\xi} \tilde{R}_N \right) \phi_{\mu} \xi = 4\alpha \phi_{\mu} \xi.$$

From this, with the assumption of $L_{\xi} \tilde{R}_N = 0$, we have $\alpha = 0$. But the principal curvature $\alpha = -2 \tan(2r)$ in Proposition $B$ never vanishes for $r \in (0, \frac{\pi}{4})$. This gives a contradiction for the case $\xi \in \mathfrak{D}$. Accordingly, we complete the proof of our Theorem 2 for $\xi \in \mathfrak{D}$ in the introduction.

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