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EXISTENCE OF PERIODIC SOLUTIONS FOR A GENERAL CLASS OF p-LAPLACIAN EQUATIONS

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ABSTRACT. The existence of T-periodic solutions for a general class of p-Laplacian equations is investigated. By using coincidence degree theory, some existence and uniqueness results, which generalize some earlier works on this topic, are presented.

1. INTRODUCTION

In this paper, we study the solvability of the following p-Laplacian type nonlinear periodic boundary value problem:

$$(\phi_p(x'))' + f(t, x') + g(t, x) = e(t) \tag{1}$$

$$x(0) = x(T), \quad x'(0) = x'(T), \tag{2}$$

where $\phi_p(u) = |u|^{p-2}u$ with $p > 1$ and f, g and e are continuous functions and are T-periodic in t with $f(t, 0) = 0$ for all $t \in \mathbb{R}$.

When $p = 2$, Eq(1) reduces to the following second order forced Rayleigh equation:

$$x'' + f(t, x') + g(t, x) = e(t). \tag{3}$$

The existence of periodic solutions of (1) and (3) has been an important research focus for the study of dynamic behaviors of nonlinear second order differential equations. See, for example, research papers [1-11] and the references therein. In [5], the authors have discussed the existence of T-periodic solutions for equation (3). By using degree theory, they have obtained the following results:

Theorem A. *Assume that the following conditions are satisfied:*

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(A_1) *There exists a* $d^* > 0$ *such that*

$$x\,(g(t,x)-e(t)) < 0 \quad \forall\, t \in \mathbb{R},\ |x| \geq d^*.$$

(A_2) *There exist* $m_1, m_2 \geq 0$ *such that*

$$m_1 < \frac{2}{T}, \quad |f(t,x)| \leq m_1|x| + m_2 \quad \forall t,\, x \in \mathbb{R} \quad \text{and} \quad f(t,0) = 0\, \forall\, t \in \mathbb{R}.$$

Then equation (3) *has at least one* T*-periodic solution. Moreover, if the following additional condition:*

(A_3) $(g(t,x_1) - g(t,x_2))\,(x_1 - x_2) < 0 \quad \forall t,\, x_1,\, x_2 \in \mathbb{R},\, x_1 \neq x_2$

holds, then equation (3) *has a unique* T*-periodic solution.*

In [9, Theorem 3.1], the authors have replaced the condition (A_2) by the following condition:

$(A_2)'$ There exist $m_1, m_2 \geq 0$ and $d^* > 0$ such that $0 \leq m_1 < 1/T^2$ and one of the following conditions holds:

(*i*) $f(t,x) \geq 0\, \forall\, t, x \in \mathbb{R}$ and $g(t,x) - e(t) \geq -m_1 x - m_2\, \forall\, t \in \mathbb{R},\, x \geq d^*$;

(*ii*) $f(t,x) \leq 0\, \forall\, t, x \in \mathbb{R}$ and $g(t,x) - e(t) \leq -m_1 x + m_2\, \forall\, t \in \mathbb{R},\, x \leq -d^*$.

They have obtained the existence of at least one T-periodic solution for equation (3). Moreover, they have shown that if in addition the condition (A_3) holds, then (3) has a unique T-periodic solution.

In this paper, we will obtain some sufficient conditions for the existence of at least one T-periodic solution for equation (1). Moreover, under the additional condition (A_3), we will show that (1) has a unique T-periodic solution. Our results cover and improve the results of [5] and [9]. Also, they are new and more general even in case $p = 2$.

2. Main Results

The following are the main results of this paper.

Theorem 1. *Suppose that the functions* f, g *and* e *are continuous and* T*-periodic in* t. *Assume that* (A_1) *and one of the following conditions hold:*

(*i*) *There exist* $m_1, m_2 \geq 0$ *and* $0 \leq \alpha \leq p-1$ *such that* $|f(t,u)| \leq m_1|u|^\alpha + m_2\, \forall\, t, u \in \mathbb{R}$ *and* $f(t,0) = 0\, \forall\, t \in \mathbb{R}$.

(*ii*) $f(t,x)$ *is of constant sign* $\forall\, t, x \in \mathbb{R}$.

Then the problem (1)-(2) *has at least one solution. If in addition the condition* (A_3) *holds and* $p \geq 2$, *then the problem* (1)-(2) *has a unique solution.*

If $p = 2$, then we get immediately from Theorem 1 the following result:

Corollary 1. *Assume that* (A_1) *and one of the following conditions hold:*

(i) There exist $m_1, m_2 \geq 0$ *and* $0 \leq \alpha \leq 1$ *such that* $|f(t,u)| \leq m_1|u|^\alpha + m_2 \,\forall\, t, u \in \mathbb{R}$;

(ii) $f(t,x)$ *is of constant sign* $\forall\, t, x \in \mathbb{R}$.

Then the problem (1)-(2) *has at least one* T*-periodic solution. If in addition the condition* (A_3) *holds, then the problem* (1)-(2) *has a unique* T*-periodic solution.*

Remark 1. Note that Corollary 1 generalizes Theorem A in [5] and the results of [9]. Moreover, it removes the restriction $m_1 < 2/T$ on $f(t,x)$ in [5] and the restriction on $g(t,x)$ in [9].

3. Preliminary Results

In this section, we introduce a well-known theorem about p-Laplacian-like operators and a Bellman-type inequality, which will be used in the proof of Theorem 1.

Let $X = C_T^1$ the space of all C^1-functions which are T-periodic, i.e.,

$$X = C_T^1 = \left\{x(t) \in C^1(\mathbb{R}, \mathbb{R}) : x(0) = x(T),\ x'(0) = x'(T)\right\}.$$

Lemma 1 ([8, Theorem 3.1]). *Consider the following problem*

$$(\phi_p(u'))' = h(t, u, u'), \qquad u(0) = u(T), \quad u'(0) = u'(T), \tag{4}$$

where $\phi_p(u) = |u|^{p-2}u$ *with* $p > 1$ *and* h *is a Caratheory function and* T*-periodic in* t.

Let Ω *be an open bounded set in* C_T^1. *Suppose that the following conditions hold:*

(i) For each $\lambda \in (0,1)$, *the problem*

$$(\phi_p(u'))' = \lambda h(t, u, u'), \quad u \in C_T^1 \tag{5}$$

has no solution on $\partial\Omega$.

(ii) The equation

$$H(a) := \frac{1}{T}\int_0^T h(t, a, 0)dt = 0$$

has no solution on $\partial\Omega \cap \mathbb{R}$.

(iii) The Brouwer degree

$$d_B\,[H,\, \Omega \cap \mathbb{R},\, 0] \neq 0.$$

Then problem (4) *has a solution in* Ω.

Lemma 2 (Generalized Bellman's inequality). *Consider the following inequality:*

$$|y(t)| \le C + M \int_0^t |y(s)|^\beta ds.$$

where C, M, β *are nonnegative constants and* $t > 0$. *If* $\beta \le 1$, *then for* $t \in (0, T_0]$, *we have* $|y(t)| \le D$, *where*

$$D = \begin{cases} Ce^{MT_0}, & \text{if } \beta = 1, \\ \left(C^{1-\beta} + MT_0(1-\beta)\right)^{\frac{1}{1-\beta}}, & \text{if } \beta < 1. \end{cases}$$

4. Proof of Theorem 1

Let $h(t, x, x') = e(t) - f(t, x') - g(t, x)$. Then (4) reduces to (1)-(2) and (5) reduces to

$$(\phi_p(x'))' + \lambda f(t, x') + \lambda[g(t, x) - e(t)] = 0, \quad \lambda \in (0, 1). \tag{6}$$

We first show that the set of all possible T-periodic solutions of (6) is bounded. Let $x(t)$ be an arbitrary T-periodic solution of (6). Assume that

$$x(t_1) = \max_{t\in[0,T]} x(t), \qquad x(t_2) = \min_{t\in[0,T]} x(t), \quad t_1, t_2 \in [0, T].$$

Then we have

$$x'(t_1) = x'(t_2) = 0, \quad x''(t_1) \le 0, \quad x''(t_2) \ge 0.$$

Setting $t = t_1$ and $t = t_2$ in (6) respectively and using $f(t, 0) = 0$ and $\phi_p(x'(t)))' = (p-1)|x'(t)|^{p-2}x''(t)$, then we have

$$g(t_1, x(t_1)) - e(t_1) \ge 0 \qquad \text{and} \quad g(t_2, x(t_2)) - e(t_2) \le 0.$$

Then from the assumption (A_1), we must have $x(t_1) < d^*$ and $x(t_2) > -d^*$. Hence by the periodicity of $x(t)$, we get

$$-d^* < x(t_2) \le x(t) \le x(t_1) < d^* \quad \forall\, t \in \mathbb{R}.$$

This implies that $x(t)$ is bounded and

$$\|x\|_\infty := \max_{t\in\mathbb{R}} |x(t)| < d^*.$$

Next we show that $x'(t)$ is bounded. Let $y(t) = \phi_p(x'(t))$. Then $x' = \phi_q(y)$, where $q = p/(p-1) > 1$ is the exponent conjugate to p. Then it follows from (6) that

$$y'(t) = -\lambda f(t, \phi_q(y(t))) - \lambda g(t, x(t)) + \lambda e(t).$$

First suppose that the condition (i) of Theorem 1 holds. Let $x(t_1) = \max_{t\in[0,T]} x(t)$. Then $x'(t_1) = 0$ and hence $y(t_1) = 0$. Thus from the above expression and the assumption (i), for any $t \in [0,T]$ with $0 \le t_1 \le t$, we can write

$$
\begin{aligned}
|y(t)| &= |y(t_1) + \textstyle\int_{t_1}^t y'(s)ds| \le \int_{t_1}^t |y'(s)|ds \le \int_0^t |y'(s)|ds \\
&= \textstyle\int_0^t |\lambda f(s,\phi_q(y(s))) + \lambda[g(s,x(s)) - e(s)]|ds \\
&\le \textstyle\int_0^t |f(s,\phi_q(y(s)))|ds + \int_0^T |g(t,x(t)) - e(t)|dt \\
&\le T\max_{t\in[0,T],|x|\le d^*} |g(t,x) - e(t)| + Tm_2 + m_1 \textstyle\int_0^t |y(s)|^{(q-1)\alpha}ds.
\end{aligned} \tag{7}
$$

Let $D_1 = T\max_{t\in[0,T],|x|\le d^*} |g(t,x) - e(t)| + Tm_2$ and $\beta = (q-1)\alpha = \alpha/(p-1)$. Then from (7) we get

$$
|y(t)| \le D_1 + m_1 \int_0^t |y(s)|^\beta ds. \tag{8}
$$

Note that $0 \le \beta \le 1$ since $0 \le \alpha \le p-1$ from the assumption (i). Now from Lemma 2, we obtain for $t \in [t_1, T]$,

$$
|y(t)| \le D_2,
$$

where

$$
D_2 = \begin{cases} D_1 e^{m_1 T}, & \text{if } \alpha = p-1, \\ \left(D_1^{\frac{p-1-\alpha}{p-1}} + \frac{m_1 T(p-1-\alpha)}{p-1} \right)^{\frac{p-1}{p-1-\alpha}}, & \text{if } \alpha < p-1. \end{cases}
$$

If $0 \le t \le t_1$, we have $0 \le t_1 \le t+T \le 2T$ and from the T-periodicity of $y(t)$, we obtain

$$
\begin{aligned}
|y(t)| &= |y(t+T)| = |y(t_1) + \textstyle\int_{t_1}^{t+T} y'(s)ds| = |\int_{t_1}^{t+T} y'(s)ds| \\
&\le \textstyle\int_{t_1}^{t+T} |y'(s)|ds \le \int_0^{t+T} |y'(s)|ds \\
&= \textstyle\int_0^{t+T} |\lambda f(s,\phi_q(y(s))) + \lambda[g(s,x(s)) - e(s)]|ds \\
&\le \textstyle\int_0^{t+T} |f(s,\phi_q(y(s)))|ds + \int_0^{2T} |g(t,x(t)) - e(t)|dt \\
&\le 2T\max_{t\in[0,T],|x|\le d^*} |g(t,x) - e(t)| + 2Tm_2 + m_1 \textstyle\int_0^{t+T} |y(s)|^{(q-1)\alpha}ds.
\end{aligned} \tag{9}
$$

From the above inequality, we obtain for $0 \le t \le t_1$,

$$
|y(t)| = |y(t+T)| \le 2D_1 + m_1 \int_0^{t+T} |y(s)|^\beta ds. \tag{10}
$$

Again from Lemma 2 this implies that for $0 \le t \le t_1$,

$$
|y(t)| \le D_3,
$$

where

$$D_3 = \begin{cases} 2D_1 e^{m_1 T}, & \text{if } \alpha = p-1, \\ \left((2D_1)^{\frac{p-1-\alpha}{p-1}} + \frac{m_1 T(p-1-\alpha)}{p-1}\right)^{\frac{p-1}{p-1-\alpha}}, & \text{if } \alpha < p-1. \end{cases}$$

Since $D_2 \le D_3$, the above inequalities imply that

$$\|y\|_\infty := \max_{t\in[0,T]} |y(t)| \le D_3.$$

Next suppose that the condition (ii) holds. Integrating (6) from 0 to T, we get

$$\int_0^T f(t, x'(t))dt + \int_0^T [g(t, x(t)) - e(t)]dt = 0.$$

This implies that

$$\begin{aligned} \textstyle\int_0^T |f(t, x'(t))|dt &= (sgnf) \textstyle\int_0^T f(t, x'(t))dt \\ &= -(sgnf) \textstyle\int_0^T [g(t, x(t)) - e(t)]dt \\ &\le \textstyle\int_0^T |g(t, x(t)) - e(t)|dt \\ &\le T \max_{t\in[0,T], |x|\le d^*} |g(t, x) - e(t)| \\ &\le D_1. \end{aligned} \tag{11}$$

From (11), we obtain for any $t \in [0, T]$,

$$\begin{aligned} |y(t)| &= |y(t_1) + \textstyle\int_{t_1}^t y'(s)ds| = |\textstyle\int_{t_1}^t y'(s)ds| \\ &\le |\textstyle\int_{t_1}^t |y'(s)|ds| \le \textstyle\int_0^T |y'(s)|ds \\ &\le \textstyle\int_0^T |\lambda f(s, x'(s)) + \lambda[g(s, x(s)) - e(s)]|ds \\ &\le \textstyle\int_0^T |f(s, x'(s))|ds + \textstyle\int_0^T |g(s, x(s)) - e(s)|ds \\ &\le 2D_1. \end{aligned} \tag{12}$$

It follows that

$$\|y\|_\infty \le 2D_1.$$

Since $2D_1 \le D_3$, we get in any case, $\|y\|_\infty \le D_3$. This implies that $\|x'\|_\infty \le D_3^{q-1} = D_3^{1/(p-1)}$. We have therefore

$$\|x\| := \|x\|_\infty + \|x'\|_\infty < d^* + D_3^{1/(p-1)} =: M.$$

Thus we have shown that the set of all T-periodic solutions $x(t)$ of (6) is bounded, i.e., $\|x(t)\| < M$. Now set $\Omega = \{x \in X : \|x\| < M\}$. Then equation (6) has no solution on $\partial\Omega$ for $\lambda \in (0, 1)$. Thus the condition (i) of Lemma 1 is satisfied. For $x = \pm M \in \partial\Omega \cap \mathbb{R}$, since $M > d^*$ and $f(t, 0) = 0$, it follows from the assumption

(A_1) that $H(\pm M) = \frac{1}{T}\int_0^T h(t, \pm M, 0)dt = \frac{1}{T}\int_0^T [e(t) - g(t, \pm M))dt \neq 0$. Hence the condition (ii) of Lemma 1 is also satisfied. Since $\Omega \cap \mathbb{R} = (-M, M) \subset \mathbb{R}$ and $H(-M)H(M) < 0$, by Brouwer degree theory, $d_B[H, \Omega \cap \mathbb{R}, 0] = \pm 1 \neq 0$. This shows that the condition (iii) of Lemma 1 is again satisfied. Now Lemma 1 implies that problem (1)-(2) has a solution in Ω.

To prove uniqueness of a solution of (1)-(2), we further assume that the condition (A_3) holds and $p \geq 2$. Let $x_1(t)$ and $x_2(t)$ be any two solutions of (1)-(2). Let $u(t) = x_1(t) - x_2(t), v(t) = y_1(t) - y_2(t) =: \phi_p(x_1'(t)) - \phi_p(x_2'(t))$. Then it follows from (1) and $x' = \phi_q(y)$ with $q = p/(p-1)$ that

$$\begin{cases} u'(t) = \phi_q(y_1(t)) - \phi_q(y_2(t)), \\ v'(t) = [f(t, \phi_q(y_2(t))) - f(t, \phi_q(y_1(t)))] + [g(t, x_2(t)) - g(t, x_1(t))]. \end{cases} \tag{13}$$

We first show that $u(t) \leq 0 \, \forall t \in [0, T]$. If, on the contrary, there exits a $t_0 \in [0, T)$ such that $u(t_0) = \max_{t \in [0,T]} u(t) = x_1(t_0) - x_2(t_0) > 0$, then

$$u'(t_0) = \phi_q(y_1(t_0)) - \phi_q(y_2(t_0)) = 0,$$

which implies that $y_1(t_0) = y_2(t_0)$ and $u''(t_0) \leq 0$.

Note that since $p \geq 2$, we have $q = p/(p-1) \leq 2$. We define $|u|^{q-2} = 1$ for $u = 0$, $q = 2$ and $|u|^{q-2} = +\infty$ for $u = 0$, $q < 2$. Then it follows from (A_3), (13) and $y_1(t_0) = y_2(t_0)$ that

$$\begin{aligned} u''(t_0) &= (\phi_q(y_1(t)))'|_{t=t_0} - (\phi_q(y_2(t)))'|_{t=t_0} \\ &= (q-1)|y_1(t_0)|^{q-2}y_1'(t_0) - (q-1)|y_2(t_0)|^{q-2}y_2'(t_0)) \\ &= (q-1)|y_1(t_0)|^{q-2}[y_1'(t_0) - y_2'(t_0)] \\ &= (q-1)|y_1(t_0)|^{q-2}v'(t_0) \\ &= (q-1)|y_1(t_0)|^{q-2}[f(t_0, \phi_q(y_2(t_0))) - f(t_0, \phi_q(y_1(t_0)))] \\ &\quad +(q-1)|y_1(t_0)|^{q-2}[g(t_0, x_2(t_0)) - g(t_0, x_1(t_0))] \\ &= (q-1)|y_1(t_0)|^{q-2}[g(t_0, x_2(t_0)) - g(t_0, x_1(t_0))] \\ &> 0(\text{or} = +\infty), \end{aligned}$$

which is a contradiction. Hence $\max_{t \in [0,T]} u(t) \leq 0$. Similarly, exchanging the role of x_1 and x_2, we can show that $\max_{t \in [0,T]} u(t) \geq 0$. This implies that $u(t) \equiv 0$. Therefore, the problem (1)-(2) has at most one solution. The proof of Theorem 1 is now complete. □

Remark 2. If $1 < p < 2$, we can only show the existence of at least one T-periodic solution for problem (1)-(2), but do not know if the uniqueness still holds even if in addition (A_3) is satisfied. This is an open question for readers.

Example 1. For $p \geq 2$, consider the following equation

$$(\phi_p(x'))' + (1 + \sin^2 x')|x'|^{\alpha-1}x' - x^3 = \sin t, \tag{14}$$

where $0 \leq \alpha \leq p-1$. Then $T = 2\pi$, $|f(t,u)| = (1+\sin^2 u)|u|^\alpha \leq 2|u|^\alpha$, $g(t,x) = -x^3$ and $e(t) = \sin t$. Theorem 1 now implies that equation (14) has a unique 2π-periodic solution. But in this example, the assumption $m_1 < 2/2\pi$ in (A_2) of Theorem A does not hold. Therefore, Theorem 1 improves Theorem A even in case $p = 2$.

Example 2. For $p \geq 2$, let $f(t,x') = |x'| \sin^2 x'$, $g(t,x) = -\phi_l(x)$ with $l > 1$, and $e(t) = \cos t$. Then $f(t,x')$ is of constant sign and satisfies $f(t,0) = 0$. Theorem 1 implies that problem (1)-(2) has a unique 2π-periodic solution. In this case, the assumption $m_1 < 2/2\pi$ in (A_2) again does not hold.

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