ON THE LEBESGUE SPACE OF VECTOR MEASURES

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Abstract. In this paper we study the Banach space $L^1(G)$ of real valued measurable functions which are integrable with respect to a vector measure $G$ in the sense of D. R. Lewis. First, we investigate conditions for a scalarly integrable function $f$ which guarantee $f \in L^1(G)$. Next, we give a sufficient condition for a sequence to converge in $L^1(G)$. Moreover, for two vector measures $F$ and $G$ with values in the same Banach space, when $F$ can be written as the integral of a function $f \in L^1(G)$, we show that certain properties of $G$ are inherited to $F$; for instance, relative compactness or convexity of the range of vector measure. Finally, we give some examples of $L^1(G)$ related to the approximation property.

1. Introduction

Integration of real valued measurable functions with respect to Banach space valued countably additive vector measures was introduced and studied by Lewis in [7] and [8]. The Banach space $L^1(G)$ of real valued measurable functions integrable with respect to a Banach space valued countably additive vector measure $G$ was studied from the aspect of Banach lattice by Curbera in [1], [2] and [3]. In particular, Curbera characterized $L^1(G)$ as order continuous Banach lattices with weak order unit.

In this paper we focus on different aspects of $L^1(G)$. Let $X$ be a Banach space and $G$ be a $X$-valued countably additive vector measure. After fixing notation and definitions in Preliminaries, we study in Section 3 conditions for a scalarly integrable function $f$ which guarantee $f \in L^1(G)$.

In Section 4 we consider the convergence in $L^1(G)$ and for an integrable function $f$ the induced vector measure $F$ given by the integration

$$F(E) = \int_E f dG.$$ 

We give a sufficient condition for a sequence which guarantees the convergence in $L^1(G)$. Then this result improves Lewis’s results. And, as a consequence
of the Orlicz-Pettis theorem, the induced vector measure $F$ turns out to be a countably additive vector measure. Now, assuming that $F$ is induced in the above fashion, we show that if $G$ has a relatively compact range, then so does $F$. In the same vein we prove that if $G$ satisfies the Lyapunov convexity theorem, then so does $F$. In the final section we consider the approximation property for $L^1(G)$. We give some examples; we illustrate these examples using Szakowski’s counterexample of a Banach lattice which lacks in the compact approximation property.

## 2. Preliminaries

Throughout this paper by $X$ and $Y$ we denote real Banach spaces. By $B_X$ we mean the closed unit ball of $X$ and $X^*$ is the dual of $X$. For a measurable space $(\Omega, \Sigma)$ and a countably additive vector measure $G : \Sigma \to X$ we define its semivariation $\|G\|$ by

$$\|G\|(E) = \sup_{x^* \in B_{X^*}} |x^*G|(E).$$

Here $|x^*G|$ is the variation of the signed measure $x^*G$. It is well-known that $\|G\|(\Omega) < \infty$, $\|G\|$ is monotone and subadditive; refer to [5]. Let’s denote by $ca(\Sigma)$ the set of signed measures $\lambda : \Sigma \to \mathbb{R}$. Nikodym’s convergence theorem states that if $(\mu_n)$ is a sequence from $ca(\Sigma)$ for which

$$\lim_{n \to \infty} \mu_n(E) = \mu(E)$$

exists for each $E \in \Sigma$, then $\mu \in ca(\Sigma)$; refer to [4]. Recall that $ca(\Sigma)$ is a Banach space if we define $\|\lambda\| = |\lambda|(\Omega)$. By $\lambda \in ca^+(\Sigma)$, we mean that $\lambda : \Sigma \to [0, \infty)$ is a finite countably additive measure. $G : \Sigma \to X$ is a weakly countably additive vector measure if for each $x^* \in X^*$, $x^*G$ is a signed measure. Orlicz-Pettis theorem states that a weakly countably additive vector measure on $\Sigma$ is countably additive; refer to [5, Corollary I.4.4]. If $G : \Sigma \to X$ is a countably additive vector measure, then $\{G(E) : E \in \Sigma\}$ is relatively weakly compact; refer to [5, Corollary I.3.7].

For two vector measures $F : \Sigma \to X$ and $G : \Sigma \to Y$ we say that $F$ is absolutely continuous with respect to $G$, in symbol $F \ll G$, if given $\varepsilon > 0$ there is $\delta > 0$ such that $\|F(E)\| < \varepsilon$ whenever $E \in \Sigma$ and $\|G\|(E) < \delta$. If $F : \Sigma \to X$ is a countably additive vector measure and $\lambda \in ca(\Sigma)^+$, then it is known that $F \ll \lambda$ if and only if $F(E) = 0$ whenever $\lambda(E) = 0$. Vitali-Hahn-Saks theorem states that if $\mu$ is a finitely additive nonnegative real-valued measure on $\Sigma$ and $(F_n)$ is a sequence of $X$-valued $\mu$-continuous vector measures on $\Sigma$ such that

$$\lim_{n \to \infty} F_n(E)$$

exists for each $E \in \Sigma$, then

$$\lim_{\mu(E) \to 0} F_n(E) = 0$$

uniformly in $n$; refer to [5]. A Rybakov control measure for $G$ is a measure of the form $|x^*G|$ such that $G \ll |x^*G|$. If $G : \Sigma \to X$ is a countably additive vector measure, then, according to the famous theorem of Rybakov, $G$ has a
Rybakov control measure. Moreover, if \(|x^*_G| \) is a Rybakov control measure for \(G \) and \(x^* \in X^* \), then \(|\alpha x^* + (1 - \alpha)x^*_G| \) are Rybakov control measures for \(G \) except for countably many \(\alpha \in \mathbb{R} \); refer to [13].

Following D. R. Lewis [7] we define the Lebesgue space \(L^1(G) \) by the set of measurable functions \(f : \Omega \to \mathbb{R} \) such that

(i) \(\int_{\Omega} |f| d|x^*G| < \infty \) for each \(x^* \in X^* \) (scalarly integrable); and

(ii) for each \(E \in \Sigma \) there is a vector in \(X \), denoted by \(\int_E f dG \), satisfying

\[
\int_E f dG = \int_E f \cdot x^* dG
\]

for all \(x^* \in X^* \). We regard two functions \(f \) and \(g \) in \(L^1(G) \) equal if there is \(E \in \Sigma \) such that \(|G|(E) = 0 \) and \(f = g \) on \(E^c \). We endow each \(f \in L^1(G) \) with its norm

\[
\|f\|_{L^1(G)} = \sup_{|x^*| \leq 1} \int_{\Omega} |f| d|x^*G|.
\]

It is well known that \(\|f\|_{L^1(G)} < \infty \) for all \(f \in L^1(G) \). For a quicker way of checking this observe that \(\|f\|_{L^1(G)} \leq 2\|f\| \) where

\[
\|f\| = \sup_{E \in \Sigma} \left\| \int_E f dG \right\|
\]

And by the Orlicz-Pettis theorem the vector measure \(E \to \int_E f dG \) is countably additive, hence it is bounded.

In [1] Curbera shows that \(L^1(G) \) is an order continuous Banach lattice with weak order unit over \((\Omega, \Sigma, \mu) \) for any Rybakov control measure \(\mu \) for \(G \). In particular, from the order continuity it follows that the simple functions are dense in \(L^1(G) \).

### 3. Scalarly integrable functions

In Preliminaries, we observed that \(\|f\|_{L^1(G)} < \infty \) if \(f \in L^1(G) \). Stefansson proved that \(\|f\|_{L^1(G)} < \infty \) if \(f \) is scalarly integrable only; refer to [14]. We repro the same result independently. We start this section by showing the following definition.

**Definition 3.1.** Let \(f : \Omega \to \mathbb{R} \) be measurable with respect to \(\Sigma \). We define a space \(\mathcal{M}(f) \) of set functions as follows. For each \(k \in \mathbb{N} \) write \(\Omega_k = \{ \omega \in \Omega : |f(\omega)| > \frac{1}{k} \} \) and put \(\Sigma_k = \{ E \in \Sigma : \Omega_k \cap E \} \). We let \(\mathcal{M}(f) \) be the set of all set functions \(\lambda : \bigcup_{k=1}^\infty \Sigma_k \to \mathbb{R} \) such that \(\lambda \downarrow \Sigma_k \in ca(\Sigma_k) \) for all \(k \in \mathbb{N} \) and

\[
\sup_{k \geq 1} \int_{\Omega_k} |f| d|\lambda| < \infty.
\]

Observe that in the above integral we adapted the convention of writing \(\lambda \) in place of \(\lambda \downarrow \Sigma_k \). We endow each \(\lambda \in \mathcal{M}(f) \) with its norm

\[
\|\lambda\| = \sup_{k \geq 1} \int_{\Omega_k} |f| d|\lambda|.
\]
It is easy to check that \( \| \cdot \| \) is a norm.

On our way to the proof of completeness of \( \mathcal{M}(f) \) we will need the following type of Fatou’s lemma.

**Lemma 3.2.** If \( \mu, \mu_n \in \text{ca}^+(\Sigma) \) for \( n = 1, 2, \ldots \) and \( \mu(E) = \lim_{n \to \infty} \mu_n(E) \) for each \( E \in \Sigma \), then for each measurable \( g : \Omega \to [0, \infty) \), we have \( \int_{\Omega} g \, d\mu \leq \lim \inf_{n \to \infty} \int_{\Omega} g \, d\mu_n. \)

**Proof.** We observe that for each simple \( \varphi \), \( \int_{\Omega} \varphi \, d\mu = \lim_{n \to \infty} \int_{\Omega} \varphi \, d\mu_n. \)

We recall that \( \int_{\Omega} g \, d\mu = \sup_{0 \leq \varphi \leq g} \int_{\Omega} \varphi \, d\mu \) where \( \varphi \)'s are simple measurable functions. Since \( \mu_n \) is a nonnegative measure, for each \( 0 \leq \varphi \leq g \) and \( n \in \mathbb{N} \), we have

\[
\int_{\Omega} \varphi \, d\mu_n \leq \int_{\Omega} g \, d\mu_n.
\]

Hence \( \int_{\Omega} \varphi \, d\mu \leq \lim \inf_{n \to \infty} \int_{\Omega} g \, d\mu_n \) and \( \int_{\Omega} g \, d\mu \leq \lim \inf_{n \to \infty} \int_{\Omega} g \, d\mu_n. \) \( \square \)

**Theorem 3.3.** The space \( \mathcal{M}(f) \) is a Banach space.

**Proof.** Let \( (\lambda_n) \) be a Cauchy sequence in \( \mathcal{M}(f) \). Then given \( \varepsilon > 0 \), there is \( n_0 \in \mathbb{N} \) such that if \( n, m \geq n_0 \), then \( \sup_{k} \int_{\Omega_k} |f| |\lambda_n - \lambda_m| < \varepsilon. \) For each \( k \)

\[
\int_{\Omega_k} |f| |\lambda_n - \lambda_m| \geq \frac{1}{k} |\lambda_n - \lambda_m| (\Omega_k),
\]

hence \( (\lambda_n \mid \Sigma_k) \) is a Cauchy sequence in \( \text{ca}(\Sigma_k) \). Since \( \text{ca}(\Sigma_k) \) is a Banach space, we obtain \( \lambda^k \) in \( \text{ca}(\Sigma_k) \) such that \( \lambda_n \mid \Sigma_k \) converges to \( \lambda^k \) in \( \text{ca}(\Sigma_k) \), hence \( \lambda^k(E) = \lim_{n \to \infty} \lambda_n(E) \) for all \( E \in \Sigma_k \). Then we can define \( \lambda : \bigcup_{k=1}^{\infty} \Sigma_k \to \mathbb{R} \) by putting \( \lambda = \lambda^k \) on \( \Sigma_k \). Since any Cauchy sequence is bounded, there is \( M > 0 \) such that for all \( n, \) \( \sup_{k} \int_{\Omega_k} |f| |\lambda_n| \leq M. \) Now we check that \( \lambda_n \to \lambda \) in \( \mathcal{M}(f) \). Since for each \( k \) and a fixed \( n \geq n_0 \), \( \lim_{n \to \infty} |\lambda_n - \lambda_m|(E) = |\lambda_n - \lambda(E)| \) for all \( E \in \Sigma_k \), we have, by virtue of Lemma 3.2, that \( \int_{\Omega_k} |f| |\lambda_n - \lambda| \leq \lim \inf_{n \to \infty} \int_{\Omega_k} |f| |\lambda_n - \lambda_m| \leq \varepsilon \) for all \( k \). Hence if \( n \geq n_0 \), then \( \sup_{k} \int_{\Omega_k} |f| |\lambda_n - \lambda| \leq \varepsilon. \) So, \( \sup_{k} \int_{\Omega_k} |f| |\lambda| \leq \inf_{\Omega_k} |f| |\lambda - \lambda_{n_0}| + \int_{\Omega_k} |f| |\lambda_{n_0}| < \varepsilon + M. \) That is, \( \lambda \in \mathcal{M}(f) \) and \( \lambda_n \to \lambda \) in \( \mathcal{M}(f). \) \( \square \)

**Corollary 3.4.** If \( \int_{\Omega} |f| |x^*G| < \infty \) for all \( x^* \in X^* \), then

\[
\sup_{\|x^*\| \leq 1} \int_{\Omega} |f| |x^*G| < \infty.
\]

**Proof.** By the assumption, for all \( x^* \in X^* \), we have

\[
\sup_{k \geq 1} \int_{\Omega_k} |f| |x^*G| = \int_{\Omega} |f| |x^*G| < \infty,
\]

hence \( x^*G \in \mathcal{M}(f) \). So we can define an operator \( T : X^* \to \mathcal{M}(f) \) by \( T(x^*) = x^*G \). Clearly \( T \) is a linear operator. It is enough to show that \( T \) is bounded.
Suppose \( x_n^* \to x^* \) in \( X^* \) and \( Tx_n^* \to \lambda \) in \( M(f) \). Then \( \sup_k \int_{\Omega_k} |f||x_n^*G - \lambda| \to 0 \). So for each \( k \), \( |x_n^*G - \lambda|(\Omega_k) \to 0 \) and
\[
\lambda(E) = \lim_{n \to \infty} x_n^*G(E)
\]
for all \( E \in \Sigma_k \). On the other hand \( x^*G(E) = \lim_{n \to \infty} x_n^*G(E) \) for all \( E \in \Sigma_k \). Hence we obtain \( \lambda = x^*G \) on \( \Sigma_k \) for each \( k \). So \( \lambda = x^*G \) on \( \bigcup_k \Sigma_k \). By the Closed Graph Theorem \( T \) is bounded. \( \square \)

The following corollary is just Proposition 2 in [14]. We provide our proof.

**Corollary 3.5.** Let \( \mu \in \text{ca}^+(\Sigma) \) such that \( G \ll \mu \). Suppose that \( \int_{\Omega} |f||x^*G| < \infty \) for all \( x^* \in X^* \). Define \( S : X^* \to L^1(\mu) \) by \( Sx^* = f \cdot \frac{dx^*G}{d\mu} \). Then \( S \) is a bounded linear operator.

**Proof.** Since \( G \ll \mu \), for each \( x^* \in X^* \), we have that \( x^*G \ll \mu \), hence the Radon-Nikodym derivative \( \frac{dx^*G}{d\mu} \in L^1(\mu) \) and \( \frac{dx^*G}{d\mu} = |\frac{dx^*G}{d\mu}| \). Thus
\[
\int_{\Omega} |Sx^*|d\mu = \int_{\Omega} |f||\frac{dx^*G}{d\mu}|d\mu = \int_{\Omega} |f||\frac{dx^*G}{d\mu}|d\mu
\]
\[
= \int_{\Omega} |f||x^*G| < \infty,
\]
which shows that \( Sx^* \in L^1(\mu) \). In view of Corollary 3.4 we obtain that
\[
\sup_{\|x^*\| \leq 1} \int_{\Omega} |Sx^*|d\mu = \sup_{\|x^*\| \leq 1} \int_{\Omega} |f||x^*G| < \infty.
\]
This proves our corollary. \( \square \)

**Theorem 3.6.** Suppose that \( \int_{\Omega} |f||x^*G| < \infty \) for all \( x^* \in X^* \). Let \( \mu \in \text{ca}^+(\Sigma) \) such that \( G \ll \mu \). Let \( S : X^* \to L^1(\mu) \) be the operator given by \( Sx^* = f \cdot \frac{dx^*G}{d\mu} \) as in Corollary 3.5. Then the following are equivalent:

(a) \( f \in L^1(G) \).

(b) \( S \) is weak*-weak continuous.

(c) There is \( g \in L^1(G) \) such that \( \int_{\Omega} |f||x^*G| \leq \int_{\Omega} |g||x^*G| \) for each \( x^* \in X^* \).

**Proof.** (a) \( \iff \) (b) It is a known result proved by Stefansson in [14, Theorem 4].

Clearly (a) implies (c) with \( g = f \). It remain to check that (c) implies (a).

Fix \( E \in \Sigma \) and consider \( \int_{E} \frac{fdG}{x^*G} : X^* \to \mathbb{R} \) given by \( (\int_{E} \frac{fdG})x^* = \int_{E} \frac{fdx^*G}{x^*G} \). Then we have \( \int_{E} \frac{fdG} \in X^{**} \). We claim that \( \int_{E} \frac{fdG} \in 2QW \) in \( X^{**} \), where \( W = \mathbb{R}\{\pm \int_{A} gdG : A \in \Sigma \} \) and \( Q : X \to X^{**} \) is the canonical embedding. Since \( A \to \int_{A} gdG \) is countably additive, we have that \( W \) is weakly compact. Hence \( QW \) is weak* compact in \( X^{**} \). If \( \int_{E} \frac{fdG} \notin 2QW \), then by the separation theorem we have an \( x^* \in X^* \) and \( \alpha \in \mathbb{R} \) such that \( 2x^*(x) \leq \alpha < \int_{E} \frac{fdG}x^* \) for all \( x \in W \). Then one has \( A \in \Sigma \) such that
\[
\int_{\Omega} |g||x^*G| = x^* \left( \int_{A} gdG \right) + x^* \left( -\int_{\Omega \setminus A} gdG \right).
\]
\[ \leq \alpha < x^* \left( \int_E f \, dG \right) \leq \int_{\Omega} |f| d|x^*G|; \]
a contradiction. \( \square \)

4. The convergence in \( L^1(G) \) and vector measures induced by integration

Lewis proved the following theorem in his classical paper; refer to [7].

**Theorem 4.1.** Let \((\varphi_n)\) be a sequence in \( L^1(G) \) which converges pointwise to \( f \) on \( \Omega \) and \( g \) be in \( L^1(G) \) such that \(|\varphi_n| \leq |g| \) for each \( n \). Then we have

\[ \int_E f \, dG = \lim_{n \to \infty} \int_E \varphi_n \, dG \]

uniformly for all \( E \in \Sigma \).

By the above theorem, we observe that \((\varphi_n)\) converges to \( f \) in \( L^1(G) \) under the hypotheses of Theorem 4.1. That is, Theorem 4.1 gives a sufficient condition which guarantees convergence in \( L^1(G) \). The following theorem gives a new sufficient condition which guarantees convergence in \( L^1(G) \).

**Theorem 4.2.** Suppose that there exists a sequence \((\varphi_n)\) in \( L^1(G) \) for which \((\int_E \varphi_n \, dG)\) is Cauchy for each \( E \in \Sigma \) and \((\varphi_n)\) is Cauchy in measure with respect to a Rybakov control measure \(|x_0^*G|\). Then there exists \( f \in L^1(G) \) such that \((\varphi_n)\) converges to \( f \) in \( L^1(G) \).

**Proof.** First we have that the set function \( F : \Sigma \to X \) given by

\[ F(E) = \lim_{n \to \infty} \int_E \varphi_n \, dG \]
defines a countably additive vector measure. Indeed, by the Nikodym convergence theorem, \( F \) is a weakly countably additive vector measure. By the Orlicz-Pettis theorem \( F \) is a countably additive vector measure. Since \( F \ll G \ll |x_0^*G| \), by the Radon-Nikodym theorem there exists a function \( f_0 \in L^1(|x_0^*G|) \) such that \( x_0^*F(E) = \int_E f_0 \, d|x_0^*G| \) for all \( E \in \Sigma \). Since for each \( E \in \Sigma \),

\[ x_0^*F(E) = \lim_{n \to \infty} \int_E \varphi_n \, d|x_0^*G|, \]

it follows that

\[ \lim_{n \to \infty} \int_E \varphi_n \, d|x_0^*G| = \int_E f_0 \, d|x_0^*G| \]

for each \( E \in \Sigma \). Here \( h \) is the Radon-Nikodym derivative of \( x_0^*G \) with respect to \( |x_0^*G| \); observe that \( |h(\omega)| = 1 \) for \( |x_0^*G| \)-almost all \( \omega \in \Omega \). So \( \varphi_n h \to f_0 \) weakly in \( L^1(|x_0^*G|) \). Then \( \{\varphi_n h : n \in \mathbb{N}\} \) is relatively weakly compact in \( L^1(|x_0^*G|) \), so \( \{\varphi_n h : n \in \mathbb{N}\} \) is uniformly integrable in \( L^1(|x_0^*G|) \) and the same with \( \{\varphi_n\} \).

In particular, we have

\[ \lim_{|x_0^*G|(E) \to 0} \int_E |\varphi_n| \, d|x_0^*G| = 0 \]
uniformly on \( n \in \mathbb{N} \). Since \((\varphi_n)\) is Cauchy in measure with respect to \(|x^*_n|G|\), by Vitali’s convergence theorem, there is \( f \in L^1(|x^*_n|G|) \) such that \( \varphi_n \to f \) in \( L^1(|x^*_n|G|) \). Hence \( \varphi_n \to f \) in measure with respect to \(|x^*_n|G|\). Now fix \( x^* \in X^* \). Since \(|x^*| \ll |x^*_n|G|\), we have \( \varphi_n \to f \) in \(|x^*|\)-measure. Since \( x^*F \ll x^*|G| \), we can check as above that \((\varphi_n)\) is uniformly integrable in \( L^1(|x^*|G|) \). Again, by Vitali’s convergence theorem, we have \( f \in L^1(|x^*|G|) \) and

\[
\lim_{n \to \infty} \int_\Omega |\varphi_n - f| \, dx^*G = 0.
\]

Now let \( E \in \Sigma \) and put

\[
\int_E f \, dG = \lim_{n \to \infty} \int_E \varphi_n \, dG.
\]

Then we have \( \int_E f \, dx^*G = x^* \int_E f \, dG \) for each \( x^* \in X^* \) because \( \varphi_n \to f \) in \( L^1(|x^*|G|) \) for each \( x^* \in X^* \). Thus we obtain \( f \in L^1(G) \).

Finally we show that \((\varphi_n)\) converges to \( f \) in \( L^1(G) \). First, for each \( m \in \mathbb{N} \), define \( F_m \) on \( \Sigma \) by \( F_m(E) = \int_E \varphi_m \, dG \). Then \((F_m)\) is a sequence of \( X \)-valued \(|x^*_m|\)-continuous vector measures on \( \Sigma \) such that \( \lim_{m \to \infty} F_m(E) \) exists for each \( E \in \Sigma \). By Vitali-Hahn-Saks theorem for vector measure (see [5, Corollary 1.4.10]), we obtain that

\[
\lim_{|x^*_m|G|E| \to 0} F_m(E) = 0
\]

uniformly in \( m \). Let \( \varepsilon > 0 \). Then there exists \( \delta > 0 \) such that \( \|F_m(E)|E| \| < \varepsilon \) for all \( m \) whenever \(|x^*_m|G|E| < \delta \). Put \( E_n = \{\omega : |f(\omega) - \varphi_n| \geq \varepsilon\} \). Since \( \varphi_n \) converges to \( f \) in measure with respect to \(|x^*_n|G|\), there exists \( N \in \mathbb{N} \) such that \(|x^*_n|G|(E_n) < \delta \) whenever \( n > N \). Now take any \( n > N \) and \( E \in \Sigma \). Since \(|x^*_n|G|(E \cap E_n) \leq |x^*_n|G|(E_n) < \delta \), we have

\[
\|F_m(E \cap E_n)\| < \varepsilon
\]

uniformly in \( m \), so \( \|F(E \cap E_n)\| \leq \varepsilon \). Then we have

\[
\left| \int_E (\varphi_n - f) \, dx^*G \right| \leq \int_{E \cap E_n} (\varphi_n - f) \, dx^*G + \left| \int_{E \cap E_n} \varphi_n \, dx^*G \right| + \left| \int_{E \cap E_n} f \, dx^*G \right|
\leq \varepsilon \|G\| \Omega \| + \left| \int_{E \cap E_n} \varphi_n \, dx^*G \right| + \left| \int_{E \cap E_n} f \, dx^*G \right|
\leq \varepsilon \|G\| \Omega \| + \left| x^* \left( \int_{E \cap E_n} \varphi_n \, dG \right) \right| + \left| x^* \left( \int_{E \cap E_n} f \, dG \right) \right|
\leq \varepsilon \|G\| \Omega \| + \|F_n(E \cap E_n)\| + \|F(E \cap E_n)\|
\leq \varepsilon \|G\| \Omega \| + 2\varepsilon
\]

for all \( x^* \in B_{X^*} \). Hence we obtain that

\[
\int_E f \, dG = \lim_{n \to \infty} \int_E \varphi_n \, dG
\]
uniformly for all $E \in \Sigma$. Thus we conclude that $(\varphi_n)$ converges to $f$ in $L^1(G)$.

\[\square\]

Remark 4.3. By the proof of Theorem 4.1, the uniform bounded condition of a sequence $(\varphi_n)$ in Theorem 4.1 implies that $(\int_E \varphi_n dG)$ is Cauchy for each $E \in \Sigma$. Then Theorem 4.2 improves Lewis’s result. Moreover, Theorem 4.2 improves [7, Theorem 2.4].

For the next two theorems let’s assume that $F$ and $G$ are two countably additive vector measures where $F$ is given by the integration with respect to $G$; that is, there is $f \in L^1(G)$ such that

$$F(E) = \int_E f dG$$

for all $E \in \Sigma$.

**Theorem 4.4.** If $G(\Sigma)$ is relatively compact, then $F(\Sigma)$ is relatively compact.

**Proof.** In case $f = \chi_A$, then $F(\Sigma) = \{G(E \cap A) : E \in \Sigma \} \subset G(\Sigma)$; hence $F(\Sigma)$ is compact. If $f = \sum_{i=1}^n a_i \chi_{A_i}$ is a simple function, then $F(\Sigma) \subset \sum_{i=1}^n a_i G(\Sigma)$ is compact.

Now let $f \in L^1(G)$ and $\varepsilon > 0$. Since simple functions are dense in $L^1(G)$, we find a simple function $h$ such that $\|f - h\|_{L^1(G)} < \varepsilon$. If we write $H(E) = \int_E h dG$, then

$$\|F(E) - H(E)\| = \sup_{\|x^*\| \leq 1} \left| \int_E (f - h) d^*x G \right| \leq \|f - h\|_{L^1(G)} < \varepsilon.$$ 

Thus $F(\Sigma) \subset H(\Sigma) + \varepsilon B_X$ where $H(\Sigma)$ is compact by the previous paragraph. Thus $F(\Sigma)$ is compact. \[\square\]

**Theorem 4.5.** If $\{G(E \cap A) : A \in \Sigma\}$ is weakly compact convex for each $E \in \Sigma$, then the same holds for $F$.

In order to prove Theorem 4.5 we rely on Knowles’s version of the Lyapunov convexity theorem; see [5], [6] and [10].

**Theorem 4.6.** Let $(\Omega, \Sigma)$ be a measurable space, $X$ a Banach space and $G : \Sigma \to X$ a countably additive vector measure. Suppose $\lambda \in ca^+(\Sigma)$ with $G \ll \lambda$. Then the following are equivalent.

(a) $\{G(E \cap A) : A \in \Sigma\}$ is weakly compact convex for each $E \in \Sigma$.

(b) If $h \neq 0$ in $L^\infty(\lambda)$, then $\int_{h \neq 0} ghdG = 0$ for some $g \in L^\infty(\lambda)$ with $gh \neq 0$ in $L^\infty(\lambda)$.

**Proof of Theorem 4.5.** First fix a Rabakov control measure $\lambda = |x_0^* G|$ for $G$ where $\|x_0^*\| \leq 1$. Observe that $F$ is a countably additive vector measure and $F \ll \lambda$.

Assume that $\{G(E \cap A) : A \in \Sigma\}$ is weakly compact convex for each $E \in \Sigma$. In order to show that $\{F(E \cap A) : A \in \Sigma\}$ is weakly compact convex for each $E \in \Sigma$, we check (b) of Theorem 4.6.
Let $h \neq 0$ in $L^\infty(\lambda)$. In view of Theorem 3.6 we have $fh \in L^1(G)$. In case $fh = 0$ $\lambda$-a.e. we may take $g = \chi_{\Omega}$; then $gh \neq 0$ in $L^\infty(\lambda)$ and

$$\int_{\Omega} ghdF = \int_{\Omega} fghdG = 0.$$ 

Now assume that $fh \neq 0$ in $L^1(G)$. Then there exists $E \in \Sigma$ such that $fh \chi_E \in L^\infty(\lambda)$ and $fh \chi_E \neq 0$ in $L^\infty(\lambda)$. We apply Theorem 4.6 to $G$ to obtain $g_1 \in L^\infty(\lambda)$ such that $fh \chi_E g_1 \neq 0$ in $L^\infty(\lambda)$ but

$$\int_{\Omega} fh \chi_E g_1 dG = 0.$$ 

Put $g = \chi_E g_1$. Then $gh \neq 0$ in $L^\infty(\lambda)$ and

$$\int_{\Omega} ghdF = \int_{\Omega} fghdG = \int_{\Omega} fh \chi_E g_1 dG = 0.$$ 

This proves Theorem 4.5. \hfill \Box

## 5. Approximation property of $L^1(G)$

In this section, we consider the approximation property of $L^1(G)$. We ask some questions: Does $L^1(G)$ have the approximation property in general? And if $G$ is a $X$-valued countably additive vector measure, then are $X$ and $L^1(G)$ the same from the aspect of the approximation property? By illustrating some examples, we answer these questions.

**Definition 5.1.** A Banach space $X$ is said to have the approximation property if for every compact subset $K$ of $X$ and $\epsilon > 0$, there is a finite rank operator $T$ on $X$ such that $\|Tx - x\| < \epsilon$ for all $x \in K$.

Since $L^1(\mu)$ has the approximation property whenever $\mu$ is a nonnegative real valued measure, $L^1(G)$ which is order isomorphic to $L^1(\mu)$ has the approximation property. But $L^1(G)$ does not have the approximation property in general as the following example shows. For this example we need following facts.

**Theorem 5.2** ([1, Theorem 8]). Let $X$ be an order continuous Banach lattice with weak order unit. Then there exists a countably additive vector measure $G$ such that $X$ is order isometric to $L^1(G)$.

**Theorem 5.3** ([11, Theorem 2.4.15]). If $X$ is a reflexive Banach lattice, then $X$ has the order continuous norm.

**Example 5.4.** There exists a countably additive vector measure $G$ such that $L^1(G)$ does not have the approximation property. Thanks to Szankowski there exists a uniformly convex Banach lattice $E$ with weak order unit which fails to have the approximation property; refer to [15] or [10, Theorem 1.g.2]. Since every uniformly convex Banach space is reflexive [9, Proposition 1.e.3], $E$ is a reflexive Banach lattice. Hence, by Theorem 5.3, $E$ has order continuous norm.
By Theorem 5.2, there exists a countably additive vector measure $G$ such that $E$ is order isometric to $L^1(G)$. Since $E$ fails to have the approximation property, $L^1(G)$ does not have the approximation property.

The above example gives natural questions: if $G$ is a $X$-valued vector measure and $L^1(G)$ has the approximation property, then does $X$ have the approximation?, or if $X$ has the approximation property, then does $L^1(G)$ have the approximation property? Unfortunately, we don’t know much about this: we can give a negative answer for the former and we do not know the answer for the latter. We need the following theorem; refer to [12].

**Theorem 5.5.** Let $X$ be an infinite dimensional Banach space. Then there exists an $X$-valued countably additive vector measure $G$ with finite variation which satisfy $L^1(\vert G\vert) = L^1(G)$.

**Example 5.6.** There exists an $X$-valued countably additive vector measure $G$ such that $X$ does not have the approximation property even though $L^1(G)$ has the approximation property. In Example 5.4, there exists a uniformly convex Banach lattice $X$ with weak order unit which fails to have the approximation property. By Theorem 5.5, there exists an $X$-valued countably additive vector measure $G$ with finite variation which satisfy $L^1(\vert G\vert) = L^1(G)$. Since $L^1(\vert G\vert)$ has the approximation property, $L^1(G)$ has the approximation property.

Now we give the following problem.

**Problem.** If $G$ is an $X$-valued countably additive vector measure and $X$ has the approximation property, then does $L^1(G)$ have the approximation property?

If the above problem had an affirmative answer, then we would have a sufficient condition for a Banach space which guarantees the approximation property for $L^1(G)$ which is not isomorphic to $L^1(\mu)$.

References


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