GENERALIZED IDEAL ELEMENTS IN le-Γ-SEMIGROUPS

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Abstract. In this paper we introduce and give some characterizations of \((m,n)\)-regular \(le\)-\(Γ\)-semigroup in terms of \((m,n)\)-ideal elements and \((m,n)\)-quasi-ideal elements. Also, we give some characterizations of subidempotent \((m,n)\)-ideal elements in terms of \(r_\alpha\)- and \(l_\alpha\)-closed elements.

1. Introduction and preliminaries

In 1981, Sen [19] introduced the concept and notion of the \(Γ\)-semigroup as a generalization of semigroup and of ternary semigroup. Many classical notions and results of the theory of semigroups have been extended and generalized to \(Γ\)-semigroups. We [1, 2, 5] introduced and gave several other properties and characterizations in \(le\)-\(Γ\)-semigroups in general and regular \(le\)-\(Γ\)-semigroups in particular. The concept of generalized ideals and \((m,n)\)-ideals elements in semigroups have been introduced by Lajos in [14] as a generalization of one-sided (left or right) ideals in semigroups and it was studied by several authors such as [7], [10], [11], [15], [16], [17], [18] and others. Kehayopulu [8, 9] generalized this concept and several results related in \(poe\)-and \(le\)-semigroups. In this paper we extend these concepts and results in \(le\)-\(Γ\)-semigroups. During this paper we introduce and give some characterizations of \((m,n)\)-regular \(le\)-\(Γ\)-semigroup in terms of \((m,n)\)-ideal elements and \((m,n)\)-quasi-ideal elements. Also, we give some characterizations of subidempotent \((m,n)\)-ideal elements in terms of \(r_\alpha\)- and \(l_\alpha\)-closed elements with respect to appropriate elements.

We introduce below necessary notions and present a few auxiliary results that will be used throughout the paper.

In 1986, Sen and Saha [20] defined \(Γ\)-semigroup as a generalization of semigroup and ternary semigroup as follows:

Definition 1.1. Let \(M\) and \(Γ\) be two non-empty sets. Denote by the letters of the English alphabet the elements of \(M\) and with the letters of the Greek alphabet the elements of \(Γ\). Then \(M\) is called a \(Γ\)-semigroup if
Example 1.2. Let $M$ be a semigroup and $\Gamma$ be any non-empty set. If we define $a\gamma b = ab$ for all $a, b \in M$ and $\gamma \in \Gamma$. Then $M$ is a $\Gamma$-semigroup.

Example 1.3. Let $M$ be a set of all negative rational numbers. Obviously $M$ is not a semigroup under usual product of rational numbers. Let $\Gamma = \{ -\frac{1}{p} : p$ is prime$\}$. Let $a, b, c \in M$ and $\alpha \in \Gamma$. Now if $aab$ is equal to the usual product of rational numbers $a, \alpha, b$, then $aab \in M$ and $(aab)bc = aa(bbc)$. Hence $M$ is a $\Gamma$-semigroup.

Example 1.4. Let $M = \{-i, 0, i\}$ and $\Gamma = M$. Then $M$ is a $\Gamma$-semigroup under the multiplication over complex numbers while $M$ is not a semigroup under complex number multiplication.

These examples show that every semigroup is a $\Gamma$-semigroup and $\Gamma$-semigroups are a generalization of semigroups.

A $\Gamma$-semigroup $M$ is called a commutative $\Gamma$-semigroup if for all $a, b \in M$ and $\gamma \in \Gamma$, $a\gamma b = b\gamma a$. A nonempty subset $K$ of a $\Gamma$-semigroup $M$ is called a sub-$\Gamma$-semigroup of $M$ if for all $a, b \in K$ and $\gamma \in \Gamma$, $a\gamma b \in K$.

Example 1.5. Let $M = [0, 1]$ and $\Gamma = \{ \frac{1}{n} | n$ is a positive integer$\}$. Then $M$ is a $\Gamma$-semigroup under usual multiplication. Let $K = [0, 1/2]$. We have that $K$ is a nonempty subset of $M$ and $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$. Then $K$ is a sub-$\Gamma$-semigroup of $M$.

Different examples can be found in [1, 3, 4, 19, 20].

Definition 1.6. A po-$\Gamma$-semigroup (: ordered $\Gamma$-semigroup) is an ordered set $M$ at the same time a $\Gamma$-semigroup such that for all $a, b, c \in M$ and for all $\gamma \in \Gamma$

$$a \leq b \Rightarrow a\gamma c \leq b\gamma c, c\gamma a \leq c\gamma b.$$  

A poe-$\Gamma$-semigroup is a po-$\Gamma$-semigroup $M$ with a greatest element “$e$” (i.e., $e \geq a, \forall a \in M$).

In a po-$\Gamma$-semigroup $M$, $a$ is called a right (resp. left) ideal element if $aab \leq a$ (resp. $baa \leq a$) for all $b \in M$ and for all $\alpha \in \Gamma$. And $a$ is called an ideal element if it is both a right and left ideal element. In a poe-$\Gamma$-semigroup $M$, $a$ is called right (resp. left) ideal element if $aae \leq a$ (resp. $eaa \leq a$) for all $\alpha \in \Gamma$.

Examples of ordered $\Gamma$-semigroups can be found in [3, 4, 6].

For nonempty subsets $A$ and $B$ of $M$ and a nonempty subset $\Gamma'$ of $\Gamma$, let $A\Gamma'B = \{ a\gamma b : a \in A, b \in B$ and $\gamma \in \Gamma' \}$. If $A = \{ a \}$, then we also write $\{ a \}\Gamma'B$ as $a\Gamma'B$, and similarly if $B = \{ b \}$ or $\Gamma' = \{ \gamma \}$.

Let $T$ be a sub-$\Gamma$-semigroup of $M$. For $A \subseteq T$ we denote
An element $a$ of a po-$\Gamma$-semigroup $M$ is called regular if there exists $b \in M$ such that $a \leq aab\beta a$ for some $\alpha, \beta \in \Gamma$. A po-$\Gamma$-semigroup $M$ is called regular if every element of $M$ is regular. The following are equivalent:

1. For every $A \subseteq M$, $A \subseteq (\alpha\Gamma M \Gamma A]$, and for some $a \in A$.
2. For every element $a \in M$, $a \in (a\Gamma M \Gamma a]$. 

An element $a$ of a po-$\Gamma$-semigroup is called a quasi-ideal element if $ea^\ast \subseteq ae$ exists and $ea^\ast \subseteq ea^\ast$ for all $a \in A$. We denote by $q(a)$ the quasi-ideal element of $M$ generated by $a$, i.e., the least quasi-ideal element of $M$ containing $a$. We say that $a \in M$ is a bi-ideal element of $M$ if and only if $a\alpha\beta a \leq a$, $\forall a, \beta \in \Gamma$.

**Definition 1.7.** Let $M$ be a semilattice under $\lor$ with a greatest element $e$ and at the same time a po-$\Gamma$-semigroup such that for all $a, b, c \in M$ and for all $\gamma \in \Gamma$

$$a\gamma(b \lor c) = a\gamma b \lor a\gamma c$$

and

$$(a \lor b)\gamma c = a\gamma c \lor b\gamma c.$$

Then $M$ is called a $\lor\Gamma$-semigroup.

A $\lor\Gamma$-semigroup which is also a lattice is called an $le\Gamma$-semigroup.

The usual order relation $\leq$ on $M$ is defined in the following way

$$a \leq b \iff a \lor b = b.$$ 

Then we can show that for any $a, b, c \in M$ and $\gamma \in \Gamma$, $a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$.

**Example 1.8** ([1]). Let $(X, \leq)$ and $(Y, \leq)$ be two finite chains. Let $M$ be the set of all isotone mappings from $X$ into $Y$ and $\Gamma$ be the set of all isotone mappings from $Y$ into $X$. Let $f, g \in M$ and $\alpha \in \Gamma$. We define $f\alpha g$ to denote the usual mapping composition of $f, \alpha$ and $g$. Then $M$ is a $\Gamma$-semigroup. For $f, g \in M$, the mappings $f \lor g$ and $f \land g$ are defined by letting, for each $a \in X$

$$(f \lor g)(a) = \max\{f(a), g(a)\}, \quad (f \land g)(a) = \min\{f(a), g(a)\}$$

(the maximum and minimum are considered with respect to the order $\leq$ in $X$ and $Y$). The greatest element $e$ is the mapping that sends every $a \in X$ to the greatest element of finite chains $(Y, \leq)$. Then $M$ is an $le\Gamma$-semigroup.

**Example 1.9** ([1]). Let $M$ be a po-$\Gamma$-semigroup. Let $M_1$ be the set of all ideals of $M$. Then $(M_1, \leq, \cap, \cup)$ is an $le\Gamma$-semigroup.

**Example 1.10** ([1]). Let $M$ be a po-$\Gamma$-semigroup. Let $M_1 = P(M)$ be the set of all subsets of $M$ and $\Gamma_1 = P(\Gamma)$ the set of all subsets of $\Gamma$. Then $M_1$ is a
Let $I$ be an element of $U \subset S$. In [6] we deal with lattice-ordered Rees matrix-semigroups. It is easy verified that $(I, \wedge)$ and $(\alpha, \wedge)$ are related to the $I$-semigroup $\bigvee_{\beta \in \Gamma} \beta$. For other definitions and terminologies not given in this paper, the reader is referred to [1], [19], [20].

De nition 2.1. Let $G$ be a group, $I$, $\wedge$ two index sets and $\Gamma$ the collection of some $\wedge \times I$ matrices over $G^\omega = G \cup \{0\}$, the group with zero. Let $\mu^\omega$ be the set of all elements $(a)_{\lambda \beta}$ where $i \in I$, $\lambda \in \wedge$, and $(a)_{\lambda \beta}$ the $I \times \wedge$ matrix over $G^\omega$ having $a$ in the $i$-th row and $\lambda$-th column, its remaining entries being zero. The expression $(0)_{\lambda \beta}$ will be used to denote the zero matrix. For any $(a)_{\lambda \beta}, (b)_{\mu j}, (c)_{k \nu} \in \mu^\omega$ and $\alpha = (p_{\lambda j}), \beta = (q_{\lambda j}) \in \Gamma$ we define $(a)_{\lambda \alpha \beta} b_{\mu} = (a p_{\lambda j} b_{\mu j})_{\beta}$. Then it is easy verified that $[a_{\lambda \alpha \beta} b_{\mu j}]_{\beta} c_{k \nu} = (a_{\lambda \alpha \beta} b_{\mu j}) c_{k \nu}$. Thus $\mu^\omega$ is a $\Gamma$-semigroup. We call $\Gamma$ the sandwich matrix set and $\mu^\omega$ the Rees $I \times \wedge$ matrix $\Gamma$-semigroup over $G^\omega$ with sandwich matrix set $\Gamma$ and denote it by $\mu^\omega(G : I, \wedge, \Gamma)$. In [6] we deal with lattice-ordered Rees matrix $\Gamma$-semigroups.

If $M$ is a $\forall e\Gamma$-semigroup, then every map $\phi : M \rightarrow M$ is called a topology on $M$ [1]. A topology $\phi$ on $M$ is said to be an:

1. $S$-topology on $M$ if and only if $a_1, a_2 \in M$, with $a_1 \leq a_2$ implies $\phi(a_1) \leq \phi(a_2)$.
2. $I$-topology on $M$ if and only if $\phi(a) \in M$, implies $a \leq \phi(a)$.
3. $U$-topology on $M$ if and only if $\phi(\phi(a)) = \phi(a)$ for every $a \in M$.

An element $a \in M$ is called a closed element of $M$ related to a topology $\phi$ (or $\phi$-closed) if and only if $\phi(a) = a$. The set of all closed element of $M$ related to $\phi$ will be denoted by $F_\phi$.

In a $\forall e\Gamma$-semigroup $M$, we define two mappings $r_\alpha$ and $l_\beta$ for each $\alpha, \beta \in \Gamma$ as follows:

$$r_\alpha : M \rightarrow M, r_\alpha(a) = a a e \vee a,$$

$$l_\beta : M \rightarrow M, l_\beta(a) = c a \vee a$$

for all $a \in M$.

It is clear that $r_\alpha$ and $l_\beta$, for $\alpha, \beta \in \Gamma$, are $I$- and $S$-topologies on $M$. If $M$ is an $le\Gamma$-semigroup, then $r_\alpha$ and $l_\beta$ are $U$-topologies on $M$.

For other definitions and terminologies not given in this paper, the reader is referred to [1], [19], [20].

2. On $(m, n)$-ideal elements in $le\Gamma$-semigroups

Let $M$ be a $\forall e\Gamma$-semigroup and $m, n \in \mathbb{Z}^+$. The set of all elements $(a)_{\lambda \beta}$ where $i \in I$, $\lambda \in \wedge$, and $(a)_{\lambda \beta}$ the $I \times \wedge$ matrix over $G^\omega$ having $a$ in the $i$-th row and $\lambda$-th column, its remaining entries being zero. The expression $(0)_{\lambda \beta}$ will be used to denote the zero matrix. For any $(a)_{\lambda \beta}, (b)_{\mu j}, (c)_{k \nu} \in \mu^\omega$ and $\alpha = (p_{\lambda j}), \beta = (q_{\lambda j}) \in \Gamma$ we define $(a)_{\lambda \alpha \beta} b_{\mu} = (a p_{\lambda j} b_{\mu j})_{\beta}$. Then it is easy verified that $[a_{\lambda \alpha \beta} b_{\mu j}]_{\beta} c_{k \nu} = (a_{\lambda \alpha \beta} b_{\mu j}) c_{k \nu}$.

De nition 2.1. An element $a \in M$ is called an $(m, n)$-ideal element of $M$ if

$$(a_{\gamma_1 a_{\gamma_2 a \ldots a_{\gamma_{m-1} a}} \alpha \beta (a p_{\gamma_1 a p_{\gamma_2 a} \ldots a p_{\gamma_{m-1} a}}) \leq a$$

for all $\alpha, \beta \in \Gamma$ and for some $\gamma_1, \gamma_2, \ldots, \gamma_{m-1}, \rho_1, \ldots, \rho_{n-1} \in \Gamma$.
a^0 is defined such that a^0 \gamma b = b \gamma a^0 = b (b \in M, \gamma \in \Gamma).

For \( m = 0, n = 1 \) (resp. \( m = 1, n = 0 \)) Definition 2.1 gives us the trivial case of left (resp. right)-ideal elements. It is clear that the right (resp. left)-ideal elements are \((m, 0)\) (resp. \((0, n)\))-ideal elements for every \( m \geq 1 \) (resp. \( n \geq 1 \)).

An \( \epsilon\Gamma \)-semigroup is called subidempotent if \( a \alpha a \leq a \) for all \( a \in M, \alpha \in \Gamma \).

In the sequel of the paper, for the sake of simplicity, we denote \( a^n = a\gamma_1a\gamma_2a\ldots\gamma_{m-1}a \) for some \( \gamma_1, \gamma_2, \ldots, \gamma_{m-1} \in \Gamma \) and for some \( m \in \mathbb{Z}^+ \).

It can be easily proved the following statements:

1. Let \( M \) be a \( \epsilon\Gamma \)-semigroup and \( a, b \) two \((m, n)\)-ideal elements of \( M \).

Then the intersection \( a \land b \) if exists is an \((m, n)\)-ideal element of \( M \).

2. Let \( M \) be a \( \epsilon\Gamma \)-semigroup. Then the \( k \)-power \((k \geq 1)\) of an \((m, n)\)-ideal element is also an \((m, n)\)-ideal element.

3. Let \( M \) be a \( \epsilon\Gamma \)-semigroup. If \( a \) is an \((m, n)\)-ideal element of \( M \) \((m, n \geq 1)\), then for every \( b \in M, \alpha \in \Gamma \) such that \( aab \leq a \) (resp. \( baa \leq a \)), the products \( aab \) and \( baa \) are \((m, n)\)-ideal elements of \( M \).

For \( m = n = 1 \) we have: If \( a \) is a bi-ideal element, then \( a\gamma b \) and \( b\gamma a \), \( \gamma \in \Gamma \), are bi-ideal elements, for all \( b \in M \). Clearly, \( \forall \alpha, \beta, \gamma \in \Gamma \),

\[
(a\gamma b)a\epsilon\beta(a\gamma b) = (a\gamma b)\alpha(e\beta a)\gamma b \leq (a\gamma b)a\epsilon\beta b = a\gamma(ba\epsilon\beta y) \leq a\gamma b,
\]
similarly \( (b\gamma a)a\epsilon\beta(b\gamma a) \leq b\gamma a \).

4. Let \( M \) be a \( \forall \epsilon \Gamma \)-semigroup and \( a, b \) two left (resp. right)-ideal elements of \( M \). Then the union \( a \lor b \) is a subidempotent \((m, n)\)-ideal element, \( \forall m \geq 0, \forall n \geq 1 \) (resp. \( n \geq 0, m \geq 1 \)).

In the following, \( \langle a \rangle_{(m,n)} \) will be denoted the principal \((m, n)\)-ideal element of \( M \) generated by \( a \), i.e., the least \((m, n)\)-ideal element of \( M \) containing \( a \) and by \( I_{(m,n)} \) the set of all \((m, n)\)-ideal elements of \( M \).

**Definition 2.2.** Let \( M \) be a \( \forall \epsilon \Gamma \)-semigroup and \( m, n \in \mathbb{Z}^+ \). \( M \) is called \((m, n)\)-regular if

\[
a \leq (a\gamma_1a\gamma_2a\ldots\gamma_{m-1}a)\epsilon\beta(\alpha_1\beta_2a\ldots\beta_{n-1}a)
\]

for all \( a \in M, \alpha, \beta, \gamma \in \Gamma \) and for some \( \gamma_1, \gamma_2, \ldots, \gamma_{m-1}, \alpha_1, \ldots, \beta_{n-1} \in \Gamma \).

**Lemma 2.3.** Let \( M \) be a \( \forall \epsilon \Gamma \)-semigroup and \( m, n, k \in \mathbb{Z}^+ \). Then the following hold true:

1. \( (a \lor a^m\epsilon\beta a^k)^m \gamma c = a^m \gamma c, \forall a \in M, \alpha, \beta, \gamma \in \Gamma \).
2. \( e\gamma(a \lor a^k\epsilon\beta a^k)^n = e\gamma a^n, \forall a \in M, \alpha, \beta, \gamma \in \Gamma \).
3. \( \langle a \rangle_{(m,n)} = a \lor a^m\epsilon\beta a^n, \forall a \in M, \alpha, \beta, \epsilon \in \Gamma \).

**Proof.** We prove the first two conditions in case \( m, n \geq 1 \), since the case \( m = 0 \) or \( n = 0 \) is obvious. For \( a \in M, \alpha, \beta, \rho \in \Gamma \), we have

\[
(1) (a \lor a^m\epsilon\beta a^k)^m \gamma c = ((a \lor a^m\epsilon\beta a^k)\rho)^m(a \lor a^m\epsilon\beta a^k)\gamma c = ((a \lor a^m\epsilon\beta a^k)\rho)^m-1((a\gamma c)\epsilon\beta a^k)\gamma c = (((a\gamma c)\epsilon\beta a^k)\rho)^m-1a\gamma c = (((a\gamma c)\epsilon\beta a^k)\rho)^m-1a\gamma c = \cdots = a^m \gamma c.
\]

\[
(2) e\gamma(a \lor a^k\epsilon\beta a^k)^n = e\gamma a^n = \cdots = a^n.
\]
The proof of (2) is analogous to that of (1).

(3) Let \( m, n \geq 0, \alpha, \beta \in \Gamma \). Then, \( (a \lor a^m a \epsilon a^n)^m a \epsilon a^n (a \lor a^m a \epsilon a^n) = a^m a \epsilon a^n \) (by (1) and (2)), so that \( a \lor a^m a \epsilon a^n \in I_{m,n} \). Now, if \( b \) is an \((m,n)\)-ideal element of \( M \) containing \( a \), then \( a \lor a^m a \epsilon a^n \leq b \). This finishes the proof. \( \square \)

**Theorem 2.4.** \( \forall e \Gamma \)-semigroup \( M \) is \((m,n)\)-regular if and only if

\[(\ast) \quad a^m a \epsilon a^n = a, \; \forall a \in I_{m,n}, \forall \alpha, \beta \in \Gamma.\]

**Proof.** \((\Rightarrow)\) This statement is obvious.

\( (\Leftarrow) \) Let \( a \in M, \alpha, \beta \in \Gamma \). Since \( \langle a \rangle_{(m,n)} \in I_{m,n} \), we have:

\[
(\langle a \rangle_{(m,n)}^m a \epsilon a^n (\langle a \rangle_{(m,n)}^n = \langle a \rangle_{(m,n)}
\]

But, \( (\langle a \rangle_{(m,n)}^m a \epsilon a^n = a^m a \epsilon a^n \) (by Lemma 2.3(3),(1)), and \( e \beta (\langle a \rangle_{(m,n)}^n = e \beta a^n \) (by Lemma 2.3(3),(2)).

Thus, \( a \leq a^m a \epsilon a^n \) and \( M \) is \((m,n)\)-regular. \( \square \)

**Theorem 2.5.** Let \( M \) be a subidempotent \( le \)-\( \Gamma \)-semigroup. Then \( M \) is \((m,n)\)-regular if and only if

\[(\ast\ast) \quad a \land b = a^m a \epsilon b^n, \; \forall a \in I_{m,0}, b \in I_{0,n}, \alpha, \beta \in \Gamma.\]

**Proof.** \((\Rightarrow)\) Let \( M \) be a \((m,n)\)-regular \( le \)-\( \Gamma \)-semigroup. Let \( a \in I_{m,0} \) and \( b \in I_{0,n} \). Then, \( a^m a \epsilon \leq a \) and \( e \beta b^n \leq b \), hence \( a^m a \epsilon \land e \beta b^n \leq a^m a \epsilon \land e \beta b^n \leq a \land b \). On the other hand, \( a \land b \leq (a \land b)^m a \epsilon a^n (a \land b)^n \leq a^m a \epsilon a^n \leq a^m a \epsilon b^n \leq a^m a \epsilon b^n \), similarly, \( a \land b \leq a^m a \epsilon b^n \), hence \( a \land b \leq a^m a \epsilon b^n \).

\( (\Leftarrow) \) Since \( M \) is a \((0,0)\)-regular \( le \)-\( \Gamma \)-semigroup, the the statement is true for \( m = n = 0 \).

Let \( m \neq 0, n = 0 \). If \( a \in I_{m,0} \), then since \( e \) is a \((0,0)\)-ideal element of \( M \), we have by \((\ast\ast)\), that \( a = a^m a \epsilon, \alpha \in \Gamma \), so that \( M \) is \((m,0)\)-regular (Theorem 2.4). The proof in the case \( m = 0, n \neq 0 \) is analogues.

Now, let \( m \neq 0, n \neq 0 \). Then, \( M \) has the property:

\[(\ast\ast\ast) \quad a \land b \leq a a b, \; \forall a \in I_{m,0}, \forall b \in I_{0,n}, \forall \alpha \in \Gamma.\]

Indeed: let \( a \in I_{m,0} \). Then \( \forall a \in \Gamma \),

\[a \land b = a^m a \epsilon \land a a b^n \leq a a b \quad (m \geq 1, n \geq 1).\]

Now, let \( a \in M \). Since \( \langle a \rangle_{(m,n)} \in I_{m,0} \) and \( e \) is a \((0,n)\)-ideal element of \( M \), we have by \((\ast\ast)\), that \( \forall a, \beta \in \Gamma \),

\[
\langle a \rangle_{(m,0)} = \langle a \rangle_{(m,0)}^m a \epsilon \land \langle a \rangle_{(m,0)}^n \beta e^n
\leq \langle a \rangle_{(m,0)}^n a \epsilon = a^m a \epsilon \quad (by \text{Lemma 2.3(3),(1)}),
\]

thus \( a = a^m a \epsilon a^n \). Similarly, \( a = e \beta a^n \). On the other hand,

\[a \leq \langle a \rangle_{(m,n)} \land \langle a \rangle_{(0,n)} \leq \langle a \rangle_{(m,n)} \gamma \langle a \rangle_{(0,n)}, \forall \gamma \in \Gamma (by (\ast\ast\ast))\]

\[= a^m a \epsilon ^2 \beta a^n \leq a^m a \epsilon a^n.\]
Therefore, $M$ is $(m,n)$-regular, and this finishes the proof. 

**Theorem 2.6.** Let $M$ be a poe-$\Gamma$-semigroup and $m, n \in \mathbb{Z}^+$ with $m + n \geq 1$. Let be the following statements for $a \in M$,

1. $\exists a_i \in F_{r_\rho}(a_{i-1})$, $i = 1, 2, \ldots, m$ and $\exists b_j \in F_{l_\rho}(b_{j-1})$, $j = 1, 2, \ldots, n$ where $a_0 = e, b_0 = a_m$ and $b_n = a$ (resp. $b_0 = e, a_0 = b_n$ and $a_m = a$).

2. $a$ is a subidempotent $(m,n)$-ideal element of $M$.

Then, (1) $\Rightarrow$ (2). In particular, if $M$ is a $\forall e$-$\Gamma$-semigroup, then (1) $\iff$ (2).

**Proof.** (1) $\Rightarrow$ (2). In fact $a$ is subidempotent, and for $\alpha, \beta, \rho \in \Gamma$, we have:

$$b_n^m \alpha e \beta b_n^m = (b_n^m \alpha e \beta b_n^m \leq (b_n^m \alpha e \beta b_n^m)_{n=1}^{m-1} \alpha e \beta b_n^m$$

$$\leq (b_n^m \alpha e \beta b_n^m)_{n=1}^{m-2} \alpha e \beta b_n^m \leq \cdots \leq (b_n^m \alpha e \beta b_n^m)_{n=1}^{m-(m-1)} \alpha e \beta b_n^m$$

$$= b_n \alpha e b_n \leq a_m \alpha e b_n \leq a_m \beta b_n$$

$$= a_m \beta b_n \beta b_n^{n-1} \leq a_m \beta b_n \beta b_n^{n-1} \leq b_1 b_n \beta b_n^{n-2} \leq b_2 b_n^{n-2}$$

$$\leq \cdots \leq b_1 b_n \beta b_n^{n-(n-1)} = b_1 b_n \leq b_n.$$ 

(2) $\Rightarrow$ (1). Let $M$ be a $\forall e$-$\Gamma$-semigroup and let $a$ be a subidempotent $(m,n)$-ideal element of $M$. We put:

$$a_i = \langle a \rangle_{i,0}, \ i = 0, 1, 2, \ldots, m \text{ and } b_j = \langle a \rangle_{j,0}, \ j = 0, 1, 2, \ldots, n.$$ 

Then, by Lemma 2.3(3), we have $\forall \alpha, \rho \in \Gamma$,

$$a_i = \langle \alpha \rangle_{i,0} = a \vee a' \alpha e = a \vee (a \alpha)^{i-1} a \alpha e \leq a \vee a^{i-1} \alpha e$$

$$= \langle \alpha \rangle_{i-1,0} = a_{i-1}, \ i = 1, 2, \ldots, m$$

and $\forall \delta, \gamma, \rho \in \Gamma$

$$b_j = \langle \delta \rangle_{j,0} = a \vee a' \delta e a' \gamma e a \gamma a = a \vee a' \delta e a \gamma a^{j-1} \leq a \vee a' \delta e a \gamma a^{j-1}$$

$$= \langle \delta \rangle_{j-1,0} = b_{j-1}, j = 1, 2, \ldots, n.$$ 

Also, $a_0 = e, b_0 = a_m$ and $b_n = a$. Moreover,

$$a_{i,0} = \langle a \rangle_{i,0} \rho(a)_{i-1,0} = (a \vee a' \alpha e) \rho(a \vee a^{i-1} \alpha e)$$

$$= a^2 \vee a' \alpha e a' \alpha e a \vee a' \alpha e a \alpha e a^{i-2} \alpha e \alpha e a^{i-1} \alpha e \leq a \vee a' \alpha e$$

$$= \langle a \rangle_{i,0} = a, \ i = 1, 2, \ldots, m$$

that is,

$$a_i \in F_{r_\rho}(a_{i-1}), \ i = 1, 2, \ldots, m.$$
Also, \( \forall \delta, \gamma, \rho \in \Gamma \)
\[
\langle a \rangle_{(m,j-1)} \rho \langle a \rangle_{(m,j)} \\
= (a \lor a^m \delta e \gamma a^{j-1}) \rho (a \lor a^m \delta e \gamma a^j) \\
= a^2 \lor a^m \delta e \gamma a^j \lor a^{m+1} \delta e \gamma a^j \lor a^m \delta e \gamma a^{j-1} + m \delta e \gamma a^j \\
\leq a \lor a^m \delta e \gamma a^j = \langle a \rangle_{(m,j)}, j = 1, 2, \ldots, n.
\]
Therefore,
\[
b_{j-1} \rho b_j \leq b_j, j = 1, 2, \ldots, n
\]
that is,
\[
b_j \in \mathcal{F}_\nu^{\langle b_j \rangle}, j = 1, 2, \ldots, n.
\]
The other case can be proved similarly. In that case, for \((2) \Rightarrow (1)\) we put:
\[
b_j = \langle a \rangle_{(0,j)}, j = 0, 1, 2, \ldots, n
\]
and
\[
a_i = \langle a \rangle_{(i,n)}, i = 0, 1, 2, \ldots, m.
\]
Let \( M \) be a poe-\( \Gamma \)-semigroup. An element \( a \in M \) is called \( r_\alpha l_\alpha \)-closed, \( \alpha \in \Gamma \), if there exists a right-ideal element \( b \in M \) such that \( a \) is \( l_\alpha \)-closed with respect to \( b \). Similarly, \( a \) is called \( l_\alpha r_\alpha \)-closed, if there exists a left-ideal element \( b \) with \( a \in \mathcal{F}_\nu^{b} \).

**Corollary 2.7.** Let \( M \) be a poe-\( \Gamma \)-semigroup. Then all \( r_\alpha l_\alpha \)-closed and \( l_\alpha r_\alpha \)-closed elements are subidempotent bi-ideal elements. In particular, if \( M \) is a \( \forall e \)-semigroup, the preceding three classes of elements are the same.

3. On \((m, n)\)-quasi-ideal elements in \( le \)-\( \Gamma \)-semigroups

**Definition 3.1.** Let \( M \) be a poe-\( \Gamma \)-semigroup. An element \( q \) of \( M \) is called an \((m, n)\)-quasi-ideal element of \( M \) if \( q^m a e \land e b q^n \) exists and \( q^m a e \land e b q^n \leq q \), \( \alpha, \beta \in \Gamma \).

**Remark 3.2.** Every quasi-ideal element \( q \) of a poe-\( \Gamma \)-semigroup \( M \) is an \((m, n)\)-quasi-ideal element of \( M \) for all \( m, n \in \mathbb{Z}^+ \) such that \( q^m a e \land e b q^n \) exists. For \( m, n \in \mathbb{Z}^+ \), every \((m, n)\)-quasi-ideal element is an \((m, n)\)-ideal element of \( M \). If \( \{q_i; i \in I\} \) is a nonempty family of \((m, n)\)-quasi-ideal elements of \( M \), then \( \bigwedge_{i \in I} q_i \) is an \((m, n)\)-quasi-ideal element if \( (\bigwedge_{i \in I} q_i) a e \land e c a (\bigwedge_{i \in I} q_i), \forall a \in \Gamma \) exists.

**Remark 3.3.** Quasi-ideal elements are subidempotent. In poe-\( \Gamma \)-semigroups, quasi-ideal elements are subidempotent bi-ideal elements. In \( \forall e \Gamma \)-semigroups, quasi-ideal elements are \( r_\alpha l_\alpha \) (resp. \( l_\alpha r_\alpha \))-closed. In regular \( le \)-\( \Gamma \)-semigroups, the converse of the last statement also holds (by Corollary 2.7 above and Corollary 2.3 [1]).
Let we denote by $Q_{(m,n)}$ the set of all $(m,n)$-quasi-ideal elements of $M$.

In distributive $le$-$\Gamma$-semigroups [2], the quasi-ideal elements are exactly the intersections of the left- and right- ideal elements. The following theorem shows that the analogues property is true for the $(m,n)$-quasi-ideal elements, too.

**Theorem 3.4.** Let $M$ be a distributive $le$-$\Gamma$-semigroup. Then, an element $q$ is an $(m,n)$-quasi-ideal element of $M$ if and only if there exists an $(m,0)$-ideal element $a$ and a $(0,n)$-ideal element $b$ of $M$ such that

$$q = a \land b.$$ 

**Proof.** ($\Rightarrow$) Let $a \in I_{(m,0)}$ and $b \in I_{(0,n)}$. Then, since $a, b \in Q_{(m,n)}$, we have

$$a \land b \in Q_{(m,n)}.$$ 

($\Leftarrow$) Let $q \in Q_{(m,n)}$. Then, by Lemma 2.2(3), we have

$$q = q \lor (q^m \alpha e \land e \beta q^n) = (q \lor q^m \alpha e) \land (q \lor e \beta q^n)$$

$$= \langle q \rangle_{(m,0)} \land \langle q \rangle_{(0,n)}$$

where $\langle q \rangle_{(m,0)} \in I_{(m,0)}$ and $\langle q \rangle_{(0,n)} \in I_{(0,n)}$. 

It is clear that $(m,0)$ (resp. $(0,n)$)-ideal elements and $(m,0)$ (resp. $(0,n)$)-quasi-ideal elements are the same. So, the following theorem is true:

**Theorem 3.5.** Let $M$ be a distributive $le$-$\Gamma$-semigroup. Then, an element $q$ is an $(m,n)$-quasi-ideal element of $M$ if and only if there exists an $(m,0)$-quasi-ideal element $a$ and a $(0,n)$-quasi-ideal element $b$ of $M$ such that

$$q = a \land b.$$ 

Since the $poe$-$\Gamma$-semigroups are semilattices under $\land$, we have the following theorem.

**Theorem 3.6.** Let $M$ be a $poe$-$\Gamma$-semigroup. Then the element

$$q = a \land b$$

where $a$ is an $(m,0)$-ideal element and $b$ an $(0,n)$-ideal element of $M$, is a $(m,n)$-quasi-ideal element of $M$.

We denote by $(a)_{(m,n)}$ the $(m,n)$-quasi-ideal element of $M$ generated by $a \in M$.

**Lemma 3.7.** Let $M$ be an $le$-$\Gamma$-semigroup, $a \in M, \alpha, \beta, \gamma, \rho \in \Gamma$ and $m, n \in \mathbb{Z}^+$. The following hold true:

1. $(a \lor (a^m \alpha e \land e \beta a^n))^m \gamma e \leq a^m \gamma e$.
2. $e \gamma (a \lor (a^m \alpha e \land e \beta a^n))^m \leq e \gamma a^n$.
3. $(a)_{(m,n)}$ exists and $(a)_{(m,n)} = a \lor (a^m \alpha e \land e \beta a^n)$.
Proof. (1) Since for $m = 0$ it is clear. Let $m \geq 1$. Then we have
\[(a \lor (a^m \alpha e \land e b a^n))^m \gamma e\]
\[\leq (a \lor (a^m \alpha e \land e b a^n))^{m-1} \rho(a \lor (a^m \alpha e \land e b a^n)) \gamma e\]
\[= (a \lor (a^m \alpha e \land e b a^n))^{m-1} \rho(a \gamma e \lor (a^m \alpha e^2 \land e b a^n \gamma e))\]
\[= (a \lor (a^m \alpha e \land e b a^n))^{m-1} \rho(a \gamma e)\]
\[= (a \lor (a^m \alpha e \land e b a^n))^{m-2} \rho(a b \gamma e)\]
\[\leq (a \lor (a^m \alpha e \land e b a^n))^{m-2} \rho(a^2 \gamma e)\]
\[\leq (a \lor (a^m \alpha e \land e b a^n))^{m-2} \rho(a b \gamma e)\]
\[\leq a^m \gamma e.
\]
(2) It is proved similarly.
(3) Let $m, n \geq 0$. From (1) and (2) we have $\forall \alpha, \beta \in \Gamma$
\[(a \lor (a^m \alpha e \land e b a^n))^{m \alpha e \land e b (a \lor (a^m \alpha e \land e b a^n))^n \leq a^m \alpha e \land e b a^n\]
hence $a \lor (a^m \alpha e \land e b a^n)$ is an $(m, n)$-quasi-ideal element of $M$ containing $a$. Now, if $b$ is an $(m, n)$-quasi-ideal element of $M$ such that $b \geq a$, then $a \lor (a^m \alpha e \land e b a^n) \leq b$. □

Remark 3.8. In general in le-$\Gamma$-semigroups, $\langle a \rangle_{(m, n)} \leq \langle a \rangle_{(m, n)}$. In particular, $\langle a \rangle_{(m, 0)} = \langle a \rangle_{(m, 0)}$ and $\langle a \rangle_{(0, n)} = \langle a \rangle_{(0, n)}$.

Theorem 3.9. Let $M$ be an le-$\Gamma$-semigroup and $m, n \in \mathbb{Z}^+$. Then the following are equivalent:

1. $M$ is $(m, n)$-regular.
2. $a^m \alpha e \beta a^n = a$, $\forall a \in I_{(m, n)}$, $\alpha, \beta \in \Gamma$.
3. $q^m \alpha e \beta q^n = q$, $\forall q \in Q_{(m, n)}$, $\alpha, \beta \in \Gamma$.
4. $((a)_{(m, n)})^{m \alpha e \beta ((a)_{(m, n)})^n = (a)_{(m, n)}$, $\forall a \in M$, $\alpha, \beta \in \Gamma$.
5. $((a)_{(m, n)})^{m \alpha e \beta ((a)_{(m, n)})^n = (a)_{(m, n)}$, $\forall a \in M$, $\alpha, \beta \in \Gamma$.

Proof. (1) $\Rightarrow$ (2). It is obvious by Theorem 2.4.
(2) $\Rightarrow$ (3). It is obvious by Remark 3.2.
(3) $\Rightarrow$ (1). Let $a \in M$. By Theorem 3.5 it follows that the element $\langle a \rangle_{(m, 0)} \land \langle a \rangle_{(0, n)}$ is an $(m, n)$-quasi-ideal element of $M$. Thus, by (3) and Lemma 2.3,
\[a \leq \langle a \rangle_{(m, 0)} \land \langle a \rangle_{(0, n)}\]
\[= ((a)_{(m, 0)} \land (a)_{(0, n)})^{m \alpha e \beta ((a)_{(m, 0)} \land (a)_{(0, n)})^n}\]
\[\leq ((a)_{(m, 0)})^{m \alpha e \beta ((a)_{(0, n)})^n}\]
\[= a^{m \alpha e \beta a^n}.
\]
(2) $\Rightarrow$ (4). It is clear.
(4) $\Rightarrow$ (2). If $a \in I_{(m, n)}$, then $\langle a \rangle_{(m, n)} = a$ and by 4), we have $a^{m \alpha e \beta a^n} = a$. 
(3) $\Rightarrow$ (5). The proof is similar with the previous case of $(m, n)$-ideal elements. □

Remark 3.10. From all above, it is clear that the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) hold in poe-$\Gamma$-semigroups, and the equivalence (2) $\Leftrightarrow$ (4) in $\vee\wedge$-$\Gamma$-semigroups in general.

Remark 3.11. In regular poe-$\Gamma$-semigroups, we have $I_{(m, n)} = Q_{(m, n)}$. For every element $a$ of a poe-$\Gamma$-semigroup $M$ and $\alpha, \beta, \delta, \rho \in \Gamma$, we have

$$a^m\alpha e \land e \beta a^n \leq (a^m\alpha e \land e \beta a^n)e\rho(a^m\alpha e \land e \beta a^n) \leq (a^m\alpha e\rho(e\beta a^n) \leq a^m\alpha e \land e \beta a^n.$$ 

So, in regular poe-$\Gamma$-semigroups, we have $a^m\alpha e\beta a^n = a^m\alpha e \land e \beta a^n$.

References

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