On Partitioning and Subtractive Ideals of Ternary Semirings

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ABSTRACT. In this paper, we introduce a partitioning ideal of a ternary semiring which is useful to develop the quotient structure of ternary semiring. Indeed we prove: 1) The quotient ternary semiring $S/I_Q$ is essentially independent of choice of $Q$. 2) If $f : S \to S'$ is a maximal ternary semiring homomorphism, then $S/\ker f_Q \cong S'$. 3) Every partitioning ideal is subtractive. 4) Let $I$ be a $Q$-ideal of a ternary semiring $S$. Then $A$ is a subtractive ideal of $S$ with $I \subseteq A$ if and only if $A/I_{(Q \cap A)} = \{q + I \in Q \cap A\}$ is a subtractive ideal of $S/I_Q$.

1. Introduction

The literature of ternary algebraic system was introduced by D. H. Lehmer [12] in 1932. The notion of ternary semiring was introduced by T. K. Dutta and S. Kar [4] in 2003 and has since then been studied many analogous to several well known results in the theory of rings and semirings (See, for example, [4], [5], [6], [7], [11]). Ternary semiring constitute a fairly natural generalization of semirings. In this paper we introduce the notion of partitioning ideal of a ternary semiring, maximal ternary semiring homomorphism which are useful to develop the quotient structure of ternary semiring and the main part is devoted to stating and proving analogous to several well known results in the theory of semirings (See, section 2 and 3). For the sake of completeness, we state some definitions and notations used throughout. For the definition of semiring we refer [8]. Throughout this paper $\mathbb{Z}_0^+$, $\mathbb{Z}_0^-$ will denote the set of all non-negative integers, non-positive integers respectively.

Definition 1.1. A non-empty set $S$ together with a binary operation called addition (+) and a ternary operation called ternary multiplication ($\cdot$) is called ternary semiring if it satisfies the following conditions for all $a, b, c, d, e \in S$:

1. $(a + b) + c = a + (b + c)$;
2. $a + b = b + a$;

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3. \( (a \cdot b \cdot c) \cdot d \cdot e = a \cdot (b \cdot c \cdot d) \cdot e = a \cdot b \cdot (c \cdot d \cdot e) \);

4. there exists \( 0 \in S \) such that \( a + 0 = a = 0 + a, a \cdot b \cdot 0 = a \cdot 0 \cdot b = 0 \cdot a \cdot b = 0 \);

5. \( (a + b) \cdot c \cdot d = a \cdot c \cdot d + b \cdot c \cdot d \);

6. \( a \cdot (b + c) \cdot d = a \cdot b \cdot d + a \cdot c \cdot d \);

7. \( a \cdot b \cdot (c + d) = a \cdot b \cdot c + a \cdot b \cdot d \).

Clearly, every semiring is a ternary semiring. With usual addition and ternary multiplication, \( \mathbb{Z}_0 \) forms a ternary semiring but it is not a semiring.

**Definition 1.2.** If \( S \) and \( S' \) are ternary semirings, then a function \( f \) from \( S \) to \( S' \) is a ternary semiring homomorphism if and only if the following conditions are satisfied:

1. \( f(a + b) = f(a) + f(b) \) for all \( a, b \in S \);
2. \( f(abc) = f(a)f(b)f(c) \) for all \( a, b, c \in S \).

A ternary semiring homomorphism \( f \) from a ternary semiring \( S \) to a ternary semiring \( S' \) is called an isomorphism if \( f \) is one to one and onto.

**Definition 1.3.** A non-empty subset \( I \) of a ternary semiring \( S \) is called an ideal of \( S \) if the following conditions are satisfied:

1. \( a, b \in I \) implies \( a + b \in I \);
2. \( a \in I, r, s \in S \) implies \( rsa, ras, ars \in I \).

The following lemma which is similar to lemma ([9], Lemma 1) is easy to prove.

**Lemma 1.4.** Let \( I \) be an ideal of a ternary semiring \( S \) and \( a, x \in S \) such that \( a + I \subseteq x + I \). Then

1. \( a + r + I \subseteq x + r + I \);
2. \( rsa + I \subseteq rsx + I \);
3. \( ras + I \subseteq rxs + I \);
4. \( ars + I \subseteq xrs + I \) for all \( r, s \in S \).

**2. Partitioning ideals**

In this section we extend some definitions and results of Allen [1], [2] and Atani [3] to ternary semirings.

**Definition 2.1.** An ideal \( I \) of a ternary semiring \( S \) will be called a partitioning ideal (= \( Q \)-ideal) if there exists a subset \( Q \) of \( S \) such that

1. \( S = \cup \{q + I : q \in Q \} \).
2. if \( q_1, q_2 \in Q \), then \( (q_1 + I) \cap (q_2 + I) \neq \emptyset \Leftrightarrow q_1 = q_2 \).

Since every semiring is a ternary semiring, every partitioning ideal of a semiring \( S \) is a partitioning ideal of a ternary semiring \( S \).

**Lemma 2.2.** Let \( I \) be a partitioning ideal of a ternary semiring \( S \). If \( x \in S \), then there exists a unique \( q \in Q \) such that \( x + I \subseteq q + I \). Hence \( x = q + a \) for some \( a \in I \).

**Proof.** Trivial. \( \Box \)

Now we extend a result of P. J. Allen ([2], Lemma 36) for semirings to ternary semirings.

**Lemma 2.3.** If \( I \) is a partitioning ideal of a ternary semiring \( S \), then there exists a unique \( q_0 \in Q \) such that \( I = q_0 + I \).

**Proof.** Since \( I \) is a partitioning ideal, by Lemma 2.2, there exists a unique \( q_0 \in Q \) such that \( 0 = q_0 + a_0 \) for some \( a_0 \in I \). If \( b \in I \), then by Lemma 2.2, there exists a unique \( q \in Q \) such that \( b = q + a \) for some \( a \in I \). Therefore, \( q + a = b + 0 = b + q_0 + a_0 \in q_0 + I \). Hence \( I \subseteq q_0 + I \). Again by Lemma 2.2, there exists a unique \( q \in Q \) such that \( q_0 + q_0 = q^2 + c \) for some \( c \in I \). Now \( q_0 = q_0 + 0 = q_0 + q_0 + a_0 = q^2 + c + a_0 \in q^2 + I \). Also \( q_0 \in q_0 + I \). Hence \( (q^2 + I) \cap (q_0 + I) \neq \emptyset \) and so \( q_0 = q^2 \). Thus, \( q_0 + I = q^2 + c + a_0 + I = q_0 + c + a_0 + I = c + q_0 + a_0 + I = c + I \subseteq I \). Hence \( I = q_0 + I \) where \( q_0 \in Q \) is a unique element. \( \Box \)

Let \( I \) be a partitioning ideal of a ternary semiring \( S \). Then \( S/I(Q), \oplus, \odot \) forms a ternary semiring under the following addition “\( \oplus \)” and ternary multiplication “\( \odot \)”, \( (q_1 + I) \oplus (q_2 + I) = q^2 + I \) where \( q^2 \in Q \) is a unique element such that \( q_1 + q_2 + I \subseteq q^2 + I \) and \( (q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q_4 + I \) where \( q_4 \in Q \) is a unique element such that \( q_1 q_2 q_3 + I \subseteq q_4 + I \). This ternary semiring will be called a quotient ternary semiring of \( S \) by \( I \) and denoted by \( (S/I(Q), \oplus, \odot) \) or just \( S/I(Q) \). By Lemma 2.3, there exists a unique \( q_0 \in Q \) such that \( q_0 + I = I \). This \( q_0 + I \) is the zero element of \( S/I(Q) \). If \( I \) is an ideal of a ternary semiring \( S \), then it is possible that \( I \) can be considered to be a partitioning ideal with respect to many different subsets \( Q \) of \( S \). However, the next theorem proves that the structure \( (S/I(Q), \oplus, \odot) \) is essentially independent of \( Q \).

**Theorem 2.4.** If \( I \) is a partitioning ideal with respect to two subsets \( Q_1 \) and \( Q_2 \) of a ternary semiring \( S \), then \( S/I(Q_1) \cong S/I(Q_2) \).

**Proof.** Define \( f: S/I(Q_1) \to S/I(Q_2) \) by \( f(q_1 + I) = q_2 + I \) where \( q_2 \in Q_2 \) is a unique element such that \( q_1 + I \subseteq q_2 + I \). Clearly, \( f \) is well defined.

1) Let \( q_1 + I, q_1 + I, q_1'' + I \in S/I(Q_1) \). Therefore,

\[
\begin{align*}
f((q_1 + I) \oplus (q_1' + I)) &= f(q_1'' + I) = q_2 + I, \\
\end{align*}
\]

where \( q_1'' \in Q_1 \) is a unique element such that \( q_1 + q_1' + I \subseteq q_1'' + I \) and \( q_2 \in
$Q_2$ is a unique element such that $q_1''' + I \subseteq q_2 + I$. Also

(ii) \[ f(q_1 + I) \oplus f(q_1' + I) = (q_2' + I) \oplus (q_2'' + I) = q_2'' + I, \]

where $q_2', q_2'' \in Q_2$ are unique elements such that $q_1 + I \subseteq q_2' + I$ and $q_1' + I \subseteq q_2'' + I$ and $q_2 \in Q_2$ is a unique element such that $q_2 + q_2'' + I \subseteq q_2 + I$. Now

(iii) \[ q_1 + q_1' \in q_1 + q_1' + I \subseteq q_1''' + I \subseteq q_2 + I \]

Also by Lemma 1.4,

\[ q_1 + I \subseteq q_2' + I \] and \[ q_1' + I \subseteq q_2'' + I \Rightarrow q_1 + q_1' + I \subseteq q_2' + q_2'' + I \]

\[ \subseteq q_2 + q_2'' + I \]

Therefore,

(iv) \[ q_1 + q_1' \in q_1 + q_1' + I \subseteq q_2''' + I \]

From (iii) and (iv), $q_2 = q_2'''$. Hence by (i) and (ii), $f((q_1 + I) \oplus (q_1' + I)) = f(q_1 + I) \oplus f(q_1' + I)$. Similarly, it can be shown that $f((q_1 + I) \odot (q_1' + I)) = f(q_1 + I) \odot f(q_1' + I)$.

2) Let $q_2 + I \in S/I(Q_2)$. Since $q_2 \in S$, there exists a unique $q_1 \in Q_1$ such that $q_2 + I \subseteq q_1 + I$. But then there exists a unique $q_2' \in Q_2$ such that $q_1 + I \subseteq q_2' + I$. Now $q_2 = q_2'$ implies $q_2 + I = q_2' + I$ and hence $f(q_1 + I) = q_2 + I$. So $f$ is onto.

3) Suppose that $f(q_1 + I) = f(q_1' + I) = q_2 + I$ say, where $q_2 \in Q_2$ is a unique element such that $q_1 + I \subseteq q_2 + I$ and $q_1 + I \subseteq q_2 + I$. Choose $t_1 \in Q_1$ such that $q_2 + I \subseteq t_1 + I$. But then $q_1 = t_1 = q_1$. So $q_1 + I = q_1 + I$. Thus, $f : S/I(Q_1) \rightarrow S/I(Q_2)$ is an isomorphism.

\[ \square \]

**Theorem 2.5.** If $I$ is a partitioning ideal with respect to two subsets $Q_1$ and $Q_2$ of a ternary semiring $S$, then $S/I(Q_1)$ and $S/I(Q_2)$ are equal as sets.

**Proof.** Let $q_1 + I \in S/I(Q_1)$. Then $q_1 \in Q_1 \subseteq S$ and hence by Lemma 2.2, there exists a unique $q_2 \in Q_2$ such that $q_1 + I \subseteq q_2 + I$. Again there exists a unique $q_1'$, in $Q_1$ such that $q_2 + I \subseteq q_1' + I$. Now $q_1 + I = q_1' + I = q_2 + I \in S/I(Q_2)$. So $S/I(Q_1) \subseteq S/I(Q_2)$. Similarly, $S/I(Q_2) \subseteq S/I(Q_1)$.

\[ \square \]

**Example 2.6.** In a ternary semiring $S = (\mathbb{Z}_6, +, \times)$, the ideal $I = \{ 0, 2, 4 \}$ is a partitioning ideal of $S$ with respect to three sets $Q_1 = \{ 0, 4 \}, Q_2 = \{ 0, 3 \}, Q_3 = \{ 0, 5 \}$ where $S/I(Q_1) = \{ 0 + I, 4 + I \} = \{ 0, 4 \}, S/I(Q_2) = \{ 0 + I, 3 + I \} = \{ 0, 3 \}, S/I(Q_3) = \{ 0 + I, 5 + I \} = \{ 0, 5 \}$. Here $S/I(Q_1)$, $S/I(Q_2)$ and $S/I(Q_3)$ are equal as sets. But $S/I(Q_1)$, $S/I(Q_2)$ and $S/I(Q_3)$ considered as ternary semirings are not pairwise equal because $I + I \in S/I(Q_1)$ but $I + I \notin S/I(Q_2)$.
For any is an onto ternary semiring homomorphism. Also \( q_2 \neq S \).

**Definition 2.7.** An onto ternary semiring homomorphism \( f : S \to S' \) will be called maximal if for each \( a \in S' \) there exists a unique \( q_a \in f^{-1}(\{a\}) \) such that \( x + kerf \subseteq q_a + kerf \) for each \( x \in f^{-1}(\{a\}) \) where \( kerf = \{ x \in S : f(x) = 0_{S'} \} \).

Clearly, every maximal semiring homomorphism is a maximal semiring homomorphism.

P. J. Allen ([1], Lemma 14, Lemma 15 and Theorem 16) has proved the results for semirings. However, we extend the following Lemma 2.8, Lemma 2.11 and Theorem 2.12 for ternary semirings.

**Lemma 2.8.** If \( f : S \to S' \) is a maximal ternary semiring homomorphism, then \( kerf \) is a partitioning ideal of \( S \).

**Proof.** Since \( f \) is maximal, for each \( a \in S' \) there exists a unique \( q_a \in f^{-1}(\{a\}) \) such that \( x + kerf \subseteq q_a + kerf \) for all \( x \in f^{-1}(\{a\}) \). Take \( Q = \{ q_a : a \in S' \} \). Clearly \( \cup \{ q_a + kerf : q_a \in Q \} \subseteq S \). On the other hand if \( m \in S \), then \( f(m) \in S' \). Now \( m \in f^{-1}(\{f(m)\}) \) implies \( m \in m + kerf \subseteq q_{f(m)} + kerf \). Hence \( S \subseteq \cup \{ q_a + kerf : q_a \in Q \} \). Now for \( q_a, q_b \in Q \), suppose that \( (q_a + kerf) \cap (q_b + kerf) \neq \emptyset \). Let \( q_k = q_a + k \) for some \( k \in kerf \). Now \( a = f(q_a) + f(k) = f(q_k) = f(q_a + k) = f(q_a) + f(k') = b \). Hence \( q_a = q_b \). Thus, \( kerf \) is a partitioning ideal of \( S \).

The converse of Lemma 2.8 is not true.

**Example 2.9.** Let \( S = (\mathbb{Z}_6^+, \max, \min) \) and \( S' = (\{0, 1\}, \max, \min) \). Then \( S, S' \) are ternary semirings. Define \( f : S \to S' \) by

\[
 f(x) = \begin{cases} 
 0 & \text{if } x \leq 5 \\
 1 & \text{if } x > 5 
\end{cases}
\]

Then, \( f \) is an onto semiring homomorphism([1], Example 11). Hence, \( f \) is an onto ternary semiring homomorphism. Also \( kerf = \{ 0, 1, 2, 3, 4, 5 \} \) is a partitioning ideal of \( S \) with \( Q = \{ 0, 6, 7, \ldots \} \). For \( 1 \in S' \) there cannot exists any \( q_1 \in f^{-1}(\{1\}) \) such that \( x + kerf \subseteq q_1 + kerf \) for all \( x \in f^{-1}(\{1\}) \). So \( f \) is not a maximal ternary semiring homomorphism.

**Example 2.10.** Let \( S = (\mathbb{Z}_6^+, +, \cdot) \), \( S' = (\mathbb{Z}_6, +_6, \times_6) \). Then \( S, S' \) are ternary semirings. Define \( f : S \to S' \) by \( f(x) = r \) where \( x \equiv r \pmod{6} \), \( 0 \leq r \leq 5 \). Clearly, \( f \) is an onto ternary semiring homomorphism. Also \( kerf = \{ 0, -6, -12, -18, \ldots \} \). For any \( \pi \in S' \) there exists a unique \( q_{-a} = -a \in f^{-1}(\{\pi\}) \) such that \( x + kerf \subseteq q_{-a} + kerf \) for all \( x \in f^{-1}(\{\pi\}) \). Hence \( f \) is a maximal ternary semiring homomorphism.

**Lemma 2.11.** Let \( S, S', f \) and \( Q \) be as stated in Lemma 2.8. Let \( q_a, q_b, q_c, q_d, q_t \in Q \), then
(i) If \( q_a + q_b + ker f \subseteq q_c + ker f \), then \( a + b = r \).

(ii) If \( q_a q_b q_c + ker f \subseteq q_t + ker f \), then \( abc = t \).

**Proof.** (i) Since \( q_a + q_b \in q_a + q_b + ker f \subseteq q_r + ker f \), \( q_a + q_b = q_r + k \) for some \( k \in ker f \). Now \( a + b = f(q_a) + f(q_b) = f(q_a + q_b) = f(q_r + k) = f(q_r) + f(k) = r \).

(ii) Since \( q_a q_b q_c + ker f \subseteq q_t + ker f \), \( q_a q_b q_c = q_t + k' \) for some \( k' \in ker f \). Now \( abc = f(q_a) f(q_b) f(q_c) = f(q_a q_b q_c) = f(q_t + k') = f(q_t) + f(k') = f(q_t) = t \).

\( \square \)

**Theorem 2.12.** If \( f : S \to S' \) is a maximal ternary semiring homomorphism, then \( S/kern f(Q) \cong S' \) where \( Q \) is as stated in Lemma 2.8.

**Proof.** By Lemma 2.8, \( kern f \) is a partitioning ideal of \( S \). Define \( \overline{f} : S/kern f(Q) \to S' \) by \( \overline{f}(q_a + ker f) = f(q_a) = a \) for each \( q_a \in Q \). If \( q_a + kern f, q_b + kern f \in S/kern f(Q) \), then \( \overline{f}(q_a + ker f) + \overline{f}(q_b + ker f) \Rightarrow a = b \Leftrightarrow q_a + ker f = q_b + ker f. \) Hence \( \overline{f} \) is well defined and one-one. Since \( f \) is maximal, \( \overline{f} \) is onto. For \( q_a + ker f, q_b + ker f, q_c + ker f \in S/kern f(Q) \), consider (i) \( \overline{f}((q_a + ker f) + (q_b + ker f)) = \overline{f}(q_r + ker f) = r \) where \( q_r \) is a unique element in \( Q \) such that \( q_a + q_b + ker f \subseteq q_r + ker f \).

By Lemma 2.11, \( a + b = r \). Now \( \overline{f}(q_a + ker f) + \overline{f}(q_b + ker f) = a + b = r = \overline{f}((q_a + ker f) + (q_b + ker f)). \) (ii) \( \overline{f}((q_a + ker f) \cdot (q_b + ker f)) = \overline{f}(q_r + ker f) = t \) where \( q_r \) is a unique element in \( Q \) such that \( q_a q_b q_c + ker f \subseteq q_r + ker f \).

By Lemma 2.11, \( abc = t \). Therefore, \( \overline{f}(q_a + ker f) \overline{f}(q_b + ker f) \overline{f}(q_c + ker f) = abc = t = \overline{f}((q_a + ker f) \cdot (q_b + ker f) \cdot (q_c + ker f)). \) Hence \( \overline{f} \) is a ternary semiring isomorphism. Thus, \( S/kern f(Q) \cong S' \).

\( \square \)

3. Subtractive ideals

In this section, we extend some results of S. E. Atani [3] to ternary semirings.

**Definition 3.1.** An ideal \( I \) of a ternary semiring \( S \) is called a subtractive ideal (= k-ideal) if \( a, a + b \in I, b \in S \), then \( b \in I \).

**Theorem 3.2.** Every partitioning ideal \( I \) of a ternary semiring \( S \) is subtractive.

**Proof.** Since \( I \) is a partitioning ideal, by Lemma 2.3, \( I = q_0 + I \) for some \( q_0 \in Q \). Let \( a, a + b \in I \) where \( b \in S \). Therefore \( a = q_0 + \alpha, a + b = q_0 + \beta \) for some \( \alpha, \beta \in I \). By Lemma 2.2, there exists a unique \( q' \in Q \) such that \( b = q' + \lambda \) for some \( \lambda \in I \). Now \( b + a = q' + \lambda + a \in q' + I \) and \( a + b = I = q_0 + I \). Hence \( \cap (q_0 + I) \neq \emptyset \) and so \( q_0 = q' \). Thus, \( b = q' + \lambda \in q' + I = q_0 + I = I \).

The converse of Theorem 3.2 is not true.

**Example 3.3** ([10], p.182). In a ternary semiring \( S = (Z_0^+, \gcd, lcm) \), the ideal \( 2Z_0^+ = \{0, 2, 4, 6, \ldots \} \) of \( S \) is subtractive but not partitioning.

S. E. Atani ([3], Lemma 2.1, Proposition 2.2 and Theorem 2.3) has proved the results for semirings. However, we extend the following Lemma 3.4, Theorem 3.5 and Theorem 3.6 to ternary semirings.
Lemma 3.4. Let I be a Q-ideal of a ternary semiring S. If A is a subtractive ideal of S such that I ⊆ A, then I is a Q ∩ A-ideal of A.

Proof. It is sufficient to show that A = ∪{q + I : q ∈ Q ∩ A}. Clearly, ∪{q + I : q ∈ Q ∩ A} ⊆ A. On the other hand, let x ∈ A. Since I is a Q-ideal, by Lemma 2.2, x = q + a for some q ∈ Q, a ∈ I ⊆ A. Then q ∈ Q ∩ A, since A is a subtractive ideal. So we have an equality. □

Theorem 3.5. Let I be a Q-ideal, A be a subtractive ideal of a ternary semiring S with I ⊆ A. Then A/I_{Q \cap A} = \{q + I : q ∈ Q \cap A\} is a subtractive ideal of S/I_{Q}.

Proof. By Lemma 2.3, let q_{0} ∈ Q be unique such that q_{0} + I is the zero element of S/I_{Q}. First, we show that q_{0} + I ∈ A/I_{Q \cap A}. Let a + I ∈ A/I_{Q \cap A} ⊆ S/I_{Q} where a ∈ Q ∩ A. Then (a + I) ⊕ (q_{0} + I) = a + I where a ∈ Q is a unique such that a + q_{0} + I ⊆ a + I. So a + q_{0} = a + b for some b ∈ I ⊆ A. Since A is a subtractive ideal, q_{0} ∈ A. Thus, q_{0} + I ∈ A/I_{Q \cap A}. Next, suppose that q_{1} + I, q_{2} + I ∈ A/I_{Q \cap A}, where q_{1}, q_{2} ∈ Q ∩ A. Then (q_{1} + I) ⊕ (q_{2} + I) = q_{3} + I, where q_{3} ∈ Q is a unique such that q_{1} + q_{2} + I ⊆ q_{3} + I. So q_{1} + q_{2} = q_{3} + c for some c ∈ I ⊆ A. Hence q_{3} ∈ Q ∩ A, since A is a subtractive ideal. Now (q_{1} + I) ⊕ (q_{2} + I) = q_{3} + I ∈ A/I_{Q \cap A}. Now let r + I, s + I ∈ S/I_{Q}, q + I ∈ A/I_{Q \cap A}. Then (r + I) ⊕ (s + I) ⊕ (q + I) = q_{4} + I, where q_{4} ∈ Q is a unique such that rsq + I ⊆ q_{4} + I. So rsq = q_{4} + d for some d ∈ I ⊆ A. Hence q_{4} ∈ Q ∩ A, since A is a subtractive ideal. Thus, (r + I) ⊕ (s + I) ⊕ (q + I) = q_{4} + I ∈ A/I_{Q \cap A}. Similarly, it can be shown that (r + I) ⊕ (s + I) ⊕ (q + I) = q_{4} + I ∈ A/I_{Q \cap A}. Finally, assume that a + I, (a + I) ⊕ (b + I) = c + I ∈ A/I_{Q \cap A} where a, c ∈ Q ∩ A, b ∈ Q and a + b + I ⊆ c + I. Then a + b = c + e for some e ∈ I ⊆ A. Now b ∈ Q ∩ A, since A is a subtractive ideal. Thus, b + I ∈ A/I_{Q \cap A} as needed. □

Theorem 3.6. Let I be a Q-ideal of a ternary semiring S and L be a subtractive ideal of S/I_{Q}. Then L = P/I_{Q \cap P} for some subtractive ideal P of S with I ⊆ P.

Proof. By Lemma 2.3, let q_{0} + I = I be the zero element of S/I_{Q} where q_{0} ∈ Q. Denote P = \{a ∈ S : there exists a unique q ∈ Q such that a + I ⊆ q + I ∈ L\}. We show that P is a subtractive ideal of S with I ⊆ P and L = P/I_{Q \cap P}.

(1) Let a ∈ I. Then a + I ⊆ I = q_{0} + I ∈ L, so a ∈ P. Thus, I ⊆ P.

(2) Let a, b ∈ P. Then there are unique elements q_{1}, q_{2} ∈ Q such that a + I ⊆ q_{1} + I ∈ L and b + I ⊆ q_{2} + I ∈ L. Now (q_{1} + I) ⊕ (q_{2} + I) = q_{3} + I ∈ L where q_{3} ∈ Q is a unique such that q_{1} + q_{2} + I ⊆ q_{3} + I. By Lemma 1.4, a + I ⊆ q_{1} + I and b + I ⊆ q_{2} + I implies a + b + I ⊆ q_{1} + q_{2} + I ⊆ q_{3} + I ⊆ q_{3} + I ∈ L. Hence a + b ∈ P. Similarly, if r, s ∈ S, a ∈ P, then rsa, ras, ras ∈ P. Thus, P is an ideal of S.

(3) Let a, b ∈ P where b ∈ S. Then there are unique elements q_{1}, q_{2}, q_{3} ∈ Q such that a + I ⊆ q_{1} + I ∈ L, a + b + I ⊆ q_{2} + I ∈ L, b + I ⊆ q_{3} + I ∈ S/I_{Q}. Since q_{1} + I, q_{3} + I ∈ S/I_{Q}, there exists a unique element q_{4} ∈ Q such that (q_{1} + I) ⊕ (q_{3} + I) = q_{4} + I where q_{1} + q_{3} + I ⊆ q_{4} + I. By Lemma 1.4, a + I ⊆ q_{1} + I and b + I ⊆ q_{3} + I implies a + b + I ⊆ q_{1} + b + I ⊆ q_{1} + q_{3} + I ⊆ q_{4} + I. Hence
\[a + b \in (q_2 + I) \cap (q_4 + I).\] So \(q_4 + I = q_2 + I \in L.\] Since \(L\) is a subtractive ideal, \(q_3 + I \in L.\) Now \(b + I \subseteq q_3 + I \in L.\) So \(b \in P.\) Hence \(P\) is a subtractive ideal of \(S.\)

(4) By Lemma 3.4, \(I\) is a \(Q \cap P\)-ideal of \(P.\) If \(q + I \in L\) where \(q \in Q\) then \(q \in P.\) So \(q \in Q \cap P,\) hence \(q + I \in P/I(Q \cap P).
Thus, \(L \subseteq P/I(Q \cap P).\) On the other hand, if \(q + I \in P/I(Q \cap P),\) then \(q \in Q \cap P \subseteq P.\) So \(q + I \subseteq q' + I \in L\) for some unique \(q' \in Q.\) Therefore, \(q + I = q' + I \in L.\) Thus, \(P/I(Q \cap P) \subseteq L.\)

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References


