ON NIL-EXTENSIONS OF LEFT STRONGLY SIMPLE
po-SEMIGROUPS

Qing Shun Zhu

ABSTRACT. In this paper, we first introduce the concept of left strongly
simple *po*-semigroups, then we discuss properties and characterizations
nil-extensions of left strongly simple *po*-semigroups and semilattices of left
strongly simple *po*-semigroups. Finally, we give some characterizations of
the chain of left strongly simple *po*-semigroups.

1. Introduction and preliminaries

The ideal extension problem for semigroups is as follows: Given a semigroup
S and a semigroup Q with zero such that $S \bigcap Q^* = \emptyset$ (where $Q^* = Q \setminus \{0\}$),
construct all the semigroups V which have an ideal K which is isomorphic
to S and the Rees quotient V/K is isomorphic to Q. For ideal extensions
of semigroups (without order) have been considered by A. H. Clifford in [4]
who gave the first general structure theorem in the case when semigroups is
weakly reductive (Theorem 4.21 of [4]). A detailed exposition of the ideal
extensions of semigroups can be found in [5, 13]. Extensions of weakly reductive
semigroups, strict and pure extensions, retract extensions, dense extensions,
equivalent extensions have been also considered in [13]. Ideal extensions of
ordered semigroups have been studied in [10], and nil-extensions of simple
ordered semigroups in [3]. We are often interested in building more complex
semigroups, lattices, ordered sets, and ordered or topological semigroups out of
some of "simple" structure and this can be sometimes achieved by constructing
the ideal extensions. The aim of this paper is to study the structure of left
strongly simple *po*-semigroups referring to nil-extension and semilattice of this
type of *po*-semigroups.

Throughout this paper, Z^+ will denote the set of positive integers. As in
[1], a *po-semigroup* (: ordered semigroup) is an ordered set $(S, \leq)$ at the same
time a semigroup such that:

$$a \leq b \Longrightarrow xa \leq xb \text{ and } ax \leq bx, \forall a, b, x \in S.$$

Definition 1.1. Let S be a *po*-semigroup and $\emptyset \neq A \subseteq S$. A is called a *left (right) ideal* of S if

(1) $SA \subseteq A$ ($SA \subseteq A$);

(2) If $a \in A, b \leq a$ with $b \in S$, then $b \in A$.

A is called an *ideal* of S if A is both a left and right ideal of S. S is *left (right) simple* if it does not contain proper left (right) ideals. S is *simple* if S does not contain proper ideals.

Let S be a *po*-semigroup. For $H \subseteq S$, let $(H] := \{t \in S | t \leq h \text{ for some } h \in H\}$. For $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$ ($a \in S$). Let $I(a), L(a)$ and $R(a)$ denote the ideal, the left ideal and the right ideal of S generated by an element a of S, respectively. One can easily prove that:

$$L(a) = (a \bigcup Sa],$$
$$R(a) = (a \bigcup aS],$$
$$I(a) = (a \bigcup Sa \bigcup aS \bigcup SaS].$$

We defined the Green's relation $\mathcal{J} := \{(x,y) | I(x) = I(y)\}$ and $\mathcal{U}_{\mathcal{J}}(S)$ will denote the union of all $\mathcal{J}$-classes of S which are subsemigroups of S.

An element a of a *po*-semigroup S is *semisimple* if $a \in (SaSaS]$. A set of all semisimple elements of S will be denoted by $Semis(S)$. A semigroup S is called semisimple if $S = Semis(S)$.

Lemma 1.1. *Let S be a po-semigroup. Then*

(1) S *is left (right) simple if and only if* $S = (Sa] (S = (aS])$ *for every* $a \in S$ *(see [6])*.

(2) S *is simple if and only if* $S = (SaS]$ *for every* $a \in S$ *(see [6])*.

(3) S *is semisimple if and only if* $I(a) = (I^2(a)]$ *for every* $a \in S$.

Proof. (3) is easy to verify by the definition of semisimple. $\square$

Definition 1.2 ([7, 11])**.** Let Y be a semilattice. Then Y with the natural order on Y defined by: for $\alpha, \beta \in Y$

$$\alpha \leq \beta \Leftrightarrow \alpha\beta = \alpha$$

is a *po*-semigroup. The *po*-semigroup $(Y, \cdot, \leq)$ is called a *complete semilattice*. A *po*-semigroup S is called a *complete semilattice Y of subsemigroups $S_\alpha (\alpha \in Y)$* if S is the semilattice Y of subsemigroups $S_\alpha (\alpha \in Y)$ such that

$$(\forall \alpha, \beta \in Y)(\forall a \in S_\alpha)(\forall b \in S_\beta) a \leq b \Rightarrow \alpha \leq \beta.$$

A congruence determined by the partition $\{S_\alpha; \alpha \in Y\}$ of S is called a *complete semilattice congruence on S*.

A subsemigroup F of a *po*-semigroup S is a *filter* of S if : (1) $a, b \in S, ab \in F$ implies $a \in F$ and $b \in F$; (2) $a \in F, b \in S, a \leq b$ implies $b \in F$. The filter of S generated by $x (x \in S)$ is denoted by $N(x)$. Let $\mathcal{N}$ be an equivalence relation defined, as in [6], by

$$\mathcal{N}:=\{(x,y)\in S\times S|N(x)=N(y)\}.$$

As in [7, 11], for a po-semigroup S, $\mathcal{N}$ is the least complete semilattice congruence on S. If S is intra-regular, then we have $\mathcal{N}=\mathcal{J}$ and S is a complete semilattice Y of simple semigroups $S_\alpha(\alpha\in Y)$.

As in [2], a relation τ on a po-semigroup S is defined by:

$$(a,b)\in\tau\Leftrightarrow(\exists x,y\in S^1)b\le xay \text{ for every } a,b\in S.$$

A po-semigroup S is said to have *Putcha's property* if for $a,b\in S$, $a\tau b$ implies $a^2\tau b^m$ for some $m\in Z^+$.

Lemma 1.2 (cf. [2, Theorem 2.7]). *Let S be a po-semigroup. Then the following conditions are equivalent*:

(1) S *is a semilattice of Archimedean po-semigroup.*

(2) S *has Putcha's property.*

(3) $(\forall a,b\in S)(\exists n\in Z^+)(ab)^n\in(Sa^2S].$

(4) $\mathcal{N}$ *is the greatest semilattice congruence on S such that each of its congruence classes is an Archimedean subsemigroup.*

If S is a po-semigroup with zero 0, we will write $S=S^0$. An element a of S is called *nilpotent* if there exists $n\in Z^+$ such that $a^n=0$. The set of all nilpotents of S is denoted by $Nil(S)$. A semigroup S is a *nil-semigroup* if $S=Nil(S)$. Let K be a proper ideal of a po-semigroup S. Then

$$\rho_K=(K\times K)\bigcup 1_S$$

is a semigroup congruence on S. The quotient semigroup

$$S/\rho_K=\{K\}\bigcup\{\{x\}|x\in S\setminus K\}$$

is the *Rees factor semigroup of S modulo K* and is normally written by S/K rather than S/ρ_K. We may describe S/K as the result of collapsing K into a single (zero) element, while the elements of S outside of K retain their identity. A simple argument shows that S/K with an order "$\le_K$" defined by: for $a=\{x\},b=\{y\},x,y\in S\setminus K$,

$$a\le_K b\Leftrightarrow x\le_K y,\ K\le_K a\Leftrightarrow(\exists z\in K)z\le x$$

is a po-semigroup with zero K.

Definition 1.3. Let K be an ideal of a po-semigroup S. Then $(S/K,\cdot,\le_K)$ is called the *Rees factor semigroup of S modulo K*, and S is called an *ideal extension of K by the po-semigroup S/K*. An ideal extension S of K is called a *nil-extension* of K if S/K is a nil po-semigroup.

By Definition 1.3, we can easily prove the following lemma (see [3]).

Lemma 1.3. *Let S be a po-semigroup and K be an ideal of S. Then the following conditions are equivalent*:

(1) S *is a nil-extension of K.*

(2) $(\forall a\in S)(\exists m\in Z^+)a^m\in K.$

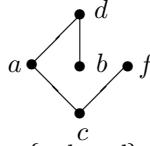**Example 1.1.** We consider the set $S=\{a,b,c,d,f\}$, defined by multiplication and the order below [8]:

.	a	b	c	d	f
a	a	b	c	d	c
b	a	b	c	d	a
c	a	b	c	d	c
d	d	d	d	d	d
f	a	b	c	d	c

$$\leq := \{(a,a),(a,d),(b,b),(b,d),(c,a),(c,c),(c,d),(c,f),(d,d),(f,f)\}.$$

For an easy way to check that S is a *po*-semigroup, we refer to [8]. We give the covering relation and the figure of S:

$$\prec := \{(a,d),(b,d),(c,a),(c,f)\}$$

The ideals of S are the sets: $\{a,b,c,d\}$ and S. Let $K = \{a,b,c,d\}$. Since $f^2 = c \in K$, by Lemma 1.3, it is obvious that S is a nil-extension of K.

Definition 1.4 ([14, 12]). An element a of a *po*-semigroup S is a *left* (*right*) *semiregular* element if $a \leq xaya$ ($a \leq ax'ay'$) for some $x,y,x',y' \in S$.

Equivalent Definition: $a \in (SaSa]$ ($a \in (aSaS]$) for every $a \in S$ [14].

The set of all left (right) semiregular elements of S will be denoted by $LSReg(S)$ ($RSReg(S)$). S is called left (resp. right) semiregular if $S = LSReg(S)$ ($S = RSReg(S)$). S is called semiregular if $S = LSReg(S) \bigcap RSReg(S)$.

Lemma 1.4 ([14]). *Let S be a po-semigroup. Then the following conditions are equivalent*:
  (1) *S is left semiregular.*
  (2) *$(L^2] = L$ for every left ideal L of S.*
  (3) *$L(a) \bigcap L(b) \subseteq (L(a)L(b)]$ for every $a,b \in S$.*

Definition 1.5. Let S be a *po*-semigroup, S is called *left* (*right*) *strongly simple* if S is simple and left (right) semiregular. S is called *strongly simple* if S is both left and right strongly simple.
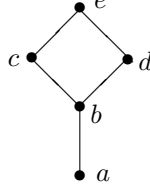
Example 1.2. We consider the ordered semigroup $S = \{a,b,c,d,e\}$, defined by multiplication and the order below [9]:

.	a	b	c	d	e
a	a	a	c	a	c
b	a	a	c	a	c
c	a	a	c	a	c
d	d	d	e	d	e
e	d	d	e	d	e

$$\leq := \{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(b,c),(b,d),$$
$$(b,e),(c,c),(c,e),(d,d),(d,e),(e,e)\}.$$

We give the covering relation and the figure of S:

$$\prec := \{(a,b),(b,c),(b,d),(c,e),(d,e)\},$$

For any $x \in S$, it is enough to prove that $(SxS] = S$. Hence, S is simple. For an easy way to check that S is left semiregular and right semiregular. Thus S is both left and right strongly simple. Hence S is a strongly simple *po*-semigroup.

Now, we generalize the concepts of (intra-, left, right) π-regular semigroups without order to *po*-semigroups. Let S be a *po*-semigroup. The set of all regular (left regular, right regular, intra-regular) elements of S is denoted by $Reg(S)$ $(LReg(S), RReg(S), Intra(S))$.

Definition 1.6. A *po*-semigroup S is called π-*regular* (*left* π-*regular, right* π-*regular, intra-*π-*regular, left* π-*semiregular*) if for every $a \in T$ there exists $n \in Z^+$ such that $a^n \in Reg(S)$ $(LReg(S), RReg(S), Intra(S), LSReg(S))$.

Definition 1.7. A *po*-semigroup S is called *Archimedean*, if for any $a, b \in S$ there exists $n \in Z^+$ such that $a^n \in (SbS]$.

2. Nil-extensions of left strongly simple *po*-semigroups

Now we are ready to prove the main results of this paper. First, we introduce a lemma which will be used frequently in this paper.

Lemma 2.1. *Let po-semigroup S be a complete semilattice Y of subsemigroups $S_\alpha (\alpha \in Y)$. Then*
(1) *the following conditions hold:*

$$LSReg(S) = \bigcup_{\alpha \in Y} LSReg(S_\alpha), \ Semis(S) = \bigcup_{\alpha \in Y} Semis(S_\alpha),$$

$$Intra(S) = \bigcup_{\alpha \in Y} Intra(S_\alpha), \ LReg(S) = \bigcup_{\alpha \in Y} LReg(S_\alpha).$$

(2) *S is left semiregular (semisimple, intra-regular, left regular) if and only if S_α is left semiregular (semisimple, intra-regular, left regular) for all $\alpha \in Y$.*
(3) *S is left π-semiregular (π-semisimple, intra π-regular, left π-regular) if and only if S_α is left π-semiregular (π-semisimple, intra π-regular, left π-regular) for all $\alpha \in Y$.*

Proof. (1) It is obvious that $LSReg(S) \supseteq \bigcup_{\alpha \in Y} LSReg(S_\alpha)$. Let $a \in LSReg(S)$. Then there exists $\alpha \in Y$ and $u \in S_\beta, v \in S_\gamma$ such that $a \in S_\alpha$ and $a \leq uava$. Since Y is a complete semilattice of S, we have $\alpha \leq \beta\alpha\gamma\alpha$. From $\beta\alpha\gamma\alpha \leq \beta\alpha\gamma \leq \beta\alpha \leq \alpha$, which implies $\alpha = \alpha\beta\gamma$. Since $a \leq uava \leq u(uava)v(uava) = (u^2av)a(vuav)a$, in which $u^2av, vuav \in S_{\beta\alpha\gamma} = S_\alpha$, we have that $a \in LSReg(S_\alpha)$. Thus it implies that $LSReg(S_\alpha) \subseteq \bigcup_{\alpha \in Y} LSReg(S_\alpha)$. Hence, the first equality in (1) holds. Similar arguments can show that the other equalities in (1) hold.

Both (2) and (3) are immediate consequences of (1). $\square$

Theorem 2.1. *The following conditions on a po-semigroup S are equivalent*:
(1) S *is left strongly simple.*
(2) $(\forall a, b \in S)a \in (SbSa]$.
(3) *Every left ideal of S is a simple po-semigroup.*
(4) S *is simple and $(L^2] = L$ for every left ideal L of S.*
(5) S *is simple and every left ideal of S is an intra-regular po-semigroup.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) It follows immediately from Lemma 1.1 and the definition of left strongly simple.

(3) $\Rightarrow$ (4) Since S is a left ideal of S itself. By hypothesis, S is simple. Let L be a left ideal of S. It is obvious that $(L^2] \subseteq L$. Let $a \in L$. Then $L(a) = (L(a)aL(a)] \subseteq (L^2(a)] \subseteq (L^2]$. It implies that $L \subseteq (L^2]$. Therefore, we have $(L^2] = L$.

(4) $\Rightarrow$ (5) Let L be a left ideal of S and $a \in L$. By hypothesis, we have $a^3 \in L$ and $S = (Sa^3S]$. Thus

$$a \in L = (L^2] \subseteq (SL] \subseteq ((Sa^3S]L] \subseteq (SLa^2L] \subseteq (La^2L]$$

which shows that L is an intra-regular *po*-semigroup.

(5) $\Rightarrow$ (1) Let $a \in S$, by hypothesis, we have $a \in L(a) = (L(a)a^2L(a)] \subseteq ((a \bigcup Sa]a^2(a \bigcup Sa]] \subseteq (SaSa]$. Thus, S is left semiregular. $\square$

Theorem 2.2. *The following conditions on a po-semigroup S are equivalent*:
(1) S *is a semilattice of left strongly simple po-semigroup.*
(2) $(\forall a \in S)a \in (Sa^nSa]$ *for every $n \in Z^+$.*
(3) $(\forall a \in S)a \in (Sa^2Sa]$.
(4) *Every left ideal of S is an intra-regular subsemigroup of S.*
(5) *Every left ideal of S is a semisimple subsemigroup of S.*

Proof. (1) $\Rightarrow$ (2) Suppose S is a semilattice of Y subsemigroups $S_\alpha(\alpha \in Y)$ which are left strongly simple. Let $a \in S_\alpha$ for some $\alpha \in Y$. Then $a^n \in S_\alpha$ for every $n \in Z^+$. Since S_α is left strongly simple, by Theorem 2.1, we have $a \in (S_\alpha a^n S_\alpha a] \subseteq (Sa^nSa]$.

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (4) Let L be a left ideal of S and $a \in L$. By hypothesis, we have

$$a \in (Sa^2Sa] \subseteq (S(Sa^4Sa^2]Sa] \subseteq (Sa^4Sa^2Sa] \subseteq (SLa^2SL] \subseteq (La^2L].$$

Thus L is an intra-regular subsemigroup of S.

(4) $\Rightarrow$ (5) In view of the hypothesis, a simple argument shows that this is evident.

(5) $\Rightarrow$ (1) Let $a \in S$. In view of the hypothesis, we have $a \in (L(a)aL(a)aL(a)] \subseteq (Sa^2Sa]$. Thus S is an intra-regular and left semiregular. It implies S is a complete semilattice Y of simple semigroup $S_\alpha(\alpha \in Y)$. By Lemma 2.1, S_α is left semiregular, so S_α is left strongly simple. $\square$

The intersection of all ideals of a *po*-semigroup S, if nonempty, is called the *kernel* of S and denoted by $K(S)$. A *po*-semigroup S is called *completely regular* if S is regular, left regular, right regular *po*-semigroup [15]. In [15], we have shown that S is completely regular if and only if $\forall a \in S, a \in (a^2Sa^2]$, equivalently, S is a union of disjoint B-simple subsemigroups of S.

Lemma 2.2 ([3]). *Let S be an Archimedean po-semigroup. If $Intra(S) \neq \emptyset$, then*

(1) S *has a kernel* $K(S)$ *such that*
$$\forall a \in Intra(S), \ K(S) = (SaS], \ Intra(S) \subseteq K(S).$$

(2) S *is a nil-extension of simple po-semigroup* $K(S)$.

Lemma 2.3 ([16]). *Let S be a po-semigroup. Then kernel $K(S)$ of S is a completely regular simple subsemigroup.*

Theorem 2.3. *The following conditions on a po-semigroup S are equivalent:*

(1) S *is a nil-extension of a left strongly simple po-semigroup.*

(2) $(\forall a, b \in S)(\exists n \in Z^+)a^n \in (Sb^mSa^n]$ *for every* $m \in Z^+$.

(3) $(\forall a, b \in S)(\exists n \in Z^+)a^n \in (SbSa^n]$.

(4) S *is Archimedean and left π-semiregular.*

Proof. (1) $\Rightarrow$ (2) Let S be a nil-extension of a left strongly simple *po*-semigroup K. Let $a, b \in S$. Then $a^n \in K$ for some $n \in Z^+$. Since K is an ideal of S, we have $a^nb^m \in KS \subseteq K$ for every $m \in Z^+$. But K is left strongly simple, by Theorem 2.1, for a^n, a^nb^m, we have $a^n \in (Sa^nb^mSa^n] \subseteq (Sb^mSa^n]$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) The implcations follow immediately.

(4) $\Rightarrow$ (1) Suppose S is Archimedean and left π-semiregular. Let $a \in S$. Then there exist $x, y \in S$ and $n \in Z^+$ such that $a^n \leq xa^nya^n \leq (xa^ny)^ma^n$ for every $m \in Z^+$. Since S is Archimedean, for xa^ny and a^{2n} there exist $k \in Z^+$ and $u, v \in S$ such that $(xa^ny)^k \leq ua^{2n}v$. Now

$$a^n \leq xa^nya^n \leq (xa^ny)^ka^n \leq ua^{2n}va^n = ua^{2n}(va^n) \Rightarrow a^n \in (Sa^{2n}S]$$

which shows that $a^n \in Intra(S)$. Hence $Intra(S) \neq \emptyset$. By Lemma 2.2, we conclude that S is a nil-extension of simple *po*-semigroup $K(S)$. By Lemma 2.3, $K(S)$ is a completely regular simple subsemigroup. A simple argument shows that $K(S)$ is left semiregular. Hence S is a nil-extension of a left strongly simple *po*-semigroup. $\square$

Lemma 2.4 ([3])**.** *Let S be a po-semigroup. Then S is Archimedean and intra-regular if and only if S is simple.*

Theorem 2.4. *The following conditions on a po-semigroup S are equivalent*:
  (1) S is a semilattice of nil-extensions of left strongly simple po-semigroups.
  (2) $(\forall a, b \in S)(\exists n \in Z^+)(ab)^n \in (Sa^m S(ab)^n]$ *for every* $m \in Z^+$.
  (3) $(\forall a, b \in S)(\exists n \in Z^+)(ab)^n \in (Sa^2 S(ab)^n]$.
  (4) S is intra π-regular and $LSReg(S) = Intra(S) = Semis(S) = \mathcal{U}_{\mathcal{J}}(S)$.

Proof. (1) $\Rightarrow$ (2) Suppose S is a semilattice Y of subsemigroups $S_\alpha (\alpha \in Y)$ which are nil-extensions of left strongly simple *po*-semigroup. Let $a \in S_\alpha, b \in S_\beta$ for some $\alpha, \beta \in Y$. Then $ab \in S_{\alpha\beta}, a^m b \in S_{\alpha\beta}$ for every $m \in Z^+$, so that there exists $n \in Z^+$ such that

$$(ab)^n \in (S_{\alpha\beta}a^m b S_{\alpha\beta}(ab)^n] \subseteq (Sa^m S(ab)^n]$$

by Theorem 2.3.

  (2) $\Rightarrow$ (3) It is obvious.

  (3) $\Rightarrow$ (4) Let $a \in S$. By hypothesis, there exist $x, y \in S$ and $n \in Z^+$ such that $a^{2n} \leq xa^2 ya^{2n} \leq x(a^2 yx)^m a^2 ya^{2n}$ for every $m \in Z^+$. But for a^2, yx there exist $u_1, v_1 \in S$ and $k_1 \in Z^+$ such that $(a^2 yx)^{k_1} \leq u_1 a^4 v_1 (a^2 yx)^{k_1}$. So we have

$$a^{2n} \leq x(a^2 yx)^{k_1}a^2 ya^{2n} \leq xu_1 a^4 v_1 (a^2 yx)^{k_1}a^2 ya^{2n} \in (Sa^4 Sa^{2n}].$$

Hence there exist $u_2, v_2 \in S$ such that $a^{2n} \leq u_2 a^4 v_2 a^{2n} \leq u_2 (a^4 v_2 u_2)^{k_2}a^4 v_2 a^{2n}$ for every $k_2 \in Z^+$. Continuing this way, then there exist $u_{2n}, v_{2n} \in S$ such that $a^{2n} \leq u_{2n}a^{4n}v_{2n}a^{2n} \in (Sa^{4n}Sa^{2n}]$, which shows that S is intra π-regular and left π-semiregular.

  Let $a \in Semis(S)$. Then $(a)_{\mathcal{J}}$ is a subsemigroup of S. In fact: Let $b, c \in (a)_{\mathcal{J}}$, then $I(a) = I(b) = I(c)$. A simple argument shows that $I(bc) \subseteq I(a)$. Conversely, since $a \in Semis(S)$, then $a \in (SaSaS] \subseteq (SaSaSaSaS] \subseteq (I^5(b)] \subseteq (SbS]$. Similarly, $a \in (ScS]$. Thus we have $a \in (SaSaS] \subseteq (S(ScS]S(SbS]S] \subseteq (ScSbS]$. Then there exist $x, y, z \in S$ such that $a \leq xcybz$. Since $a \in Semis(S)$, there exist $u, v, d \in S$ such that for every $n \in Z^+$

$$a \leq uavad \leq u^{n+1}a(vad)^{n+1} \leq u^{n+1}a(vxcybzd)^{n+1}$$
$$= u^{n+1}avx(cybzdvx)^n cybzd.$$

But for $cyb, zdvx$, by hypothesis, there exist $p, q \in S$ and $m \in Z^+$ such that

$$(cybzdvx)^m \leq p(cyb)^2 q(cybzdvx)^m = (pcy)bc(ybq)(cybzdvx)^m$$
$$\in SI(bc)S \subseteq I(bc).$$

Hence $I(a) \subseteq I(bc)$, so $I(a) = I(bc)$, it implies $bc \in (a)_{\mathcal{J}}$. Thus $(a)_{\mathcal{J}}$ is a subsemigroup of S and $Semis(S) \subseteq \mathcal{U}_{\mathcal{J}}(S)$ is obvious by $a \in (a)_{\mathcal{J}} \in \mathcal{U}_{\mathcal{J}}(S)$.

  On the other hand, let $a \in \mathcal{U}_{\mathcal{J}}(S)$, then there exists $b \in S$ such that $a \in (b)_{\mathcal{J}} \in \mathcal{U}_{\mathcal{J}}(S)$. Since $(b)_{\mathcal{J}}$ is a subsemigroup of S, then $a^4 \in (b)_{\mathcal{J}}$. Hence $I(a^4) = I(b) = I(a)$, so we have $a \in I(a^4) \subseteq (Sa^2 S] \subseteq (SaSaS]$, i.e., $a \in$

$Semis(S)$ and $a \in Intra(S)$. Thus $\mathcal{U}_{\mathcal{J}}(S) \subseteq Semis(S)$ and $\mathcal{U}_{\mathcal{J}}(S) \subseteq Intra(S)$, so that $Semis(S) = \mathcal{U}_{\mathcal{J}}(S)$. By $Semis(S) = \mathcal{U}_{\mathcal{J}}(S)$, a simple argument shows that $LSReg(S) \subseteq Semis(S) = \mathcal{U}_{\mathcal{J}}(S)$ and $Intra(S) \subseteq Semis(S) = \mathcal{U}_{\mathcal{J}}(S)$. Thus $Intra(S) = \mathcal{U}_{\mathcal{J}}(S)$.

By hypothesis, it is obvious that S has Putcha's property. In view of Lemma 1.2, $\mathcal{N}$ is the greatest semilattice congruence on S such that $(x)_{\mathcal{N}}$ is an Archimedean subsemigroup for every $x \in S$. But $\mathcal{N}$ is the least complete semilattice congruence on S. Since S is left π-semiregular and intra π-regular, by Lemma 2.1, $(x)_{\mathcal{N}}$ is left π-semiregular and intra π-regular. From this it follows by Theorem 2.3 that $(x)_{\mathcal{N}}$ is a nil-extensions of left strongly simple po-semigroup K_x for every $x \in S$. Let $c \in Intra(S)$. Then $c \in Intra((c)_{\mathcal{N}})$, i.e., there exist $u, v \in (c)_{\mathcal{N}}$ such that $c \le uc^2v \le u^kc(cv)^k$ for every $k \in Z^+$. For u, cv there exists $n \in Z^+$ such that $u^n, (cv)^n \in K_c$. Hence $c \le u^nc(cv)^n \in K_ccK_c \subseteq K_c \subseteq LSReg(S)$. Thus, $Intra(S) \subseteq LSReg(S)$. We have $Intra(S) = LSReg(S)$.

$(4) \Rightarrow (1)$ Let $a, b \in S$. By S is intra π-regular, then there exist $u, v \in S$ and $n \in Z^+$ such that $(ab)^n \le u(ab)^{2n}v \in (S(ba)^{n+1}S]$ and $(ba)^{n+1} \in (S(ab)^nS]$, which implies that $(ab)^n, (ba)^{n+1} \in ((ab)^n)_{\mathcal{J}}$. Since $(ab)^n \in Intra(S) = Semis(S)$, the process is similar to the proof of $(3) \Rightarrow (4)$, we have $((ab)^n)_{\mathcal{J}}$ is a subsemigroup of S. Then $(ba)^{n+1}(ab)^n \in ((ab)^n)_{\mathcal{J}}$. Now

$$(ab)^n \in I((ba)^{n+1}(ab)^n) = I((ba)^nba^2b(ab)^{n-1}) \subseteq (Sa^2S].$$

Thus S has Putcha's property by Lemma 1.2. By Lemma 1.2, $\mathcal{N}$ is the greatest semilattice congruence on S and the $\mathcal{N}$-class $(a)_{\mathcal{N}}$ of S containing a is an Archimedean subsemigroup for every $a \in S$. Since $\mathcal{N}$ is the least complete semilattice congruence on S, and S is intra π-regular, by Lemma 2.1, $(a)_{\mathcal{N}}$ is intra π-regular. Let $b \in (a)_{\mathcal{N}}$, then there exists $m \in Z^+$ such that $b^m \in Intra((a)_{\mathcal{N}}) \subseteq Intra(S) = LSReg(S)$. By Lemma 2.1, then $b^m \in LSReg((b^m)_{\mathcal{N}})$. Since $(a)_{\mathcal{N}}$ is a subsemigroup, we have $b^m \in (a)_{\mathcal{N}}$, thus $(b^m)_{\mathcal{N}} = (a)_{\mathcal{N}}$, so we get $(a)_{\mathcal{N}}$ is a left π-semiregular po-semigroup. Since $(a)_{\mathcal{N}}$ is Archimedean left π-semiregular po-semigroup. By Theorem 2.3, $(a)_{\mathcal{N}}$ is nil-extensions of left strongly simple po-semigroups. □

3. Chain of left strongly simple po-semigroups

Further, we will consider chains of left strongly simple po-semigroups.

For a po-semigroup S, σ a semilattice congruence on S, we denote by "$\preceq$" the order on the semigroup $S/\sigma = \{(x)_\sigma | x \in S\}$ defined by:

$$(x)_\sigma \preceq (y)_\sigma \Leftrightarrow (x)_\sigma = (xy)_\sigma$$

$(S/\sigma, \cdot, \preceq)$ is a po-semigroup.

Definition 3.1. Let S be a po-semigroup. S is called a *chain* of left strongly simple po-semigroups if there exists a semilattice congruence σ on S such that $(x)_\sigma$ is a left strongly simple po-subsemigroup of S for every $x \in S$ and $(S/\sigma, \preceq)$ is a chain.

Theorem 3.1. *Let S be a po-semigroup, $\mathcal{C}(S)$ the set of ideals of S. The following conditions are equivalent*:

(1) *S is a chain of left strongly simple semigroups.*

(2) *$(\forall a, b \in S)a \in (SabSa]$ or $b \in (SabSb]$.*

(3) *$(\forall a \in S)a \in (Sa^2Sa]$ and every ideal of S is prime.*

(4) *$(\forall a \in S)a \in (Sa^2Sa]$ and $(\mathcal{C}(S), \subseteq)$ is a chain.*

Proof. $(1) \Rightarrow (2)$ Let σ be a semilattice congruence of S such that $(x)_\sigma$ is a left strongly simple semigroup of S for every $x \in S$ and $(S/\sigma, \preceq)$ is a chain. Let $a, b \in S$. For $(a)_\sigma, (b)_\sigma$, we have $(a)_\sigma \preceq (b)_\sigma$ or $(b)_\sigma \preceq (a)_\sigma$. If $(a)_\sigma \preceq (b)_\sigma$, then $a, ab \in (a)_\sigma$, and by Theorem 2.1, we have $a \in ((a)_\sigma ab(a)_\sigma a] \subseteq (SabSa]$. If $(b)_\sigma \preceq (a)_\sigma$, similar arguments can show that $b \in (SabSb]$.

$(2) \Rightarrow (3)$ It is obvious.

$(3) \Rightarrow (4)$ Let $\mathcal{C}(S)$ be the set of ideals of S. Let $A, B \in \mathcal{C}(S)$ such that $A \not\subseteq B$. We prove that $B \subseteq A$. Let $a \in A, a \notin B$ and $b \in B$. Then $ab \in AB \subseteq A, B$. In view of hypothesis, we have $ab \in (S(ab)^2Sab] \subseteq (SabS]$. Since $(SabS]$ is an ideal of S, we have $a \in (SabS]$ or $b \in (SabS]$. If $a \in (SabS]$, then $a \in (SabS] \subseteq (SBS] \subseteq (B] = B$, impossible. Thus $b \in (SabS] \subseteq (SAS] \subseteq (A] = A$.

$(4) \Rightarrow (1)$ In view of the hypothesis, it is clear that S is intra-regular. By [13, Theorem 2], there exists a complete semilattice congruence σ of S such that $(x)_\sigma$ is a simple *po*-semigroup of S for every $x \in S$ and $(S/\sigma, \preceq)$ is a chain. Let $a \in (x)_\sigma$. By hypothesis, we have $a \in (Sa^2Sa]$. Assume that $u, v \in S$ such that $a \leq ua^2va$. Then, taking into account the fact that σ is complete, we have $(a, aua^2va) \in \sigma$. Since σ is a semilattice congruence on S, we have $(aua^2va, a^2ua^2v) \in \sigma$. Since $(a^2, a) \in \sigma$, we have $(aua^2va, aua^2v) \in \sigma$. Then, since $(u, u^2) \in \sigma$, we have $(aua^2v, au^2a^2v) \in \sigma$. Then, since $(au^2a^2v, uaua^2v) \in \sigma$, we have $(aua^2va, uaua^2v) \in \sigma$. Therefore, we get that $(a, uaua^2v) \in \sigma$ and $uaua^2v \in (a)_\sigma$. Analogously, $vaua^2v \in (a)_\sigma$. Now

$$a \leq ua^2va \leq ua(ua^2va)va \leq (uaua^2v)ava(ua^2va) = (uaua^2v)a(vaua^2v)a$$

in which $uaua^2v, vaua^2v \in (a)_\sigma$, we shows that $(x)_\sigma$ is a left semiregular subsemigroup. Thus, $(x)_\sigma$ is a left strongly simple subsemigroup. □

Theorem 3.2. *The following conditions on a po-semigroup S are equivalent*:

(1) *S is a chain of nil-extension of left strongly simple semigroups.*

(2) *$(\forall a, b \in S)(\exists n \in Z^+)a^n \in (Sa^mbSa^n]$ or $b^n \in (Sab^mSb^n]$ for every $m \in Z^+$.*

(3) *$(\forall a, b \in S)(\exists n \in Z^+)a^n \in (SabSa^n]$ or $b^n \in (SabSb^n]$.*

(4) *S is left π-semiregular and $LSReg(S)$ is a chain of left strongly simple po-semigroups.*

Proof. $(1) \Rightarrow (2)$ Let σ be a semilattice congruence of S such that $(x)_\sigma$ is a nil-extension of left strongly simple semigroup K_x of S for every $x \in S$ and $(S/\sigma, \preceq)$ is a chain. Let $a, b \in S$. For $(a)_\sigma, (b)_\sigma$, we have $(a)_\sigma \preceq (b)_\sigma$ or

$(b)_\sigma \preceq (a)_\sigma$. If $(a)_\sigma \preceq (b)_\sigma$, then $a, ab \in (a)_\sigma$, so for every $m \in Z^+$, we have $a, a^m b \in (a)_\sigma$. By Theorem 3.1, then $a^n \in ((a)_\sigma a^m b(a)_\sigma a^n] \subseteq (Sa^m b Sa^n]$. If $(b)_\sigma \preceq (a)_\sigma$, in a similar way, we obtain $b^n \in (Sab^m Sb^n]$ for every $m \in Z^+$.

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (4) It is clear that S is an Archimedean po-semigroup and $a^n \in (Sa^2 Sa^n]$ for every $a \in S$. Then there exist $x, y \in S$ such that $a^n \leq xa^2 ya^n \leq (xa^2 y)^k a^n$ for every $k \in Z^+$. For $xa^2 y, a^n$ there exist $u, v \in S$ and $m \in Z^+$ such that $(xa^2 y)^m \leq ua^n v$. Now, we have $a^n \leq (xa^2 y)^m a^n \leq ua^n va^n$, which shows that $a^n \in LSReg(S)$. Hence S is left π-semiregular.

Let $a, b \in LSReg(S)$. Then there exist $u, v \in S$ such that $a \leq uava \leq (uav)^m a$ for every $m \in Z^+$. Since S is Archimedean, for uav, ab there exist $x, y \in S$ and $k \in Z^+$ such that $(uav)^k \leq xaby$. Now we have $ab \leq (uav)^k ab \leq xabyab$, so $ab \in LSReg(S)$, i.e., $LSReg(S)$ is a subsemigroup.

On the other hand, $LSReg(S)$ is simple. In fact: Let $a \in LSReg(S)$, then there exist $u, v \in S$ such that $a \leq uava \leq u^n a(va)^n$ for every $n \in Z^+$. Since S is left π-semiregular, for u, va there exists $k \in Z^+$ such that $u^k, (va)^k \in LSReg(S)$. Hence $a \in (LSReg(S)aLSReg(S)] \subseteq LSReg(S)$, so $(LSReg(S)aLSReg(S)] = LSReg(S)$.

Let $K = LSReg(S), a, b \in K$. For $a, ab \in K$, then $K = (KabK]$. Thus there exist $x, y, u, v \in K$ such that $a \leq xaya \leq xuabvya \in (KabKa]$. For $b, ab \in K$, in a similar way, we have $b \in (KabKb]$. By Theorem 3.1, K is a chain of left strongly simple po-semigroups.

(4) $\Rightarrow$ (1) Let $a, b \in S$. Since S is left π-semiregular, then there exist $n, m \in Z^+$ such that $(ab)^n, (ba)^m \in LSReg(S)$. For $(ba)^m, (ab)^n$, by Theorem 3.1, we have

$$(ba)^m \in (LSReg(S)(ba)^m(ab)^n LSReg(S)(ba)^m] \subseteq (Sa^2 S]$$

or

$$(ab)^n \in (LSReg(S)(ba)^m(ab)^n LSReg(S)(ab)^n] \subseteq (Sa^2 S].$$

Thus $(ab)^{mn} \in (S(ab)^{mn}S(ab)^{mn}] \subseteq (Sa^2 S(ab)^{mn}] \subseteq (Sa^2 S]$. From this it follows by Lemma 1.2 that S has Putcha's property. In view of Lemma 1.2, $\mathcal{N}$ is the greatest semilattice congruence on S such that $(x)_\mathcal{N}$ is an Archimedean subsemigroup for every $x \in S$. Since $\mathcal{N}$ is the least complete semilattice congruence on S, and S is left π-semiregular, by Lemma 2.1, $(x)_\mathcal{N}$ is left π-semiregular. From this it follows by Theorem 2.3 that $(x)_\mathcal{N}$ is a nil-extensions of left strongly simple po-semigroups.

Moreover, Let $(a)_\sigma, (b)_\sigma \in S/\mathcal{N}$. Since S is left π-semiregular, then there exist $n, m \in Z^+$ such that $a^n, b^m \in LSReg(S)$. For a^n, b^m, by Theorem 3.1, we have

$$a^n \in (LSReg(S)a^n b^m LSReg(S)a^n] \subseteq (SabS] \subseteq I(ab)$$

or

$$b^m \in (LSReg(S)a^n b^m LSReg(S)b^m] \subseteq (SabS] \subseteq I(ab).$$

If $a^n \in (SabS]$, then there exist $u, v \in S$ such that $a^n \leq uabv$. Then $N(a) \ni a^n \leq uabv$, so we have $uabv \in N(a)$, it is implies $ab \in N(a)$, we get $N(ab) \subseteq N(a)$. If $b^m \in (SabS]$, in a similar way, we have $N(ab) \subseteq N(b)$. On the other hand, $ab \in N(ab)$, we have $a, b \in N(ab)$. Therefore, $N(a) \subseteq N(ab)$ and $N(b) \subseteq N(ab)$. Thus, we have $N(ab) = N(a)$ or $N(ab) = N(b)$ i.e., $(a)_\mathcal{N} = (ab)_\mathcal{N}$ or $(b)_\mathcal{N} = (ab)_\mathcal{N}$. We have $(a)_\mathcal{N} \preceq (b)_\mathcal{N}$ or $(b)_\mathcal{N} \preceq (a)_\mathcal{N}$. $\qquad\square$

References

[1] G. Birkhoff, *Lattice Theory, Vol. XXV*, American Math. Soc. Coll. Publ., Providence, 1967.

[2] Y. Cao, *On weak commutativity of po-semigroups and their semilattice decompositions*, Semigroup Forum **58** (1999), no. 3, 386–394.

[3] Y. Cao and X. Xu, *Nil-extensions of simple po-semigroups*, Comm. Algebra **28** (2000), no. 5, 2477–2496.

[4] A. H. Clifford, *Extensions of semigroups*, Trans. Amer. Math. Soc. **68** (1950), 165–173.

[5] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups. Vol. I*, Mathematical Surveys, no. 7, American Mathematical Society, Rhode Island, 1964.

[6] N. Kehayopulu, *Note on Green's relations in ordered semigroups*, Math. Japon. **36** (1991), no. 2, 211–214.

[7] _____, *Remark on ordered semigroups*, Math. Japon. **35** (1990), no. 6, 1061–1063.

[8] _____, *On intra-regular ordered semigroups*, Semigroup Forum **46** (1993), no. 1, 271–278.

[9] _____, *On completely regular poe-semigroups*, Math. Japon. **37** (1992), no. 1, 123–130.

[10] N. Kehayopulu and M. Tsingelis, *Ideal extensions of ordered semigroups*, Comm. Algebra **31** (2003), no. 10, 4939–4969.

[11] _____, *Remark on ordered semigroups*, Sovremennaja Algebra, St. Petersburg Gos. Ped. Herzen Inst., 1992, 56–63.

[12] S. K. Lee and Y. I. Kwon, *On left, right weakly prime ideals on po-semigroups*, Commun. Korean Math. Soc. **11** (1996), no. 2, 315–321.

[13] M. Petrich, *Introduction to Semigroups*, Merrill Research and Lecture Series, Charles E. Merrill Publishing, Ohio, 1973.

[14] Q. S. Zhu, *A note on left (right) semiregular po-semigroups*, Far East J. Math. Sci. **26** (2007), no. 3, 697–704.

[15] _____, *On characterizations of completely regular ordered semigroups*, Southeast Asian Bull. Math. **29** (2005), no. 4, 827–834.

[16] Q. S. Zhu and T. F. Qing, *On kernel in ordered semigroups*, Journal of Information Engineering University(China) **9** (2008), no. 4, 492–494.

DEPARTMENT OF APPLIED MATHEMATICS AND PHYSICS
INSTITUTE OF SCIENCES
INFORMATION ENGINEERING UNIVERSITY
ZHENGZHOU 450001, P. R. CHINA
E-mail address: qingshunzhu@yahoo.com.cn