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Modified (Q, r) Model for Discrete Demand*

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ABSTRACT

In the continuous review (Q, r) model one continuously monitors inventory level and places a replenishment order when the inventory position reaches the reorder point. In many business practices, however, inventory decreases in a discrete fashion. As a result, replenishment orders are usually placed after the inventory position gets far below the reorder point. This makes a chance of shortage more likely and the service level lower than designed. Such a discrepancy can be compensated for by raising the reorder point to some extent. The question is how much the reorder point should be raised in order to compensate for a potential shortage. In this study, we present experimental analyses for this question. Regression models are also proposed for on-site use.

Keywords: Inventory, Reorder Point, Discrete Demand, Service Level

1. Introduction

In the continuous review (Q, r) model, one continuously monitors inventory level and places a replenishment order of fixed amount Q when inventory position reaches a predetermined reorder point r. Order quantity and reorder points are usually calculated by considering demand rate, order cost, inventory holding cost, shortage cost, and customer service level. In this system, stockout may occur during the

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lead time, the interval between the time a replenishment order is placed and the time it is received. The reorder point must be set to cover the expected demand during the lead time plus some safety stock determined to meet specified service levels. That is, for normally distributed demand,

$$r = \mu L + z_{\alpha}\,\sigma_L \tag{1}$$

where μ is the average daily demand, σ_L is the standard deviation of demand during the lead time, and z_{α} is the z-score (called safety factor hereafter) needed to secure a specified service level. The service level is typically denoted by $(1-\alpha)$, where α is the probability of stockout during the lead time. Note that $(1-\alpha)$ is specifically called the cycle service level, which differs from the fill rate (Li, 2007).

The model described above assumes continuous change in inventory level. In many business practices, however, inventory level decreases in a discrete fashion. This is because customer orders are usually bulky. Due to this discrete nature, replenishment orders are usually placed when the inventory level falls far below the reorder point, and in turn, the target service level may not be met. This implies the need of a discrete version of the (Q, r) model.

In this paper, we present a modified (Q, r) model for discrete demand. Our aim is to determine the revised reorder point for securing a service level of $(1-\alpha)$. Our study is based on a simulation with various demand scenarios. Regression models are also proposed for on-site use. Section 2 reviews the related literature. Section 3 defines our problem and sketches simulation modeling. Section 4 summarizes experimental results and presents regression models. Concluding remarks are given in Section 5.

2. Literature Review

Probabilistic inventory models are typically classified into three categories (Schneider, 1981). The first category is the continuous review model ((s, Q) or (Q, r) model) where the inventory position (IP) is continuously monitored and an order of size Q is placed when IP ≤ s. The second category is the periodic review (or order-up-to-level) model (S, T) where IP is checked at every T time units and an order of size (S-IP) is placed. The third category is the (s, S, T) (or (R, s, Q)) model where IP is checked at every T time units but an order of size (S-IP) is placed only when IP ≤ s. Numerous studies have been conducted regarding the service level and inventory control for those models. Some of them focused on the matter of discrete (occasionally called lumpy or sporadic) demand.

For the (s, Q) model, Ward (1978) points out that the reorder point in the (s, Q) model has to be higher for slow items with lumpy demand than that under the continuous demand. He assumes compound Poisson demand and derives a reorder point as a function of shortage fraction per order cycle. He also presents a multiple regression model to obtain a reorder point, where independent variables include the coefficient of variation of demand and shortage fraction per order cycle. Williams (1982) presents approximate expressions for the optimal reorder point for lumpy demand, assuming the gamma-distributed sized orders arrive stochastically, with no more than one order during the lead time. Williams (1983) also provides a table that will aid the practical use of the method developed by Williams (1982).

For periodic review problems, Inderfurth and Minner (1998) treat the problem of determining safety stocks in multi-stage inventory systems with normally distributed demands. For periodic review with zero lead time and discrete demand, Sakaguchi and Kodama (2006) present a multi-period review policy which is based on dynamic programming that minimizes the sum of purchasing, holding, and shortage cost. Cardós *et al*. (2006) present an exact calculation of the cycle service level, which is defined as the probability of not having a stockout in a replenishment cycle.

There are some papers that address the discrete demand case in the (s, S, T) model. Hill (1988) presents an approximate result that adjusts the undershoot in the (s, S, T) model with compound Poisson demand. Janssen *et al*. (1998) present an approximation method to compute the reorder point s in the (s, S, T) model with demand modeled as a compound Bernoulli process, which is suitable for intermittent demand. Janssen *et al*. (1999) also present an approximation method to determine s in the (s, S, T) model with demand modeled as a compound renewal process such that the target service level is achieved. Moors and Strijbosch (2002) derive an exact expression for the fill rate for gamma distributed demand and constant lead time. They also propose a fast simulation program holding for general shape parameter values. Zeng and Hayya (1999) evaluate two popular service measures which are the probability of no stockout during the lead time and the fill rate.

For lumpy or sporadic demand, Williams (1984) presents a method of classifying product demand into smooth, slow-moving or sporadic, by partitioning the variance of demand during a lead time into causal parts. Stock control and forecasting methods are developed, and simulation tests are described which compare these methods with the assumption of continuous demand. Rempala (2003) deals with the joint replenishment multi-product inventory problem with continuous production and discrete demand.

Chen and Krass (2001) distinguish a minimal service level from mean service level, and study inventory models in which the stockout cost is replaced by a minimal service level constraint that requires a certain level of service to be met in every period, instead of minimizing total cost. They show that above a certain critical service level, the optimal (s, S) policy collapses to a simple base-stock or order-up-to level policy, which is independent of the cost parameters. Mattsson (2007) considers undershoot when lead time is shorter; and proposes a simulation-based enhanced model to calculate a reorder point which will reduce the gap between desired and actually attained service levels.

Our study is different from Ward (1978) in that our regression models employ not only the coefficient of variation of the demand and target service level but replenishment lead time as independent variables; and safety factor (z_α) as a dependent variable. Our study is also different from Williams (1982) in that multiple orders are allowed to arrive and that customer order size is normally distributed.

3. Determining Safe Stock for Discrete Demand

3.1 Notations

The following notations are used in the subsequent discussion:

L : replenishment lead time (in unit times)

D : customer demand per unit time, $D \sim N(\mu, \sigma^2)$.

μ : average demand per unit time

σ^2_L: variance of the demand during the lead time, $\sigma^2_L = L\sigma^2$.

CV: coefficient of variation of the unit time demand, $CV = \sigma/\mu$

α: probability of stockout in L

SL: service level, $1-\alpha$

z_α: safety factor for achieving service level of $(1-\alpha)$

r: reorder point

Q : replenishment order quantity
$f(z_\alpha)$: service level under continuous demand with safety factor z_α
$g(z_\alpha)$: service level under discrete demand with safety factor z_α

As discussed in Section 1, it is clear that $f(z_\alpha) = 1–\alpha > g(z_\alpha)$. We are aiming at finding an adjusted safety factor z_α', which satisfies $f(z_\alpha) = g(z_\alpha')$. By plugging z_α' in Equation 1 we can obtain the reorder point for discrete demand cases. To this end, we adopted simulation approaches to measure the service level for various input parameters.

3.2 Simulation

Our simulation logic is as follows: a reorder point is set as given in (1). As time elapses, the inventory level decreases as random demand occurs. A replenishment order of size Q is placed when the inventory position falls below r. The replenishment order arrives after L units of time. Any shortages during the lead time are assumed to be backlogged. Initial inventory level is set to be 0 for all the experiments. The cycle service level is calculated as the ratio of the number of inventory cycles without shortages to the number of the total inventory cycles in the simulation horizon. Care should be taken when dealing with back orders in the simulation. Each simulation experiment runs for 10,000 units of time, in days. Simulation logic is represented by pseudocode given in the appendix, and coded using C++8.0.

3.2.1 Selecting Input Parameters

We need to identify input variables for the simulation. Recall that the inventory position varies by the interplay of random demand, replenishment order quantity, and reorder point level. Random demand is characterized by the mean and variance of the demand distribution. The reorder point, as seen in Equation 1, is a function of average demand per unit time (a day), the length of lead time, and the service level represented by a safety factor.

Through some pilot simulations, we observe that the variables that had meaningful impact on the resulting service level include the lead time (L), the coefficient of variation of the unit time demand (CV), and the safety factor (z_α). The replenishment order quantity (Q) does *not* affect the simulation result. This is because stockout can occur only during the lead time, and the size of the order quantity does not affect what happens during the lead time. The length of the lead time is critical to the simulation result. This is because a larger lead time inflates demand uncertainty during the lead time, and in turn, inflates safety stock. Instead of varying the mean and variance of the demand distribution in the simulation, we use the coefficient of variation.

In the simulation, we use 4 levels of L varying from 2 to 5 unit times (days) with an increment of 1. These values are typical in large scale domestic supply chains. We employ 9 levels of CV from 0.1 to 0.5 with an increment of 0.05, and 100 levels of z from 0 to 9.9 with an increment of 0.1. Average daily demand is set to be 100 for all cases.

3.2.2 Simulation Results

Table 1 shows part of the resulting service levels under discrete demand where L = 2 and μ = 100. As discussed earlier, the service level under discrete demand, $g(z_\alpha)$, falls far below that under continuous demand, $f(z_\alpha)$. For example, by setting z_α = 2.1 in Equation 1, one can theoretically secure SL = 1-α = 98.2% from the standard normal

Table 1. SL Achieved for Varying z_α and CV (L = 2). SL between 0.8 and 0.98 are Shown. Empty Entries are Omitted

	L = 2								
z	CV = 0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
1.7									
1.8									
1.9									
2							0.803992	0.8161471	0.8110672
2.1							0.8311377	0.8393285	0.8332016
2.2						0.8026316	0.8518962	0.8569145	0.8494071
2.3						0.8205742	0.8698603	0.8780975	0.8671937
2.4						0.8405104	0.8882236	0.8948841	0.8837945
2.5					0.8161471	0.861244	0.9037924	0.911271	0.9
2.6					0.8361311	0.877193	0.9169661	0.9216627	0.913834
2.7				0.8001599	0.8529177	0.8931419	0.9257485	0.9404476	0.9249012
2.8				0.8146225	0.8717026	0.9059011	0.9369261	0.9452438	0.958498
2.9				0.8333999	0.8860911	0.9166667	0.946507	0.9548361	0.9656126
3				0.8537755	0.8976819	0.9314195	0.9520958	0.9636291	0.972738
3.1				0.8685577	0.9108713	0.9381978	0.9616766	0.9736211	0.9778744
3.2			0.8069684	0.8861366	0.9240608	0.9481659	0.9676647	0.9772182	
3.3			0.82499	0.9009189	0.9380496	0.9565391	0.9716567		
3.4			0.8406087	0.9105074	0.9476419	0.9645136	0.9736527		
3.5			0.8490188	0.9188973	0.9540368	0.9704944			
3.6			0.8622347	0.931682	0.9620304	0.9744817			
3.7			0.8762515	0.942469	0.9684253	0.9780702			
3.8			0.8906688	0.9520575	0.9692246				
3.9			0.9022827	0.9564523	0.9728217				
4		0.8068136	0.9146973	0.9636436	0.9776179				

table. In Table 1, however, setting $z_\alpha = 2.1$ results in SL = 80.4% when CV = 0.4. Again, this happens due to the discrete nature of demand.

Using the results in Table 1, we can obtain the value of z_α' for the target service level. For example, a theoretical value of z_α is 1.3 for a 90% service level (from the standard normal table). But in Table 1, z_α' has to be 3.9 when CV = 0.2; and 2.6 when CV = 0.5. In this manner, the corresponding z_α' can be found in the table.

Figure 1 depicts a chart of the values in Table 1. In the figure, we can see strong linear relationship between SL and z_α for certain ranges of SL.

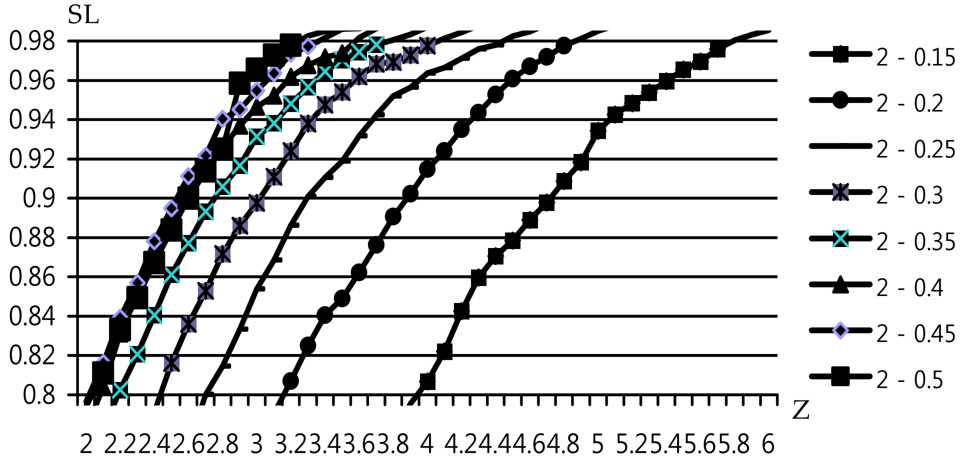

Figure 1. Service Level Achieved for Various Levels of z_α and CV (L = 2)

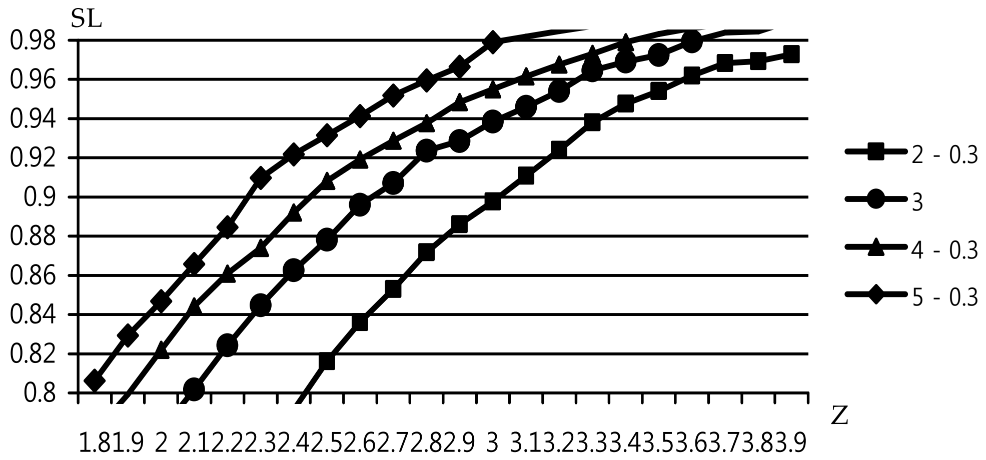

Figure 2. Service Level Achieved for Various Levels of z and L (CV = 0.3)

We can also see strong linearity between SL and z_α as depicted in Figure 2, where CV = 0.3 with various lead times. With these observations, we derive some regression equations to obtain a proper value for z_α', which will lead to the desired service level.

4. Regression Analysis

Our goal is to find a regression equation of z_α' as a function of L, CV, and SL. The model is represented as $z_\alpha' = \beta_0 + \beta_1 L + \beta_2 CV + \beta_3 SL$. In estimating the model parameters, we use four levels of L, varying from 2 to 5, and eight levels of CV, from 0.15 to 0.5 with an increment of 0.05. We confine SL to the range from 0.8 to 0.98. This is because SL below 0.8 or above 0.98 is practically meaningless.

The result of the first cut analysis is as follows. The linear relationship between z_α' and each independent variable is statistically significant as shown in Table 2.

Table 2. Coefficient of Correlation among z_α' and Three Independent Variables (L = 2 to 5, CV = 0.15 to 0.5, SL = 0.8 to 0.98)

	ZB′	L	CV	SL
ZB′	1			
L	-0.37	1		
CV	-0.62	-0.01	1	
SL	0.57	0.03	0.02	1

As can be seen in Table 3, all the independent variables have significant impact on z′. Our model is now expressed as

$$z_\alpha' = -2.731 - 0.298L - 4.819CV + 9.050SL \quad (2)$$

Table 3. Regression Output (L = 2 to 5, CV = 0.15 to 0.5, SL = 0.8 to 0.98)

regression statistic	
R-square	0.88
adj. R-square	0.88
standard error	0.30
observations	558

var.	coeff.	t	p-value
intercept	-2.73	-12.63	0.00
L	-0.30	-26.05	0.00
CV	-4.82	-42.94	0.00
SL	9.05	39.66	0.00

For example, consider a normal daily demand with a mean of 100, a standard deviation of 40, and a 5-day replenishment lead time. Then, setting $z_{.05}$ = 1.645 in Equation 1 would secure a 95% service level for the well known continuous demand case. From Equation 1 we can determine the reorder point as 647 units. For the discrete demand case, however, a 95% service level was achieved with the reorder point of 717 units in the simulation, or $z_{.05}'$ = 2.456. This value is close to $z_{.05}'$ = 2.447 calculated using Equation 2.

Using Equation 2, we can compute the adjusted safety factor z_α' which approximately satisfies $f(z_\alpha) = g(z_\alpha')$ if the values of L, CV, and SL are given. In this section, we will further refine the regression models.

It is observed in the experimental results that narrowing down the CV range helps improve the accuracy of regression models. Table 4 shows three different models (models 1 to 3) for each of the CV segments: relatively low demand variability (CV = 0.15 to 0.3), medium variability (CV = 0.25 to 0.4), and high variability (CV = 0.35 to 0.5). Models 1, 2, and 3 are better than model 0 (the model obtained in section 4.1) in terms of R-squared value and standard error of estimation.

Table 4. Refined Regression Models for z_α' (L = 2, 3, 4, 5, SL = 0.8 to 0.98)

CV range	R-squared	Standard error	Regression equation
0.15~0.3	0.93	0.24	z_α' = -2.399-0.372L-8.628CV+9.852SL (model 1)
0.25~0.4	0.95	0.09	z_α' = -1.256-0.217L-3.265CV+6.280SL (model 2)
0.35~0.5	0.94	0.10	z_α' = -2.277-0.197L-1.556CV+6.671SL (model 3)
0.15~0.5	*0.88*	*0.30*	*z_α' = -2.731-0.298L-4.819CV+ 9.050SL (model 0)*

The refined regression models in Table 4 can be useful for practical problems. Based on the CV values in question, one can select the appropriate regression equation in Table 4. By plugging the value of L, CV, and SL in the corresponding equation, one can compute z_α'. Also, by plugging the resulting z_α' into Equation 1, one can determine the modified reorder point to obtain the target service level for discrete demands.

Table 5 shows how adjusted safety factor z_α' varies across the models. Recall that the standard normal variable z_α = 1.645 yields 95% service level under continuous demand. But all the values of z_α' in Table 5 exceed 1.645. This implies that we need to keep more safety stock in the discrete demand case than in the continuous demand case. In the table, we can also see that z_α' is larger for shorter lead times and for lower demand variability. Note that larger z_α' does not necessarily mean higher safety stock, for the level of safety stock is determined mainly by the demand variability during the lead time.

Consider again the numerical example given in the previous section. Using the refined regression model 3 in Table 4, we get z_α' = 2.455. This value is closer to the simulated value z_α' = 2.456 than the one calculated using the model 0 (z_α' = 2.447) is.

Table 5. Adjusted Safety Factor z_α' (SL = 0.95)

		z_α'			
	demand variability	model 1	model 2	model 3	model 0
	CV range	0.15~0.3	0.25~0.4	0.35~0.5	0.15~0.35
CV = 0.2	L = 2	4.491	3.623	3.356	4.306
	3	4.119	3.407	3.159	4.008
	4	3.746	3.190	2.963	3.709
	5	3.374	2.973	2.766	3.411
CV = 0.3	L = 2	3.628	3.297	3.201	3.824
	3	3.256	3.080	3.004	3.526
	4	2.883	2.863	2.807	3.228
	5	2.511	2.647	2.611	2.929
CV = 0.4	L = 2	2.765	2.970	3.045	3.342
	3	2.393	2.754	2.848	3.044
	4	2.021	2.537	2.652	2.746
	5	1.648	2.320	2.455	2.447
CV = 0.5	L = 2	1.902	2.644	2.889	2.861
	3	1.530	2.427	2.693	2.562
	4	1.158	2.211	2.496	2.264
	5	0.786	1.994	2.300	1.965

5. Conclusions

In this paper, we presented a modified (Q, r) model for discrete demand. Our aim is to determine the revised reorder point for securing service level of (1-α). Our study is based on a simulation with various demand scenarios. Regression models are proposed for on-site use.

Our regression models are derived under limited experimental settings: lead times are confined up to 5 days, coefficient of variations between 0.15 and 0.5, and service levels between 0.8 and 0.98. For the cases where lead times are longer than 5 days, our models are not directly applicable. But z_α' obtained from our model can serve as an upper bound of true z_α'. For example, for the case where CV = 0.5, SL = 0.95, and L is longer than 5 days, then $z_\alpha' = 2.30$ is an upper bound of true z' as can be seen in Table 5. Note that $z_\alpha = 1.645$, the z value for the continuous demand can serve as a lower bound of z_α' in this case.

In the cases where CV is less than 0.15, our models are not directly applicable either. Simulation results give quite a large value of z_α' for lower CV cases. For example, for the cases where CV = 0.10, SL = 0.95, and L = 2, 3, 4, 5 days, corresponding z_α' values are 7.3, 6.2, 5.5, and 5.0, respectively. In other words, one needs to keep a far larger safety stock for discrete demand than for continuous demand if the demand variability is quite low. The reasoning is that since the demand variability is quite low, the reorder point is set to be quite small. A rare but quite large demand may occur even in this case, and backorder level will increase drastically. Then several consecutive inventory periods will experience stockout. This deteriorates service level.

Despite some limitations, the models we present in this paper can be used on-site when demand distribution is close to normal distribution. Further studies are anticipated for variable lead time, and for demand distributions other than the normal distribution.

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Appendix

Simulation Logic: Pseudo Code

Notations

T: day
P: experiment period
S: actual inventory
I: booked inventory
D: demand
B: Backorder
Q: Replenishment order quantity
R: Reorder point
L: Lead time

Pseudo Code

```
1. t = 0; Initial value
  A. S[t] = I[t] = 0
  B. B[t] = 0
2. for(t = 1; t < P; t++); repeat for the whole run period
  A. handling Backorder
    i. if (S[t] ≥ B[t-1]); S[t] -= B[t-1], B[t-1] = 0
    ii. else; S[t] = 0, B[t-1] -= S[t]
  B. When demand occurs
    i. adjusting actual inventory
      if (S[t] ≥ D[t]); S[t] -= D[t]
      else; S[t] = 0, B[t] = B[t-1]+D[t]-S[t]
    ii. adjusting booked inventory
      I[t] -= D[t]
  C. Inventory update
    i. actual inventory
      if (t > L); S[t] += Q[t-L]
    ii. booked inventory
```

```
        if (t > L); I[t] += Q[t-L]
    D. Placing an order
       if (S[t] ≤ R); Q[t] = Q
       else; Q[t] = 0
3. for(t = 1; t < P; t++)
    A. Counting the number of inventory cycle
       if (I[t-1] < I[t]); C++
    B. Counting the number of shortage cycles
       if (I[t-1] < 0); N ++
    C. Calculating the Service Level
    D. End
```