

CONVOLUTIONS OF WHITE NOISE OPERATORS

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ABSTRACT. Motivated by the convolution product of white noise functionals, we introduce a new notion of convolution products of white noise operators. Then we study several interesting relations between the convolution products and the quantum generalized Fourier-Mehler transforms, and study a quantum-classical correspondence.

1. Introduction

The white noise theory initiated by Hida [5] to give rigorous meaning of white noise as the time derivative of the Brownian motion has been extensively developed with wide applications to stochastic calculus, mathematical finance and mathematical physics, etc. One of the important applications of the white noise theory is to infinite dimensional analysis, specially infinite dimensional harmonic analysis, in which the convolution product [11] and Fourier-Gauss transforms [3, 11], generally with operator parameters [2], of white noise functionals were studied. The quantum white noise theory [8] has been developed based on the quantum decomposition of the (classical) white noise into the sum of the pointwise annihilation and creation operators. The quantum white noise theory plays an important role in the study of problems including singularities concerned with (white noise) operators, and then widely applied to quantum stochastic calculus [6, 10, 12, 15], see also [13] and references cited therein. On the other hand, a convolution of white noise operators was studied in [1] and the Fourier-Gauss transforms were generalized in [7] as transforms acting on white noise operators.

In this paper, motivated by the convolution product on white noise functionals and studies in [1], we introduce a new notion of convolution products of white noise operators and study several interesting relations between the convolutions and the quantum generalized Fourier-Mehler transforms of white noise operators.

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This paper is organized as follows: In Section 2 we recall basic notions and well-known results in white noise theory, which are necessary for our study, see [11, 13]. In Section 3 we introduce convolution products of white noise operators and then study relations between the convolution products and the quantum generalized Fourier-Mehler transforms, finally we study a quantum-classical correspondence including various relations between (classical) convolution products and generalized Fourier-Mehler transforms.

2. Preliminaries

2.1. Ridging of Fock space

Let H be a (complex) Hilbert space with norm $|\cdot|_0$. Let A be a selfadjoint operator in H such that $\|A^{-1}\|_{\text{OP}} < 1$ and $\|A^{-1}\|_{\text{HS}}^2 < \infty$. By the standard construction from H and A , we have a Gelfand triple:

$$E \subset H \subset E^*,$$

see [11, 14], where E^* is the strong dual space of E . For each $p \in \mathbb{R}$, $|\xi|_p \equiv |A^p \xi|_0$. The topology of E is defined by the Hilbertian norms $\{|\cdot|_p : p \in \mathbb{R}\}$, and then E becomes a countable Hilbert nuclear space. The canonical bilinear form on $E^* \times E$ is denoted by $\langle \cdot, \cdot \rangle$.

The (Boson) Fock space over H , denoted by $\Gamma(H)$, is the Hilbert space consisting of all sequences $(f_n)_{n=0}^\infty$ such that $f_n \in H^{\widehat{\otimes} n}$ the n -fold symmetric tensor product of H and $\sum_{n=0}^\infty n! |f_n|_0^2 < \infty$, where $H^{\widehat{\otimes} 0} = \mathbb{C}$. Let $\Gamma(A)$ be the second quantization of the operator A defined by

$$\Gamma(A)\phi = (A^{\otimes n} f_n)_{n=0}^\infty, \quad \phi = (f_n)_{n=0}^\infty \in \Gamma(H).$$

Then $\Gamma(A)$ is a selfadjoint operator in $\Gamma(H)$ with $\|\Gamma(A)^{-1}\|_{\text{OP}} < 1$ and $\|\Gamma(A)^{-1}\|_{\text{HS}} < \infty$. By the standard construction from $\Gamma(H)$ and $\Gamma(A)$, we have a Gelfand triple:

$$(E) \subset \Gamma(H) \subset (E)^*,$$

where (E) is the space of all $\phi = (f_n)_{n=0}^\infty$ in $\Gamma(H)$ with $f_n \in E^{\widehat{\otimes} n}$ such that $\|\phi\|_p^2 = \sum_{n=0}^\infty n! |f_n|_p^2 < \infty$ for all $p \geq 0$ and $(E)^*$ is the strong dual space of (E) . Moreover, it is known that for each $\Phi \in (E)^*$ there exists a unique sequence $\{F_n\}_{n=0}^\infty$ with $F_n \in (E^{\widehat{\otimes} n})_{\text{sym}}^*$ such that

$$\langle \Phi, \phi \rangle = \sum_{n=0}^\infty n! \langle F_n, f_n \rangle, \quad \phi = (f_n)_{n=0}^\infty \in (E).$$

For each $\xi \in E$, an *exponential vector* (or *coherent vector*) ϕ_ξ is defined by

$$\phi_\xi = \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \dots, \frac{\xi^{\otimes n}}{n!}, \dots \right).$$

It is well-known that $\{\phi_\xi : \xi \in E\}$ spans a dense subspace of (E) .

2.2. White noise operators

A continuous linear operator Ξ from (E) into $(E)^*$ is called a *white noise operator*. Let $\mathcal{L}((E), (E)^*)$ denote the space of all white noise operators equipped with the topology of bounded convergence. The *symbol* of $\Xi \in \mathcal{L}((E), (E)^*)$ is defined by

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle, \quad \xi, \eta \in E.$$

Since $\{\phi_\xi : \xi \in E\}$ spans a dense subspace of (E) , every $\Xi \in \mathcal{L}((E), (E)^*)$ is uniquely determined by its symbol by virtue of the following theorem.

Theorem 2.1 ([9, 14]). *Let $\Theta : E \times E \rightarrow \mathbb{C}$ be a \mathbb{C} -valued function. Then Θ is the symbol of an operator $\Xi \in \mathcal{L}((E), (E)^*)$ if and only if Θ is Gâteaux-entire and there exist constants $p \geq 0$ and $C, K \geq 0$ such that*

$$|\Theta(\xi, \eta)| \leq Ce^{K(|\xi|_p^2 + |\eta|_p^2)}, \quad \xi, \eta \in E.$$

Moreover, $\Xi \in \mathcal{L}((E), (E))$ if and only if for any $p \geq 0$ and $\epsilon \geq 0$, there exist constants $C \geq 0$ and $q \geq 0$ such that

$$|\Theta(\xi, \eta)| \leq Ce^{\epsilon(|\xi|_{p+q}^2 + |\eta|_{-p}^2)}, \quad \xi, \eta \in E.$$

By applying Theorem 2.1, for each given $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$, the Wick product $\Xi_1 \diamond \Xi_2 \in \mathcal{L}((E), (E)^*)$ of Ξ_1 and Ξ_2 is well-defined by

$$(1) \quad \widehat{\Xi_1 \diamond \Xi_2}(\xi, \eta) = \widehat{\Xi_1}(\xi, \eta)\widehat{\Xi_2}(\xi, \eta)e^{-\langle \xi, \eta \rangle}, \quad \xi, \eta \in E,$$

see [4].

The symbol of the Gross Laplacian $\Delta_G \in \mathcal{L}((E), (E))$ is given by

$$(2) \quad \widehat{\Delta_G}(\xi, \eta) = \langle \xi, \xi \rangle e^{\langle \xi, \eta \rangle}, \quad \xi, \eta \in E,$$

see [3, 11, 14].

On the other hand, for each $\Xi \in \mathcal{L}((E), (E)^*)$ we define *T-symbol* by

$$(3) \quad T_{\text{op}}(\Xi)(\xi, \eta) = \widehat{\Xi}(i\xi, i\eta)e^{-\frac{1}{2}(\langle \xi, \xi \rangle + \langle \eta, \eta \rangle)}, \quad \xi, \eta \in E,$$

which is motivated by the *T-transform* of white noise functionals (see Section 4).

3. Convolutions and transforms of white noise operators

In this section, we introduce a new notion of convolution products of white noise operators and study some relations between the convolutions and the quantum generalized Fourier-Mehler transforms.

3.1. Convolution products of white noise operators

Motivated by the convolution of white noise functionals (see Section 4), for each fixed $\Upsilon \in \mathcal{L}((E), (E)^*)$, we define a convolution $\Xi_1 *_{\Upsilon} \Xi_2 \in \mathcal{L}((E), (E)^*)$ of two white noise operators $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$ by

$$\Xi_1 *_{\Upsilon} \Xi_2 = \Xi_1 \diamond \Xi_2 \diamond \Upsilon.$$

Here by applying Theorem 2.1 we see that $\Xi_1 *_{\Upsilon} \Xi_2$ becomes a white noise operator.

Proposition 3.1. *Let $\Upsilon \in \mathcal{L}((E), (E)^*)$ be given. Then for any $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$,*

$$T_{\text{op}}(\Xi_1 *_{\Upsilon} \Xi_2) = T_{\text{op}}(\Xi_1)T_{\text{op}}(\Xi_2)$$

if and only if $\Upsilon = e^{\frac{1}{2}\Delta_G^} \Gamma(2I) e^{\frac{1}{2}\Delta_G}$.*

Proof. By (3) and (1) we obtain that for any $\xi, \eta \in E$,

$$\begin{aligned} (4) \quad & T_{\text{op}}(\Xi_1 *_{\Upsilon} \Xi_2)(\xi, \eta) \\ &= (\Xi_1 \diamond \Xi_2 \diamond \Upsilon)^{\wedge}(i\xi, i\eta)C(\xi, \eta) \\ &= \widehat{\Xi}_1(i\xi, i\eta)\widehat{\Xi}_2(i\xi, i\eta)\widehat{\Upsilon}(i\xi, i\eta)C(\xi, \eta)e^{2\langle \xi, \eta \rangle} \\ &= T_{\text{op}}(\Xi_1)(\xi, \eta)T_{\text{op}}(\Xi_2)(\xi, \eta)\widehat{\Upsilon}(i\xi, i\eta)C(i\xi, i\eta)e^{2\langle \xi, \eta \rangle}, \end{aligned}$$

where $C(\xi, \eta) = e^{-\frac{1}{2}(\langle \xi, \xi \rangle + \langle \eta, \eta \rangle)}$. Hence, $T_{\text{op}}(\Xi_1 *_{\Upsilon} \Xi_2) = T_{\text{op}}(\Xi_1)T_{\text{op}}(\Xi_2)$ if and only if

$$\widehat{\Upsilon}(i\xi, i\eta) = e^{-\frac{1}{2}(\langle \xi, \xi \rangle + \langle \eta, \eta \rangle) - 2\langle \xi, \eta \rangle},$$

which is equivalent to $\Upsilon = e^{\frac{1}{2}\Delta_G^*} \Gamma(2I) e^{\frac{1}{2}\Delta_G}$. □

Proposition 3.2. *Let $\Upsilon \in \mathcal{L}((E), (E)^*)$ be given. If $\Upsilon = \Gamma(I)$, then for any $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$, the convolution $\Xi_1 *_{\Upsilon} \Xi_2$ coincides with the convolution $\Xi_1 * \Xi_2$ in [1]. In this case, for $\xi, \eta \in E$, $T_{\text{op}}(\Xi_1 * \Xi_2)(\xi, \eta)$ is given by*

$$T_{\text{op}}(\Xi_1 * \Xi_2)(\xi, \eta) = T_{\text{op}}(\Xi_1)(\xi, \eta)T_{\text{op}}(\Xi_2)(\xi, \eta)e^{\frac{1}{2}\langle \xi + \eta, \xi + \eta \rangle}.$$

Proof. The proof is straightforward from (4). □

3.2. Relations between convolutions and transforms

By applying Theorem 2.1, we can easily see that there exists an operator $\mathcal{G}_{\alpha, \beta} \in \mathcal{L}((E), (E))$ such that

$$(5) \quad \mathcal{G}_{\alpha, \beta} \phi_{\xi} = \phi_{\beta\xi} \exp\{\alpha \langle \xi, \xi \rangle\}, \quad \xi \in E.$$

From (5) and (2), we can easily see that

$$(6) \quad \mathcal{G}_{\alpha, \beta} = \Gamma(\beta I)e^{\alpha \Delta_G},$$

which is called the *Fourier-Gauss transform*. The adjoint operator of $\mathcal{G}_{\alpha, \beta}$ is denoted by $\mathcal{F}_{\alpha, \beta} \in \mathcal{L}((E)^*, (E)^*)$ and called the *generalized Fourier-Mehler transform*. Then by the duality of (6), we have

$$\mathcal{F}_{\alpha, \beta} = e^{\alpha \Delta_G^*} \Gamma(\beta I).$$

If $\alpha(\theta) = \frac{1}{2} i e^{i\theta} \sin \theta$ and $\beta(\theta) = e^{i\theta}$ for $\theta \in \mathbb{R}$, then $\mathcal{F}_\theta \equiv \mathcal{F}_{\alpha(\theta), \beta(\theta)}$ is called the *Fourier-Mehler transform*. For more details, we refer to [3].

Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. The *quantum generalized Fourier-Mehler transform* $\mathcal{F}_{\alpha, \beta; \gamma, \delta}^Q \in \mathcal{L}(\mathcal{L}((E), (E)^*))$ studied in [7] is defined by

$$(7) \quad \mathcal{F}_{\alpha, \beta; \gamma, \delta}^Q(\Xi) = \mathcal{F}_{\alpha, \gamma} \Xi \mathcal{G}_{\beta, \delta}, \quad \Xi \in \mathcal{L}((E), (E)^*).$$

Theorem 3.3. *Let $\alpha_i, \beta_i \in \mathbb{C}$ and $\gamma_i, \delta_i \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ for $i = 1, 2, 3$, and $\Upsilon, \Upsilon' \in \mathcal{L}((E), (E)^*)$ be given. Then for any $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$,*

$$(8) \quad \mathcal{F}_{\alpha_1, \beta_1; \gamma_1, \delta_1}^Q(\Xi_1 * \Upsilon \Xi_2) = \mathcal{F}_{\alpha_2, \beta_2; \gamma_2, \delta_2}^Q(\Xi_1) * \Upsilon' \mathcal{F}_{\alpha_3, \beta_3; \gamma_3, \delta_3}^Q(\Xi_2)$$

if and only if $\gamma_1 = \gamma_2 = \gamma_3, \delta_1 = \delta_2 = \delta_3$ and

$$(9) \quad \Upsilon = [\Gamma(\gamma_1^{-1} I) \Upsilon' \Gamma(\delta_1^{-1} I)] \\ \times \diamond [e^{(\alpha_2 + \alpha_3 - \alpha_1) \gamma_1^{-2} \Delta_G^*} \Gamma((3 - 2(\gamma_1 \delta_1)^{-1}) I) e^{(\beta_2 + \beta_3 - \beta_1) \delta_1^{-2} \Delta_G}].$$

Proof. By (7) and (1) we obtain that for any $\xi, \eta \in E$,

$$(10) \quad \left(\mathcal{F}_{\alpha_1, \beta_1; \gamma_1, \delta_1}^Q(\Xi_1 * \Upsilon \Xi_2) \right)^\wedge(\xi, \eta) \\ = \langle\langle \Xi_1 \diamond \Xi_2 \diamond \Upsilon \phi_{\delta_1 \xi}, \phi_{\gamma_1 \eta} \rangle\rangle e^{\beta_1 \langle \xi, \xi \rangle + \alpha_1 \langle \eta, \eta \rangle} \\ = \langle\langle \Xi_1 \phi_{\delta_1 \xi}, \phi_{\gamma_1 \eta} \rangle\rangle \langle\langle \Xi_2 \phi_{\delta_1 \xi}, \phi_{\gamma_1 \eta} \rangle\rangle \langle\langle \Upsilon \phi_{\delta_1 \xi}, \phi_{\gamma_1 \eta} \rangle\rangle \\ \times e^{\beta_1 \langle \xi, \xi \rangle + \alpha_1 \langle \eta, \eta \rangle - 2 \langle \delta_1 \xi, \gamma_1 \eta \rangle},$$

and

$$(11) \quad \left(\mathcal{F}_{\alpha_2, \beta_2; \gamma_2, \delta_2}^Q(\Xi_1) \diamond \mathcal{F}_{\alpha_3, \beta_3; \gamma_3, \delta_3}^Q(\Xi_2) \diamond \Upsilon' \right)^\wedge(\xi, \eta) \\ = \langle\langle \Xi_1 \phi_{\delta_2 \xi}, \phi_{\gamma_2 \eta} \rangle\rangle \langle\langle \Xi_2 \phi_{\delta_3 \xi}, \phi_{\gamma_3 \eta} \rangle\rangle \widehat{\Upsilon}'(\xi, \eta) \\ \times e^{(\beta_2 + \beta_3) \langle \xi, \xi \rangle + (\alpha_2 + \alpha_3) \langle \eta, \eta \rangle - 2 \langle \xi, \eta \rangle}.$$

By comparing (10) and (11), we see that (8) holds if and only if $\gamma_1 = \gamma_2 = \gamma_3, \delta_1 = \delta_2 = \delta_3$ and for any $\xi, \eta \in E$,

$$(12) \quad \widehat{\Upsilon}(\delta_1 \xi, \gamma_1 \eta) = \widehat{\Upsilon}'(\xi, \eta) e^{(\alpha_2 + \alpha_3 - \alpha_1) \langle \eta, \eta \rangle + 2(\gamma_1 \delta_1 - 1) \langle \xi, \eta \rangle + (\beta_2 + \beta_3 - \beta_1) \langle \xi, \xi \rangle},$$

which is equivalent to (9). □

Corollary 3.4. *Let $\alpha_i, \beta_i \in \mathbb{C}$ for $i = 1, 2, 3, \gamma, \delta \in \mathbb{C}^*$ and $\Upsilon \in \mathcal{L}((E), (E)^*)$ be given. Then for any $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$,*

$$\mathcal{F}_{\alpha_1, \beta_1; \gamma, \delta}^Q(\Xi_1 * \Upsilon \Xi_2) = \mathcal{F}_{\alpha_2, \beta_2; \gamma, \delta}^Q(\Xi_1) \diamond \mathcal{F}_{\alpha_3, \beta_3; \gamma, \delta}^Q(\Xi_2)$$

if and only if

$$\Upsilon = e^{(\alpha_2+\alpha_3-\alpha_1)\gamma^{-2}\Delta_G^*} \Gamma((2 - (\gamma\delta)^{-1})I) e^{(\beta_2+\beta_3-\beta_1)\delta^{-2}\Delta_G}.$$

Proof. Since $\Gamma(I)$ is the identity with respect to the Wick product, then $*_{\Gamma(I)}$ coincides with the Wick product \diamond . Therefore, by applying Theorem 3.3, the proof is straightforward. \square

The following corollary is immediate from Corollary 3.4.

Corollary 3.5. *Let $\alpha, \beta \in \mathbb{C}$ and $\gamma, \delta \in \mathbb{C}^*$ and $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$. Then*

$$\mathcal{F}_{\alpha, \beta; \gamma, \delta}^Q(\Xi_1 *_{\Upsilon} \Xi_2) = \mathcal{F}_{\alpha, \beta; \gamma, \delta}^Q(\Xi_1) \diamond \mathcal{F}_{\alpha, \beta; \gamma, \delta}^Q(\Xi_2)$$

if and only if $\Upsilon = e^{\alpha\gamma^{-2}\Delta_G^*} \Gamma((2 - (\gamma\delta)^{-1})I) e^{\beta\delta^{-2}\Delta_G}$.

Corollary 3.6. *Let $\alpha_i, \beta_i \in \mathbb{C}$ for $i = 1, 2, 3$, $\gamma, \delta \in \mathbb{C}^*$ and $\Upsilon \in \mathcal{L}((E), (E)^*)$ be given. Then for any $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$,*

$$\mathcal{F}_{\alpha_1, \beta_1; \gamma, \delta}^Q(\Xi_1 \diamond \Xi_2) = \mathcal{F}_{\alpha_2, \beta_2; \gamma, \delta}^Q(\Xi_1) *_{\Upsilon} \mathcal{F}_{\alpha_3, \beta_3; \gamma, \delta}^Q(\Xi_2)$$

if and only if

$$(13) \quad \Upsilon = e^{\alpha\Delta_G^*} \Gamma((2 - \gamma\delta)I) e^{\beta\Delta_G}$$

with $\alpha = -(\alpha_2 + \alpha_3 - \alpha_1)$ and $\beta = -(\beta_2 + \beta_3 - \beta_1)$.

Proof. The proof is a simple application of Theorem 3.3 with the fact that $(e^{\diamond\Xi})^{\diamond(-1)} = e^{\diamond(-\Xi)}$, whenever the Wick exponential:

$$e^{\diamond\Xi} = \sum_{n=0}^{\infty} \frac{1}{n!} \Xi^{\diamond n}$$

is well defined as a white noise operator in $\mathcal{L}((E), (E)^*)$. Here $\Xi^{\diamond(-1)}$ stands the (Wick) inverse operator of $\Xi \in \mathcal{L}((E), (E)^*)$ with respect to the Wick product. Also, we can easily check that $\Gamma(\rho I)^{\diamond(-1)} = \Gamma((2 - \rho)I)$ for any $\rho \in \mathbb{C}$. Since we know that

$$e^{\varrho\Delta_G^*} \Gamma(\rho I) e^{\sigma\Delta_G} = e^{\varrho\Delta_G^*} \diamond \Gamma(\rho I) \diamond e^{\sigma\Delta_G}$$

for any $\rho, \varrho, \sigma \in \mathbb{C}$, from (9) we have

$$(14) \quad \Upsilon = \Gamma(\gamma I) \left[e^{\alpha\gamma^{-2}\Delta_G^*} \Gamma((2(\gamma\delta)^{-1} - 1)I) e^{\beta\delta^{-2}\Delta_G} \right] \Gamma(\delta I)$$

with $\alpha = -(\alpha_2 + \alpha_3 - \alpha_1)$ and $\beta = -(\beta_2 + \beta_3 - \beta_1)$. On the other hand, for any $\rho, \sigma \in \mathbb{C}$, we can easily check that

$$e^{\sigma\Delta_G} \Gamma(\rho I) = \Gamma(\rho I) e^{\sigma\rho^2\Delta_G}$$

and by duality,

$$\Gamma(\rho I) e^{\sigma\Delta_G^*} = e^{\sigma\rho^2\Delta_G^*} \Gamma(\rho I).$$

Therefore, from (14) we see that Υ is of the form as in (13). \square

Remark 3.7. By applying (12) we can prove Corollary 3.6 directly. However, in the proof of Corollary 3.6 we used the Wick calculus of white noise operators which is a useful tool to study equations associated with Wick product arising in the quantum field theory.

The following corollary is immediate from Corollary 3.6.

Corollary 3.8. *Let $\alpha, \beta \in \mathbb{C}$, $\gamma, \delta \in \mathbb{C}^*$ and $\Upsilon \in \mathcal{L}((E), (E)^*)$ be given. Then for any $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$,*

$$\mathcal{F}_{\alpha, \beta; \gamma, \delta}^Q(\Xi_1 \diamond \Xi_2) = \mathcal{F}_{\alpha, \beta; \gamma, \delta}^Q(\Xi_1) *_{\Upsilon} \mathcal{F}_{\alpha, \beta; \gamma, \delta}^Q(\Xi_2)$$

if and only if

$$\Upsilon = e^{-\alpha \Delta_G^*} \Gamma((2 - \gamma\delta)I) e^{-\beta \Delta_G}.$$

4. Quantum-classical correspondence

4.1. S-transform, T-transform and convolution

The *S-transform* of an element $\Phi \in (E)^*$ is a complex valued function on E defined by

$$S\Phi(\xi) = \langle\langle \Phi, \phi_\xi \rangle\rangle, \quad \xi \in E.$$

Then every $\Phi \in (E)^*$ is uniquely determined by its *S-transform* since $\{\phi_\xi : \xi \in E\}$ spans a dense subspace of (E) . The *Wick product* $\Phi \diamond \Psi$ of Φ and Ψ in $(E)^*$ is defined as the unique element of $(E)^*$ such that $S(\Phi \diamond \Psi) = S(\Phi)S(\Psi)$. Then $((E)^*, \diamond)$ is a commutative algebra. The *T-transform* $T\Phi$ of $\Phi \in (E)^*$ is defined to be a complex valued function on E satisfying

$$T\Phi(\xi) = e^{-\frac{1}{2}\langle \xi, \xi \rangle} S\Phi(i\xi), \quad \xi \in E,$$

see [5, 11]. For each $\Xi \in \mathcal{L}((E), (E)^*)$ and any $\xi \in E$, we have

$$(15) \quad T_{\text{OP}}(\Xi)(0, \xi) = \widehat{\Xi}(0, i\xi) e^{-\frac{1}{2}\langle \xi, \xi \rangle} = T(\Xi\phi_0)(\xi),$$

where $\phi_0 = (1, 0, \dots) \in (E)$ is the vacuum vector.

For fixed $\mathbf{F} \in (E)^*$, a *convolution* $\Phi *_{\mathbf{F}} \Psi$ of $\Phi, \Psi \in (E)^*$ is defined by

$$\Phi *_{\mathbf{F}} \Psi = \Phi \diamond \Psi \diamond \mathbf{F}.$$

In particular, the convolution product $*_{\mathbf{F}}$ with $\mathbf{F} = g_c$ coincides with the convolution product $*$ studied in [11], i.e., for any $\Phi, \Psi \in (E)^*$

$$\Phi *_{g_{-2}} \Psi = \Phi * \Psi \equiv \Phi \diamond \Psi \diamond g_{-2},$$

where $g_c \in (E)$ is the Gaussian (white noise) function with mean 0 and variance c such that

$$(16) \quad Sg_c(\xi) = \exp \left\{ -\frac{1}{2(1+c)} \langle \xi, \xi \rangle \right\}, \quad c \neq -1.$$

Lemma 4.1. *For each $c \in \mathbb{C}$ with $c \neq 0$ we have*

$$e^{c\Delta_G^*} \phi_0 = g_{-(1+\frac{1}{2c})}.$$

Proof. For any $\xi \in E$, we have

$$S\left(e^{c\Delta_G^*}\phi_0\right)(\xi) = \langle\langle\phi_0, e^{c\Delta_G}\phi_\xi\rangle\rangle = e^{c\langle\xi, \xi\rangle}$$

and so the proof is immediate from (16). \square

4.2. Quantum-classical correspondence

For $\Phi \in (E)^*$ and $\phi \in (E)$, the (pointwise) multiplication $\Phi\phi \in (E)^*$ is well-defined as

$$\langle\langle\Phi\phi, \psi\rangle\rangle = \langle\langle\Phi, \phi\psi\rangle\rangle, \quad \psi \in (E).$$

Here we use the fact that (E) is closed under the (pointwise) multiplication as white noise functionals (Cauchy product of vectors), see [13]. For each $\Phi \in (E)^*$, let $M_\Phi \in \mathcal{L}((E), (E)^*)$ be the multiplication operator associated with Φ , i.e., $M_\Phi(\phi) = \Phi\phi$ for any $\phi \in (E)$.

For each $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$ we can easily see that

$$(17) \quad (\Xi_1 \diamond \Xi_2)\phi_0 = (\Xi_1\phi_0) \diamond (\Xi_2\phi_0)$$

and so for each $\Phi, \Psi \in (E)^*$, $(M_\Phi \diamond M_\Psi)\phi_0 = \Phi \diamond \Psi$, more generally, we have the following lemma.

Lemma 4.2. *Let $\mathbf{F} \in (E)^*$. Then for any $\Phi, \Psi \in (E)^*$,*

$$(18) \quad M_\Phi \diamond M_\Psi = M_{\Phi \diamond \Psi}, \quad M_\Phi *_{M_{\mathbf{F}}} M_\Psi = M_{\Phi *_{\mathbf{F}} \Psi}.$$

Proof. The second identity is immediate from the definition and the first identity. The proof of the first identity is straightforward by taking operator symbols on both sides using the fact that $\phi_\xi\phi_\eta = \phi_{\xi+\eta}e^{\langle\xi, \eta\rangle}$ for any $\xi, \eta \in E$. \square

By applying Proposition 3.1 with (15) and (18) we have the following proposition.

Proposition 4.3. *Let $\mathbf{F} \in (E)^*$ be given. Then for any $\Phi, \Psi \in (E)^*$,*

$$(19) \quad T(\Phi *_{\mathbf{F}} \Psi) = T(\Phi)T(\Psi)$$

if and only if $\mathbf{F} = g_{-2}$.

Proof. By applying Proposition 3.1 with (15) and (18) we can see that (19) holds if and only if $\mathbf{F} = \Upsilon\phi_0$ with $\Upsilon = e^{\frac{1}{2}\Delta_G^*}\Gamma(2I)e^{\frac{1}{2}\Delta_G}$. On the other hand, by Lemma 4.1 we have $\Upsilon\phi_0 = e^{\frac{1}{2}\Delta_G^*}\phi_0 = g_{-2}$. \square

Theorem 4.4. *Let $\alpha_i \in \mathbb{C}$ for $i = 1, 2, 3$ and $\gamma \in \mathbb{C}^*$. Let $\mathbf{F}, \mathbf{G} \in (E)^*$ be given. Then for any $\Phi, \Psi \in (E)^*$,*

$$(20) \quad \mathcal{F}_{\alpha_1, \gamma}(\Phi *_{\mathbf{F}} \Psi) = \mathcal{F}_{\alpha_2, \gamma}(\Phi) *_{\mathbf{G}} \mathcal{F}_{\alpha_3, \gamma}(\Psi)$$

if and only if

$$(21) \quad \mathbf{F} = e^{(\alpha_2 + \alpha_3 - \alpha_1)\gamma^{-2}\Delta_G^*}\Gamma(\gamma^{-1}I)\mathbf{G}.$$

Proof. For any $\xi \in E$, we obtain that

$$S(\mathcal{F}_{\alpha_1, \gamma}(\Phi *_{\mathbf{F}} \Psi))(\xi) = e^{\alpha_1 \langle \xi, \xi \rangle} S\Phi(\gamma\xi) S\Psi(\gamma\xi) S\mathbf{F}(\gamma\xi)$$

and

$$S(\mathcal{F}_{\alpha_2, \gamma}(\Phi) *_{\mathbf{G}} \mathcal{F}_{\alpha_3, \gamma}(\Psi))(\xi) = e^{(\alpha_2 + \alpha_3) \langle \xi, \xi \rangle} S\Phi(\gamma\xi) S\Psi(\gamma\xi) S\mathbf{G}(\xi).$$

Therefore, (20) holds if and only if for any $\xi \in E$

$$S\mathbf{F}(\gamma\xi) = e^{(\alpha_2 + \alpha_3 - \alpha_1) \langle \xi, \xi \rangle} S\mathbf{G}(\xi),$$

which is equivalent to (21). \square

Remark 4.5. In Theorem 4.4, if $\alpha_2 + \alpha_3 \neq \alpha_1$, then by taking S -transform on (21), we can easily see that

$$\mathbf{F} = \Gamma(\gamma^{-1}I) \mathbf{G} \diamond g_d, \quad d = -\left(1 + \frac{1}{2(\alpha_2 + \alpha_3 - \alpha_1)\gamma^{-2}}\right).$$

Lemma 4.6. *Let $\mathbf{G} \in (E)^*$ and $\alpha, \gamma \in \mathbb{C}$. Then we have*

$$(22) \quad M_{e^{\alpha\Delta_{\mathbf{G}}^*}\Gamma(\gamma I)\mathbf{G}} = [\Gamma(\gamma I)M_{\mathbf{G}}\Gamma(\gamma I)] \diamond e^{\alpha\Delta_{\mathbf{G}}^*}\Gamma((2\alpha + 2 - \gamma^2)I) e^{\alpha\Delta_{\mathbf{G}}}.$$

Proof. For any $\xi, \eta \in E$ we have

$$\begin{aligned} M_{\widehat{e^{\alpha\Delta_{\mathbf{G}}^*}\Gamma(\gamma I)\mathbf{G}}}(\xi, \eta) &= \left\langle\left\langle e^{\alpha\Delta_{\mathbf{G}}^*}\Gamma(\gamma I)\mathbf{G}, \phi_{\xi+\eta} \right\rangle\right\rangle e^{\langle \xi, \eta \rangle} \\ &= \left\langle\left\langle \mathbf{G}, \phi_{\gamma(\xi+\eta)} \right\rangle\right\rangle e^{\alpha\langle \xi+\eta, \xi+\eta \rangle + \langle \xi, \eta \rangle} \\ &= \left\langle\left\langle \mathbf{G}, \phi_{\gamma\xi}\phi_{\gamma\eta} \right\rangle\right\rangle e^{\alpha\langle \xi, \xi \rangle + \alpha\langle \eta, \eta \rangle + (2\alpha + 2 - \gamma^2)\langle \xi, \eta \rangle - \langle \xi, \eta \rangle} \\ &= \left\langle\left\langle \Gamma(\gamma I)M_{\mathbf{G}}\Gamma(\gamma I)\phi_{\xi}, \phi_{\eta} \right\rangle\right\rangle \\ &\quad \times e^{\alpha\langle \xi, \xi \rangle + \alpha\langle \eta, \eta \rangle + (2\alpha + 2 - \gamma^2)\langle \xi, \eta \rangle - \langle \xi, \eta \rangle}, \end{aligned}$$

where we used the fact that $\phi_{\xi}\phi_{\eta} = \phi_{\xi+\eta}e^{\langle \xi, \eta \rangle}$, which implies (22). \square

For any $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $\Xi \in \mathcal{L}((E), (E)^*)$ we can easily check that

$$\mathcal{F}_{\alpha, \beta; \gamma, \delta}^Q(\Xi)\phi_0 = \mathcal{F}_{\alpha, \gamma}(\Xi\phi_0)$$

and so, by applying (17), for any $\mathbf{F} \in (E)^*$ we have

$$\mathcal{F}_{\alpha, \beta; \gamma, \delta}^Q(\Xi_1 *_{M_{\mathbf{F}}} \Xi_2)\phi_0 = \mathcal{F}_{\alpha, \gamma}(\Xi_1\phi_0 *_{\mathbf{F}} \Xi_2\phi_0)$$

for any $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$.

Theorem 4.7. *Let $\alpha_i \in \mathbb{C}$ for $i = 1, 2, 3$ and $\gamma \in \mathbb{C}^*$ with $\alpha_2 + \alpha_3 - \alpha_1 = \frac{\gamma^2 - 1}{2}$. Let $\mathbf{F}, \mathbf{G} \in (E)^*$ be given. Then for any $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$,*

$$(23) \quad \mathcal{F}_{\alpha_1, \alpha_1; \gamma, \gamma}(\Xi_1 *_{M_{\mathbf{F}}} \Xi_2) = \mathcal{F}_{\alpha_2, \alpha_2; \gamma, \gamma}(\Xi_1) *_{M_{\mathbf{G}}} \mathcal{F}_{\alpha_3, \alpha_3; \gamma, \gamma}(\Xi_2)$$

if and only if

$$(24) \quad \mathbf{F} = e^{\frac{1}{2}(1-\gamma^{-2})\Delta_{\mathbf{G}}^*}\Gamma(\gamma^{-1}I) \mathbf{G}.$$

Proof. \Rightarrow) By acting vacuum vector ϕ_0 on the both sides of (23), we have

$$\mathcal{F}_{\alpha_1, \gamma}(\Xi_1 \phi_0 *_{\mathbf{F}} \Xi_2 \phi_0) = \mathcal{F}_{\alpha_2, \gamma}(\Xi_1 \phi_0) *_{\mathbf{G}} \mathcal{F}_{\alpha_3, \gamma}(\Xi_2 \phi_0).$$

Therefore, by Theorem 4.4, (24) holds.

\Leftarrow) Suppose that (24) holds. Then by Lemma 4.6, we have

$$M_{\mathbf{F}} = \Gamma(\gamma^{-1}I) M_{\mathbf{G}} \Gamma(\gamma^{-1}I) \diamond e^{c\Delta_{\mathbf{G}}^*} \Gamma((3 - 2\gamma^{-2})I) e^{c\Delta_{\mathbf{G}}}$$

with $c = (\alpha_2 + \alpha_3 - \alpha_1)\gamma^{-2}$ and $\alpha_2 + \alpha_3 - \alpha_1 = \frac{\gamma^2 - 1}{2}$. Therefore, by Theorem 3.3, (23) holds. \square

Corollary 4.8. *Let $\alpha \in \mathbb{C}$ and $\gamma \in \mathbb{C}^*$. Let $\mathbf{F}, \mathbf{G} \in (E)^*$ be given. Then for any $\Phi, \Psi \in (E)^*$,*

$$(25) \quad \mathcal{F}_{\alpha, \gamma}(\Phi *_{\mathbf{F}} \Psi) = \mathcal{F}_{\alpha, \gamma}(\Phi) *_{\mathbf{G}} \mathcal{F}_{\alpha, \gamma}(\Psi)$$

if and only if

$$(26) \quad \mathbf{F} = e^{\alpha\gamma^{-2}\Delta_{\mathbf{G}}^*} \Gamma(\gamma^{-1}I) \mathbf{G}.$$

In particular, by (26) and Lemma 4.1, for any $\Phi, \Psi \in (E)^*$,

$$\mathcal{F}_{\alpha, \gamma}(\Phi *_{g_{-(1+\frac{\gamma^2}{2\alpha})}} \Psi) = \mathcal{F}_{\alpha, \gamma}(\Phi) \diamond \mathcal{F}_{\alpha, \gamma}(\Psi)$$

for $\alpha \neq 0$, which is a special case of (26) with $\mathbf{G} = \phi_0$.

Proof. The proof is immediate from Theorem 4.4. \square

Let $\mathbf{F}, \mathbf{G} \in (E)^*$ be given such that (26) holds. Then we have

$$\mathbf{G} = \Gamma(\gamma I) e^{-\alpha\gamma^{-2}\Delta_{\mathbf{G}}^*} \mathbf{F}.$$

On the other hand,

$$\Gamma(\gamma I) e^{-\alpha\gamma^{-2}\Delta_{\mathbf{G}}^*} = e^{-\alpha\Delta_{\mathbf{G}}^*} \Gamma(\gamma I)$$

and so

$$(27) \quad \mathbf{G} = e^{-\alpha\Delta_{\mathbf{G}}^*} \Gamma(\gamma I) \mathbf{F} = \Gamma(\gamma I) \mathbf{F} \diamond g_{\frac{1}{2\alpha} - 1},$$

for the second equality we assume that $\alpha \neq 0$. If $\mathbf{F} = \phi_0$, then for $\alpha \neq 0$ by Lemma 4.1 we have

$$\mathbf{G} = e^{-\alpha\Delta_{\mathbf{G}}^*} \Gamma(\gamma I) \phi_0 = e^{-\alpha\Delta_{\mathbf{G}}^*} \phi_0 = g_{\frac{1}{2\alpha} - 1}.$$

Therefore, by (25), we have the following corollary.

Corollary 4.9. *Let $\alpha, \gamma \in \mathbb{C}^*$. Then for any $\Phi, \Psi \in (E)^*$,*

$$\mathcal{F}_{\alpha, \gamma}(\Phi \diamond \Psi) = \mathcal{F}_{\alpha, \gamma}(\Phi) *_{g_{\frac{1}{2\alpha} - 1}} \mathcal{F}_{\alpha, \gamma}(\Psi),$$

in particular,

$$\mathcal{F}_{\theta}(\Phi \diamond \Psi) = \mathcal{F}_{\theta}(\Phi) *_{g_{-(2+i\cot\theta)}} \mathcal{F}_{\theta}(\Psi)$$

for any $\theta \in \mathbb{R}$, see [11].

Let $\mathbf{F}, \mathbf{G} \in (E)^*$ be given such that (26) holds. If $\mathbf{F} = g_c$ for $c \neq -1$ and $\gamma^2 \neq -2\alpha(1+c) \in \mathbb{C}^*$, then from (27) we have

$$(28) \quad \mathbf{G}_c = \Gamma(\gamma I) g_c \diamond g_{\frac{1}{2\alpha}-1} = g_d, \quad d = \frac{1+c}{\gamma^2 + 2\alpha(1+c)} - 1.$$

In particular, $\mathbf{G}_{-2} = g_{-(1+\frac{1}{\gamma^2-2\alpha})}$ for $\gamma^2 \neq 2\alpha \in \mathbb{C}^*$. Therefore, the following corollary is immediate from Corollary 4.8.

Corollary 4.10. *Let $\alpha, \gamma \in \mathbb{C}^*$ and $c \in \mathbb{C}$ with $\alpha \neq -1$ and $\gamma^2 \neq -2\alpha(1+c)$. Then for any $\Phi, \Psi \in (E)^*$,*

$$\mathcal{F}_{\alpha, \gamma}(\Phi *_{g_c} \Psi) = \mathcal{F}_{\alpha, \gamma}(\Phi) *_{\mathbf{G}_c} \mathcal{F}_{\alpha, \gamma}(\Psi),$$

where \mathbf{G}_c is given as in (28). In particular, for any $\Phi, \Psi \in (E)^*$,

$$\mathcal{F}_{\alpha, \gamma}(\Phi * \Psi) = \mathcal{F}_{\alpha, \gamma}(\Phi) *_{\mathbf{G}_{-2}} \mathcal{F}_{\alpha, \gamma}(\Psi)$$

for $\gamma^2 \neq 2\alpha$, specially,

$$\mathcal{F}_{\theta}(\Phi * \Psi) = \mathcal{F}_{\theta}(\Phi) *_{g_{i \tan \theta - 2}} \mathcal{F}_{\theta}(\Psi)$$

for any $\theta \in \mathbb{R}$, see [11].

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