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Mathematical Problem Solving for Everyone: A Design Experiment[1]

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An impetus for reviving research in mathematical problem solving is the recent advance in methodological thinking, namely, the design experiment ([Gorard, S. (2004). *Combining methods in educational research.* Maidenhead, England: Open University Press.]; [Schoenfeld, A. H. (2009). Bridging the cultures of educational research and design. *Educational Designer.* **1(2)**. http://www.educationaldesigner.org/ed/volume1/issue2/]). This methodological approach supports a "re-design" of contextual elements to fulfil the overarching objective of making mathematical problem solving available to all students of mathematics. In problem solving, components critical to successful design in one setting that may be adapted to suit another setting include curriculum design, assessment strategy, teacher capacity, and instructional resources. In this paper, we describe the im-

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plementation, over three years, of a problem solving module into the main mathematics curriculum of an Integrated Programme school in Singapore which had sufficient autonomy to tailor-fit curriculum to their students.

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1. INTRODUCTION

It has been more than 50 years since the publication of the first edition of Polya's *How to solve it* (Polya, 1945). Polya's goal in writing *How to solve it* was to help students of mathematics discover or invent the solution of a mathematics problem. Continuing in pursuit of this goal, Polya wrote *Mathematics and plausible reasoning* (Polya, 1954) and *Mathematics discovery* (Polya, 1961). Polya's effort caught the attention of the mathematics education community in the 1980s in the USA, when the National Council of Teachers of Mathematics (NCTM, 1980) made problem solving the focus of its Agenda for Action. Mathematical problem solving was the zeitgeist of mathematics education in the 1980s and early 1990s. The era of mathematical problem solving, its research and teaching and learning in schools ended, ambivalent on research findings and imprecise on recommendations for its teaching in schools (Schoenfeld, 1992; Lester 1994).

This paper reports an attempt by the authors to revitalize research on teaching and learning problem solving in Singapore schools. Several events juxtapose to give us the impetus to consider the re-introduction of problem solving in the mathematics curriculum (for details, see Toh, Quek, Leong, Dindyal & Tay, in press; Tay, Quek, Toh, Dong & Ho, 2007). Of these, two events were significant: the greater autonomy given to selected local schools in determining their curriculum and assessment, and the growing acceptance of the idea of applying engineering design research in education (Gorard, 2004; Schoenfeld, 2007). For these selected Singapore schools, or "Integrated Programmes" schools as they were called, one notable change was the uncoupling of the end-of-secondary-education Singapore-Cambridge GCE 'O' Level examination. The pressure to do well in this high-stakes national examination for schools has been a barrier to successful implementation of problem solving. Given the relative flexibility in the design of the curriculum and assessment system of the Integrated Programme schools, the authors worked with one such school to put in place a curriculum package for problem solving and build the teachers capacity to deliver the curriculum.

The methodology of design experiment appealed to us in that it allows for the unique demands and constraints of the schools to be met, and, at the same time, the research imposes rigour on to the design. The methodology's advocacy of an implement-research-refine iterative approach to educational design appeared to us to hold potential in dealing with the complexity of school-based innovations. A design experiment can be described as the

> "creation of an instructional intervention [in our context, a problem solving module] on the basis of a local theory regarding the development of particular understandings [Polya's model of problem solving (1954) and Schoenfeld's framework for problem solving (1985)]" (Schoenfeld, 2009)

In this paper we will discuss our design experiment.

2. A DESIGN EXPERIMENT FOR PROBLEM SOLVING

Design experiment is built on a "design-theory dualism" (Schoenfeld, 2009). In our case, it begins with a theory regarding mathematics education. Based on this theory, a design is conceptualised to improve learning. The initial design is then implemented in a suitable 'real-world' context, such as the school. Research methods are then used to examine both the "accuracy of the underlying ... theory and the power of the intervention." Based on the findings of the research, refinement is made to the design which is further implemented and the implementation-research-refinement cycle is iterated until the design-experimenter is satisfied.

> "The coherence of theory to methodology ... is of fundamental importance in the evaluation of design experiments, and is critical to explicate for any future scholar or practitioner who attempts to replicate or implement the findings of a design study." (Middleton, Gorard, Taylor & Bannan-Ritland, 2006)

We want to design a curriculum package of lessons and materials (such as a set of rich mathematical problems and guidelines for teaching problem solving) which, when a school sees fit, can be contextualized and implemented in the school's mathematics curriculum for the learning of mathematical problem solving. We based our design experiment on the methodology and terminology of Middleton, Gorard, Taylor and Bannan-Ritland (2006).

The envisaged outcome of the design experiment was to produce a workable "design" (an initiative, artefact or intervention, for instance) that can be adapted to other settings. Gorard (2004) points out that the emphasis in design experiments should be a general solution that can be 'transported' to different contexts.

Middleton, Gorard, Taylor and Bannan-Ritland (2006) used the following format to summarise the theoretical justification for a design experiment: theoretical propositions are numbered as 1, 2, 3, and so on; justifications follow after each proposition and are listed as a, b, c, and so on. We use a similar format for the theoretical justification of our design experiment on teaching and learning problem solving as follows:

1. Instruction on problem solving consists mostly of the teaching of heuristics.

   a. We know this because Silver et al. (p. 288, 2005) stated that instructional interventions intended to develop in students an inclination to "look back" at their solution to a problem in order to generate alternate solutions have been largely unsuccessful.

   b. In addition, we know that at the Singapore primary school level, there is teaching of heuristics and some problem posing (see for example, Fong, 1996; Yeap and Kaur, 1998; Fan and Zhu, 2000; Ho, Lee and Yeap, 2001) but not any teaching that covers all four stages within Polya's model.

2. The theoretical basis of Polya and Schoenfeld remain sound.

   a. Some mathematics education researchers have begun to sound the death knell for the teaching and experience of problem solving as envisaged by the traditional giants Polya and Schoenfeld. English, Lesh and Fennewald (2008) recently wrote that problem solving ala Schoenfeld was a failure and proposed instead that one should teach thinking in 'real-life' situations using a modelling approach. Other more generic models of thinking such as Understanding by Design (UbD) and Teaching for Understanding (TfU) have been proposed. These designs are relatively new and have yet to establish themselves. For example, when Hammerness, Jaramillo, Unger and Wilson (1997) reported their analysis of 38 students' understanding under the TfU programme in four different subjects (History, Physics, English and Mathematics), the mathematics class performed the worst of the four classes. In our opinion, the teaching of traditional problem solving has been successful under certain circumstances such as in Schoenfeld's undergraduate classes (Schoenfeld, 1985). The processes are sound because these are the same processes professional mathematicians use. The methods of teaching are generally not complicated. In addition, a course on problem solving is compact in terms of time and does not involve immense unsettling school changes.

3. Mathematical problem solving must include the Looking Back stage of Polya's model.

   a. The mathematicians in our project group and faculty confirm that this is what they do when they solve mathematics problems; to understand better the structure of the solution and to see where else one can go from the original problem in terms of generalisations of the problem and applicability of the solution to other problems.

b. Mathematics teachers always complain that students do not check their work and that students often do not know how to solve a slightly modified variant of the problem they had just solved.

4. Mathematics problem solving is valuable enough to be adequately assessed and must be adequately assessed to be valued.

   a. The importance of problem solving in the school mathematics curriculum is hardly a matter of debate, although, there exist subtle differences in the meaning attached to problem solving. The important role of problem solving was encapsulated in NCTM's 1989 standards (NTCM, 1989).
   b. Effective assessment practice "begins with and enacts a vision of the kinds of learning we most value for students and strive to help them achieve" (Walvoord & Anderson, 1998). Correspondingly it is common knowledge that most students will not study for curricular components which are not assessed.
   c. What we see as the root of the lack of success for previous attempts to implement problem solving in the classroom is that problem solving is not assessed. Because it is not assessed, students and teachers do not place much emphasis on the processes of problem solving; Students are more interested to learn the other components of the curriculum which would be assessed.

5. Mathematics problem solving is for every student of mathematics.

   a. To partition mathematics problem solving to a form of enrichment or optional programme for students violates the value of mathematical problem solving.
   b. As in science education where there are laboratory or practical lessons to ostensibly learn the processes of science, learning the processes of mathematics is as integral to learning mathematics as learning the mathematics content.
   c. The researchers impressed the collaborating school with the vision that "all students graduating from the school will, at the least, know how to begin to attack an unfamiliar problem, check their work, and realise the importance of extending a problem".

We begin working on the curriculum package by referencing the classical view of design where the parameters of the product are to be specified a priori (Ullman, 1992). As an analogy, in designing the newest Ferrari sports car model, the parameters are stated as follows: Acceleration (0–100km/h in 3 seconds), Cost of production (not exceeding US$100,000 per car), Aesthetics (wedge-shape with wow factor), etc. Guided and constrained by these parameters, one then designs the car with features to satisfy the parameters. One may decide on a V6 engine against a V8 engine because though a V8 engine is more powerful than a V6, the V6 fulfills both the parameters of Acceleration and Cost while the V8 fails to meet the Cost parameter.

We thus state the parameters for the design of a package for teaching problem solving. The parameters are underpinned by the theoretical justification, discussed earlier, for the design experiment.

1. Place in the curriculum: It must be part of the mainstream mathematics curriculum—Mathematical Problem Solving for Everyone (M-ProSE).
2. Model of mathematical problem solving:
   i. Polya's model – all four stages
   ii. Shoenfeld's framework – teach Heuristics and emphasise Control
3. Teacher autonomy: Teachers in school will ultimately teach the module themselves. Build teachers' capacity in problem solving and to teach it.
4. Infusion into regular mathematics content: Problem solving skills and habits learnt in the module must be infused into other mathematics modules to prevent atrophy
5. Assessment: A valued component in school assessment

The following features are then built into the first prototype of the package to satisfy the demands of the stated parameters.

From our experience with a pilot project at an Integrated Programme school (see Tay, *et al.*, 2007), we realised that we must first look for a way out of the perennial quandary of the undervaluation of assessment of problem solving. In that pilot study, faced with some lack of motivation for students to take the problem solving lessons seriously, we decided to construct a worksheet similar to those used in science practical lessons and told the students to treat the problem solving class as a mathematics 'practical' lesson. In this way, we hoped to achieve a paradigm shift in the way students looked at these 'difficult, unrelated' problems which had to be done in this 'special' problem solving class. Thus our package promoted mathematical problem solving as learning the processes of mathematics akin to the established science practical as a way to learn science processes. The mathematics 'practical' was born and the 'worksheet' assessment became its conjoined twin. Parameters 1 and 5 were addressed.

A series of lessons to teach students how to use Polya's model as a scaffold, to teach various heuristics and choice of heuristics, and to make students aware of the need for control in problem solving was developed. Parameter 2 was addressed.

To us, it appeared necessary that the teachers must make the proposed instructional approach a routine sufficiently familiar to them so that the approach becomes classroom practice. To reach this stage, it seemed essential for the teachers to adapt the researchers' ideas to make them their own, in the sense that their beliefs of mathematics and of problem solving in mathematics are transformed. Such a process would pass through a stage where the teachers negotiate and change the problem solving lesson. Finally, a community of practice would develop among the teachers to support the change process by providing opportunities to learn to engage the proposed ways – thinking, talking and reflecting on the new teaching experiences and ways of doing mathematics (Shulman & Shulman, 2004). The entire process of transforming an externally proposed instructional approach and curricular change into classroom and school practice appeared to be cyclical, incremental and emergent in nature. This was our approach to Parameter 3.

To address Parameter 4, problems that were from the regular mathematics curriculum and were rich enough for extended work would be crafted and infused into the regular schedule. The justification was the reference to science practicals going in tandem with science theory lessons, and the motivation was that these difficult problems could be used as assessment *for* learning.

The M-ProSE design research would entail conducting a systematic program of research on the learning that results from the classroom (or teacher, or school) interventions which are aimed at deep understanding of how student outcomes are related to contextual factors, and not just at uncovering and examining the relationships among the factors without deep explanation.

3. IMPLEMENTATION: REFINEMENT AND ACCOMMODATION

In this section, we describe the implementation of the problem solving module and the changes we made to it as we proceeded along with the design experiment. Refinements are made to the design based on local research, but modifications were also made to accommodate the constraints of the school and the teachers.

As this paper focuses on the zoomed-out macro elements of the overarching design experiment, the detailed substantiations of component parts of the design will not be provided here. The reader may refer to M-ProSE related publications in the reference under the same set of authors. As such, for the rest of this section, only the broad aspects of the design will be described.

At the invitation of the Principal and Head of Mathematics of the school, the researchers met with them to outline the design of the problem solving module and to seek their views on the design. The outcome was to first proceed with building the teachers' capacity in problem solving and to introduce them to the design. This was to be followed by a pilot study with a class of Year 9 students who would take the problem solving module as an elective module. The pilot class would be taught by a designer-researcher and observed by interested teachers. One of the designer-researcher would lead the lesson study sessions with these teachers to address concerns. The pilot effort would be evaluated and the school would decide on extending the problem solving module to an entire cohort of

Year 8 students.

The initial teacher training sessions did not go as expected. Firstly, there was difficulty in getting many of the teachers to buy in to the design. The training involved teachers solving non-routine problems which some of the teachers disliked or felt threatened by the mathematical demands. Other teachers were concerned about the feasibility of teaching to problem solving, saying that the students would reject such an instructional approach. Second, we found out through participant observation (our entire design team was on site) that teachers were not satisfied if problems posed in a session was not completely solved in the same session. We had planned the lessons for the teachers to be similar to those they would use with the students. The first design involved foregrounding the Polya stages. For example, in the lesson on "Understanding the Problem", a few problems were posed and the teachers were tasked to focus on understanding the problems and not to proceed to solving them. The teachers were uncomfortable with this approach and our subsequent refining of the design had us foregrounding a problem for the day with the Polya stages in the background. As such, for the subsequent designs, usually only one problem was discussed and solved in a lesson.

After the initial teacher professional development workshops, only three of the teachers sat in on the first series of lessons taught by one of the designer-researchers to the Year 8 students. One of them was the head of department. In evaluating our design experiment efforts, we realised that the Head's presence was an endorsement of sorts of the design, which was crucial to the successful implementation of the refined design into the main curriculum in the new academic year. (Remark: We say that an endorsement by school leaders may be a critical success factor because, when we tried to implement the design with another Integrated Programme (at its invitation too), we got no further than the teachers' capacity building workshops without the head of department's support). The two teachers went on to teach the first cohort of Year 8 students and they were generally positive to the design. Another teacher who did not attend the first series of lessons but were assigned to teach the module according to its design also reported positive experiences in a separate interview. Without further modification to the design, the school put a second cohort of Year 8s through the problem solving module the following year. Three other teachers conducted the second run of the module. To our surprise, one of them, who was sceptical about the approach during the first teacher workshops, was very positive about the lessons and personally, she remarked that Polya's model done on the practical worksheet was effective in her own mathematics studies at the Masters level. The problem solving module is now in its third run. For successful buy-in of the design, it is important that the "consumers" of the design be persuaded to try-out the design.

We turn now to talk about a modification of the design to meet the school's needs. The initial design of the problem solving module offered to the Year 8 cohort would consist of a series of ten 1-hour lessons to be conducted weekly. The intention was to allow students to work on at least two problems per lesson—one in the class to learn problem solving and one for homework in recognition that the students might have to persevere to solve a problem. When the school adopted the module, time constraints resulted in a reduction to a series of eight 1-hour lessons compressed within three weeks. The teachers in the school took the curriculum materials that we provided in the form of lesson plans, guidelines for conducting the lessons, and problems with solutions, and adapted them to fit into the reduced time-frame. The reduction of lesson time and compression of duration of implementation notwithstanding, the teachers were able to use all the problems in the set provided by us. The design was sufficiently robust to withstand small deviations to its implementation.

The school adopted two innovative features of the design, the practical worksheet and assessment rubric, without modification. For the students (and for the teachers as well during the training workshops), problem solving was done on the practical worksheet and was assessed by a set of rubrics. Formative and summative assessments were conducted using the mathematics practical lessons. The school fully adopted the practical worksheet with its assessment rubric. They also had a summative assessment using the practical worksheet. The teachers were able to use the scoring rubrics with high inter-rater reliability estimates. Here, the raters' discrepancies were the results of differing opinions of what constitutes a mathematical proof or an extension to a problem in the Looking Back stage of Polya's model, and this was easily resolved when the raters met to discuss.

Finally, although the problem solving module has been implemented by the school for two complete cohorts of Year 8 students, one component of the design, the infusion lessons, have not been implemented at all. Discussions on this component between researchers and school are underway.

4. CONCLUSION

Schoenfeld (2007, p. 539) insists that the current focus on problem solving research should lie in translating decades of theory building about problem solving into workable practices in the classrooms:

> That body of research—for details and summary, see Lester (1994) and Schoenfeld (1985; 1992)—was robust and has stood the test of time. It represented significant progress on issues of problem solving, but it also left some very important issues unresolved. ... The theory had been worked out; all that needed to be done was the (hard and unglamorous) work of following through in practical terms.

We have described in this paper the refinements to design from testing it out in practice and the modifications to it in order to meet the demands of the school. Results from our design experiment show that students in general have learnt about problem solving and have achieved success in going through Polya's model, including the fourth stage. Teachers also report on the usability of the design to teach problem solving. The fact that the school has put the module into its mainstream, now into its third year, indicates that the design is valued and is seen to meet their needs.

Schoenfeld's (2009) model of design experiment involved only the interaction between the designer and the researcher in a design-theory dualism. In our work, we are convinced that the teachers and school that have to implement the design have a key role in the final design itself. Black (2009) advocates this model of design experiment which includes the designer, the researcher and experienced teachers. We propose that, in effect, a design-theory-practice troika should always be considered for a designed package to be acceptable to the final users, which are the teachers and the schools.

In addition, we notice that there are distinct differences in the type of changes we made along the way in the design process. One type of change relates to what is already widely discussed in the literature on design for the purposes of developing the theory and the product. However, there is another type of change we made which is to meet the realistic constraints faced by teachers. We think it is important in discussion about design research to distinguish the two. As a proposal to advance the language of discourse on these matters, we suggest the former to be termed refinement and the latter accommodation.

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