한국생산제조시스템학회지 Vol.20 No.4 2011. 8. pp. 412~418
Korean Society of Manufacturing Technology Engineers

Comparative Study on Reliability-Based Topology Optimization

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(논문접수일 2010. 07. 13, 심사완료일 2010. 08. 18)

신뢰성 기반 위상최적화에 대한 비교 연구

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Abstract

Reliability-based Topology optimization(RBTO) is to get an optimal design satisfying uncertainties of design variables. Although RBTO based on homogenization and density distribution method has been done, RBTO based on BESO has not been reported yet. This study presents a reliability-based topology optimization(RBTO) using bi-directional evolutionary structural optimization(BESO). Topology optimization is formulated as volume minimization problem with probabilistic displacement constraint. Young's modulus, external load and thickness are considered as uncertain variables. In order to compute reliability index, four methods, i.e., RIA, PMA, SLSV and ADL(adaptive-loop), are used. Reliability-based topology optimization design process is conducted to obtain optimal topology satisfying allowable displacement and target reliability index with the above four methods, and then each result is compared with respect to numerical stability and computing time. The results of this study show that the RBTO based on BESO using the four methods can effectively be applied for topology optimization. And it was confirmed that DLSV and ADL had better numerical efficiency than SLSV. ADL and SLSV had better time cost than DLSV. Consequently, ADL method showed the best time efficiency and good numerical stability.

Key Words : Reliability-Base Topology Optimization (신뢰성기반 위상최적화), Reliability Index Approach (신뢰성 지수 접근법), Performance Measure Approach (목표 성능치 접근접), Single-Loop Single Vector (싱글루프싱글벡터), Adaptive-Loop (적응형 루프), Bi-directional Evolutionary Structural Optimization (양방향 진화적 구조최적화)

1. Introduction

A structure that is substantial to original functions, safe, and economical can be said to be the most optimal structure. The actual design of a structure contains uncertainty parameters such as material properties, external loads, and dimensions. Current design has mainly been done based on a safety factor according to the designer's experience in order to

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consider such uncertainties. However, a factor of safety cannot consider uncertainty systematically, because it is based on a theory that the maximum stress of a member developed by external loads should be less than or equal to allowable stress.

Reliability-based design optimization (RBDO) has been developed to get an optimal design satisfying such uncertainties[1]. Evaluation of probability constraint has commonly used an improved first-order second-moment (AFOSM) method. The double-loop method is classified into reliability index approach (RIA) and performance measure approach (PMA)[2] depending on the application of the AFOSM method. These methods, however, require plenty of time due to relatively excessive calculation as compared to deterministic optimization. The reason is that they involve an internal optimization process separately from the external optimization process to find the most probable failure point (MPP). Design optimization having both internal and external optimization processes is called a double-loop single-vector (DLSV) method.

In order to supplement the disadvantage of the excessive computing time of DLSV, Chen, et al.[3] proposed a method replacing the internal optimization process to find the MPP by an analytical method. The iteration process needed to obtain the MPP in the internal optimization process in DLSV is therefore not necessary any more. It is called a single-loop single-vector (SLSV) method. Therefore, the SLSV method can reduce reliability analysis time during the optimization process. However, it has a disadvantage of numerical instability.

Therefore, Choi[4] proposed an adaptive-loop (ADL) method with the rapid convergence rate of SLSV and the numerical stability of PMA. ADL is comprised of deterministic optimization, a parallel loop, and a single loop. Thus, deterministic optimization usually performs very fast from the initial design point as the first step. Reliability analysis is performed using PMA, a parallel loop with numerical stability, as the second step, and then SLSV, a single loop with a rapid convergence rate, which results in reduction of computing time cost, as the last step.

In recent years, the study of reliability-based topology optimization (RBTO)[5,6] that can consider the uncertainty variables has actively been progressing. Although RBTO based on homogenization and density distribution method[5] has been done, RBTO based on BESO has not been reported yet.

This study executed the RBTO based on BESO using the RIA, PMA, SLSV, and ADL methods. Uncertainty variables were chosen as Young's modulus, external load, and thickness in this paper. Each topology and efficiency of convergence rate and numerical stability were compared to each other.

2. Bi-directional Evolutionary Structure Optimization

The BESO method[7] maximizes stiffness of a structure by simultaneously adding and removing an element from the structure. Therefore, a topology optimization problem to maximize stiffness with a volume constraint can be formulated as a problem to minimize compliance as follows:

$$\begin{aligned} &\text{Minimize} && C - \frac{1}{2} f^T u \\ &\text{subject to} && V^* - \sum_{i=1}^{N} V_i x_i = 0 \end{aligned} \qquad (1)$$

where, **f** and **u** are applied load and displacement vector, respectively. C is known as the mean compliance. V^* means a prescribed target volume, and N is the total number of elements in a structure. A binary design variable x_i declares the absence (0) or presence (1) of an element.

When the i th element is removed from a structure, the change (elemental sensitivity number) of the mean compliance or total strain energy is equal to the elemental strain energy as

$$\alpha_i^e \triangle C_i = \frac{1}{2}\{u_i\}^T [K_i] \{u_i\} \qquad (2)$$

where, $\{u_i\}$ is the nodal displacement vector of the th element and $[K_i]$ is the element stiffness matrix. BESO[8] performs topology optimization by adding and removing elements based on the sensitivity numbers according to the criterion.

3. Reliability-based design optimization

Reliability-based design optimization aims at acquiring the target reliability as well as minimizing the objective function under the required constraint. Formulation of the RBDO is as follows.

$$\begin{aligned} &Min && f(X) \\ &S.t. && P_f = P(G \le 0) \le P_t \end{aligned} \tag{3}$$

In the above Eq. (3), G is a limit state function, meaning a constraint in the formulation of general optimum design. P_f means the failure probability of a system, and P_t means the required target failure probability.

The volume of a structure as an objective function and the displacement of a particular position as a limit state function can be expressed from Eq. (3) as follows:

$$\begin{aligned} &Min && Volume \\ &S.t. && P[G \le 0] \le P_t \\ & && G = \delta_{allow} - \delta \end{aligned} \tag{4}$$

3.1 Formulation of RIA and PMA

Given that a structure is failed when a limit state equation G < 0, the probabilistic constraint in the RBDO can be expressed as follows:

$$\begin{aligned} &P_f(X) = \\ &F_G(0) = P[G<0] = \Phi(-\beta) \le P_t = \Phi(-\beta_t) \end{aligned} \tag{5}$$

where, $\Phi(\cdot)$ represents a standard normal distribution function.

$$F_G(0) \le \Phi(-\beta_t) \tag{6}$$

Eq. (6) can be expressed in two forms as follows:

$$\begin{aligned} &\beta = -\Phi^{-1}[F_G(0)] \ge \beta_t \\ &G^* = F_G^{-1}[\Phi(-\beta_t)] \ge 0 \end{aligned} \tag{7}$$

where, G^* means a target performance value for a target reliability index, β_t, and Eq. (7) is used as a constraint in RIA and PMA[2], respectively.

Using the above equation, the formulation of the RBDO can be expressed as follows:

When using an RIA constraint,

$$\begin{aligned} &Min && V(X) \\ &S.t. && \beta = -\Phi^{-1}[F_G(0)] \ge \beta_t \end{aligned} \tag{8}$$

When using an RIA constraint,

$$\begin{aligned} &Min && V(X) \\ &S.t. && G = F_G^{-1}[\Phi(-\beta_t)] \ge 0 \end{aligned} \tag{9}$$

As above, a reliability index can be obtained from the shortest distance between the origin and the limit state function in a standard normal distribution coordinate system. The problem of sub-optimization to obtain an RIA or PMA constraint can be formulated as follows:

For an RIA constraint,

$$\begin{aligned} &Min && \| U \| = \sqrt{U^T U} \\ &S.t. && G_U(U) = 0 \end{aligned} \tag{10}$$

For a PMA constraint,

$$\begin{aligned} &Min && G_U(U) \\ &S.t. && \| U \| = \beta_t \end{aligned} \tag{11}$$

where, U is an uncertainty variable transformed into a standard normal distribution coordinate system.

3.2 Formulation of a Single-loop Single-vector method

The most probable failure point Z^* satisfying a limit state function can be defined as the shortest distance between a probability variable Z and a mean point μ_Z, which is a problem to minimize the distance, being defined as follows:

$$\begin{aligned} &Min && D(Z) \\ &S.t. && G(Z) = 0 \end{aligned} \tag{12}$$

The vector from the mean point to the limit state function can be expressed as follows:

$$Z^* - \mu_Z \pm \beta_t \alpha^* \tag{13}$$

where, α^* is a unit normal vector at the MPP, being expressed as follows:

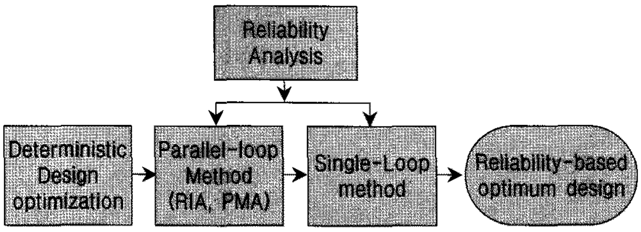


Fig. 1 Adaptive-loop method for reliability analysis

$$\alpha^* = \frac{\nabla G(Z)}{\| \nabla G(Z) \|} \tag{14}$$

where, α^* lies on the same line as $Z^* - \mu_Z$, a positive number in case of the same direction, and a negative number in case of the opposite direction. If the safe region of $G(Z)$ is negative, a limit state equation is given in a type of $G(Z) < 0$, then Z^* can be obtained as follows:

$$Z^* = \mu_Z + \beta_t \alpha^* \tag{15}$$

On the contrary, if the safe region is positive in a type of $G(Z) > 0$, then Z^* can be obtained as follows:

$$Z^* = \mu_Z - \beta_t \alpha^* \tag{16}$$

The optimization problem considering this can be formulated as follows:

$$\begin{aligned} &Min && V(X) \\ &S.t. && G(Z^{(k)}) \geq 0 \\ & && where\ Z^{*(k)} = \mu_Z^{(k)} - \beta_t \alpha^{*(k-1)} \end{aligned} \tag{17}$$

The SLSV method can calculate the MPP analytically, so it can reduce the calculation time more than a double-loop single-vector method. However, this method has a disadvantage[3] in that it has numerical instability in the process of finding the MPP when the design variation is relatively large during an external optimization process, because it uses the sensitivity in a limit state equation at the previous step.

3.3 Adaptive-loop method

ADL[6] is a reliability analysis method having the advantages of the numerical stability that a double-loop single-vector has and the rapid convergence rate that a single-loop single-vector has, as shown in Fig. 1.

The condition connecting the deterministic optimization with the reliability analysis is as follows:

$$q = \frac{|G(X^0)|}{|G(X^0 + X^*) - G(X^0)|} \gg 1 \tag{18}$$

where, q is a distance index, which plays a role of identifying reliability analysis from deterministic optimization, usually based on 1 according to the designer's experience. Deterministic optimization is performed when a distance index is larger than 1, and reliability analysis is performed when it is less than 1. However, in this paper, the DTO process in the ADL method was controlled to converge at around 30% of the initial distance index q, so it was set to be 0.5 for good time efficiency, and then PMA and SLSV methods were implemented as mentioned above.

3.3 Sensitivity analysis of probability constraint

A limit state function in reliability analysis can be defined as follows:

$$G = \delta_{allow} - \delta \tag{19}$$

Sensitivity can be obtained using a direct variation method (DVM) as follows:

$$\begin{aligned} \frac{\partial G}{\partial E} &= -\frac{\partial u}{\partial E} = \frac{u}{E} \\ \frac{\partial G}{\partial P} &= -\frac{\partial u}{\partial P} = -\frac{u}{P} \\ \frac{\partial G}{\partial t} &= -\frac{\partial u}{\partial t} = \frac{u}{t} \end{aligned} \tag{20}$$

A finite difference method (FDM) was used to verify the sensitivity derived in the above. A cantilever beam was used as a verification model, and the result of verification using a 1% forward finite difference method confirmed that it was a reliable level.

4. Numerical Examples

4.1 RBTO of a short cantilever beam

For a cantilever beam as shown in Fig. 2, topology optimization considering a probability constraint was performed, and the obtained optimal topologies by DTO were compared with those by RBTO. Dimensions of a cantilever beam are $L=8mm$, $h=50mm$, and $t=1mm$. Material properties were Young's modulus $E=100GPa$, Poisson's ratio $\nu=0.3$, and the load $P=100N$. The design domain was divided into 80×50 rectangular elements to perform topology optimization using BESO. Reliability analysis considered Young's modulus E, thickness t, and load P as uncertainty variables, and $\delta_{allow}=0.06$ was set.

As reliability analysis methods, RIA, PMA, SLSV, and ADL were used. It was assumed that uncertainty variables had a normal distribution, were probabilistically independent from each other, and had 10% variance from the mean values.

The target reliability index was defined as $\beta_t=3$ corresponding to $P_f=0.125\%$. The above reliability-based topology optimization problem can be expressed as follows:

$$
\begin{aligned}
&Min && V(X) \\
&S.t. && P_f=P(G\le 0)\le 0.125\%(\beta_t=3) \\
& && G=0.06-\delta
\end{aligned}
\qquad (21)
$$

Deterministic topology optimization can be formulated as follows:

$$
\begin{aligned}
&Min && V(X) \\
&S.t. && G=0.06-\delta\le 0
\end{aligned}
\qquad (22)
$$

In Table 2, the results of DTO and RBTO were arranged with optimal volume (V), maximum displacement (δ), and reliability index (β).

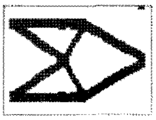
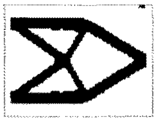
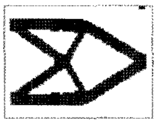
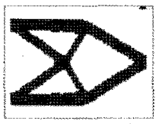
It can be seen that the reliability-based topology optimization used about 5% more materials than DTO, but its reliability index is 3.02, which is much better than DTO's reliability index, 0.05.

ADL method's ratio was 1.06 when the distance index

Table 1 Comparative table of sensitivity

uncertainty	DVM	1% FDM	ERROR (%)
Young's Modulus	2.3278e-07	2.3000e-07	1.2
Load	-2.3278e-0-04	-2.3300e-04	0.1

Table 2 Comparative table of sensitivity analysis

	Topology	V (%)	δ	β	Ratio
DTO		31.5	0.06	0.05	1
PMA		36.4	0.0507	3.02	2.80
SLSV		36.4	0.0507	3.02	1.96
ADL		36.4	0.0507	3.02	1.06

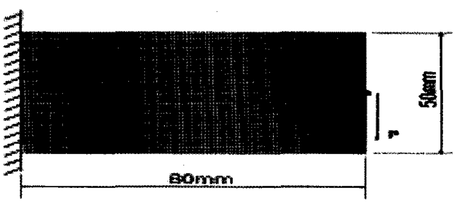


Fig. 2 Initial design of a catilever beam

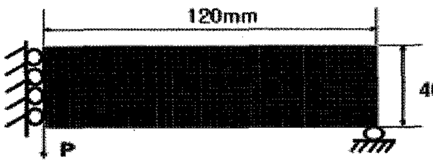
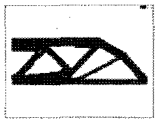
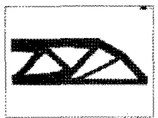
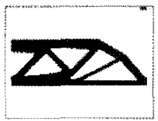
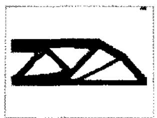


Fig. 3 Initial design of a MBB beam

Table 3 Comparison between DTO and RBTO obtained from a MBB beam .

	Topology	V (%)	δ	β	Ratio
DTO		39.3	0.230	0.33	1
PMA		47.7	0.1948	3.00	1.89
SLSV		47.5	0.1948	3.01	1.36
ADL		47.7	0.1948	3.00	1.01

was set to be 0.5, around 30% of the initial distance index. However, The ratio was 1.85 when set to be 1 as recommended. As shown above, the distance index around 30% of the initial distance index had better time efficiency than the distance index 1 as recommended.

4.2 RBTO of a MBB beam

The second example was performed by RBTO using RIA, PMA, SLSV, and ADL for a MBB beam of length $120mm$, height $40mm$ and thickness $1mm$ as shown in Fig. 3, and each result was compared with that of the DTO. Material properties were Young's modulus $E=100GPa$, Poisson's ratio $\nu=0.3$, and external load $P=100N$.

The design domain was divided into 4800 elements of 120×40, Young's modulus E, load P, and thickness t were considered as uncertainty variables, and $\delta_{allow}=0.23$ was set. Each uncertainty variable was assumed as a normal distribution with 10% variance from the mean values. A target reliability index was set as $\beta_t=3$ corresponding to $P_f=0.125\%$.

Table 3 shows the results of DTO and RBTO. RIA and PMA converged to the same volume, but the reliability index differed in the third decimal place. The reliability index, however, expresses the failure probability to the second decimal place, so RIA and PMA indicate the same failure probability. SLSV shows different reliability indices of RIA and PMA. It is known that SLSV indicates numerical instability but rapid convergence rate compared to other methods. It can be seen that the reliability-based topology optimization used about 8.5% more materials than DTO, but its reliability index is 3.0, which is much better than DTO's reliability index, 0.33.

The ADL method shows the best time efficiency among the methods, and almost the same time cost as DTO. The DTO process in the ADL method was also controlled to converge at around 30% of the initial distance index q as the cantilever beam, so it was set to be 0.5 with good time efficiency, and then PMA and SLSV methods were implemented as mentioned above. Also, it is known that the ADL method shows very good numerical stability.

The ADL method`s ratio was 1.01 when the distance index was set to be 0.5, around 30% of the initial distance index. However, the ratio was 1.33 when set to be 1 as recommended. As shown above, the distance index around 30% of the initial distance index had better time efficiency than the distance index 1 as recommended.

5. Conclusions

The following conclusions can be obtained from the results of the reliability-based topology optimization using RIA, PMA, SLSV, and ADL for two examples.

(1) Optimal topologies based on BESO satisfying the target reliability index for all reliability methods were obtained.
(2) It was confirmed that RIA and PMA had lower time efficiency than SLSV, but had better efficiency in numerical stability. Conversely, SLSV showed numerical instability but rapid convergence rate compared to other methods.
(3) It was confirmed that the ADL method showed the best time efficiency among the methods, and almost the same time cost as DTO. Also, it showed very good numerical stability.
(4) It was seen that the adaptive loop had time efficiency depending on the value setting of the distance index relying on experience, and the distance index around 30% of the initial distance index had better time efficiency than the distance index 1 as recommended.

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