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STRONG CONVERGENCE THEOREMS FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS AND INVERSE-STRONGLY MONOTONE MAPPINGS

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Abstract. In this paper, we consider an iterative scheme for finding a common element of the set of fixed points of a asymptotically quasi-nonexpansive mapping and the set of solutions of the variational inequality for an inverse strongly monotone mapping in a Hilbert space. Then we show that the sequence converges strongly to a common element of two sets. Using this result, we consider the problem of finding a common fixed point of a asymptotically quasi-nonexpansive mapping and a strictly pseudocontractive mapping and the problem of finding a common element of the set of fixed points of a asymptotically quasi-nonexpansive mapping and the set of zeros of an inverse-strongly monotone mapping.

1. Introduction and preliminaries

Let C be a closed convex subset of a real Hilbert space H and let P_C be the metric projection of H onto C.

A mapping A of C into H is called monotone if for all $x, y \in C, \langle x-y, Ax-Ay\rangle \geq 0$.

The variational inequality problem is to find a $u \in C$ such that $\langle v-u, Au\rangle \geq 0$ for all $v \in C$; see [1,2,4,6,11]. The set of solutions of the variational inequality is denoted by $VI(C, A)$.

A mapping A of C into H is called inverse-strongly monotone if there exists a positive real number α such that $\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2$ for all $x, y \in C$; see [3,5,7,8]. For such a case, A is called α-inverse-strongly monotone.

A mapping S of C into itself called asymptotically nonexpansive if there exists a sequence $\{k_n\}, k_n \geq 1$ of positive real numbers with $\lim_{n\to\infty} k_n = 1$

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and such that
$$\|S^n x - S^n y\| \le k_n \|x - y\|$$
for all integers $n \ge 1$ and $x, y \in C$. S is called uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $\forall x, y \in C$, the following inequality holds:
$$\|S^n x - S^n y\| \le L\|x - y\|.$$
A point $x \in C$ is a fixed point of S provided $Sx = x$. Denote by $F(S)$ the set of fixed points of S; that is, $F(S) = \{x \in C : Sx = x\}$.

The map S is called asymptotically quasi-nonexpansive if $F(S) \neq \emptyset$ and there exists a sequence $\{k_n\}, k_n \ge 1$ of positive real numbers with $\lim_{n\to\infty} k_n = 1$ and such that
$$\|S^n x - x^*\| \le k_n \|x - x^*\|$$
for all integers $n \ge 1$ and $x \in C, \forall x^* \in F(S)$. It is clear from this definition that every asymptotically nonexpansive mapping with a fixed point is asymptotically quasi-nonexpansive(see[16]).

In this paper, we introduce an iterative scheme for finding a common element of the set of fixed points of a asymptotically quasi-nonexpansive mapping and the set of solutions of the variational inequality for an inverse-strongly monotone mapping in a real Hilbert space. Then we show that the sequence converges strongly to a common element of two sets. Using this result, we first obtain a strong convergence theorem for finding a common fixed point of a asymptotically quasi-nonexpansive mapping and a strictly pseudocontractive mapping. Further, we consider the problem of finding a common element of the set of fixed points of a asymptotically quasi-nonexpansive mapping and the set of zeros of an inverse-strongly monotone mapping.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let C be a closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$ for all $y \in C$. P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies
$$\langle x - y, P_C x - P_C y \rangle \ge \|P_C x - P_C y\|^2 \tag{1.1}$$
for every $x, y \in H$. Moreover, $P_C x$ is characterized by the properties: $P_C x \in C$ and $\langle x - P_C x, P_C x - y \rangle \ge 0$ for all $y \in C$. In the context of the variational inequality problem, this implies that
$$u \in VI(C, A) \Longleftrightarrow u = P_C(u - \lambda A u), \quad \forall \lambda > 0. \tag{1.2}$$
A mapping $T : C \to C$ is said to be semi-compact if, for any sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some x^* in C .

If A is an α-inverse-strongly monotone mapping of C into H, then it is obvious that A is $1/\alpha$-Lipschitz continuous.We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned}\|(I-\lambda A)x-(I-\lambda A)y\|^2 &= \|(x-y)-\lambda(Ax-Ay)\|^2\\ &= \|x-y\|^2-2\lambda\langle x-y, Ax-Ay\rangle+\lambda^2\|Ax-Ay\|^2\\ &\leq \|x-y\|^2+\lambda(\lambda-2\alpha)\|Ax-Ay\|^2.\end{aligned} \tag{1.3}$$

So, if $\lambda \leq 2\alpha$, then $I-\lambda A$ is a nonexpansive mapping of C into H.

Lemma 1.1. (K. Goebel and W. A. Kirk [15]) *Let K be a nonempty, closed, convex and bounded subset of a uniformly convex Banach space X, and let $F : K \to K$ be asymptotically nonexpansitve. Then F has a fixed point.*

2. The convergence theorem

In this section, we prove a strong convergence theorem for asymptotically quasi-nonexpansive mappings and inverse-strongly monotone mappings using the idea of [13] and [14].

Theorem 2.1. *Let C be a bounded closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H and let S be a uniformly L-Lipschitzian, asymptotically quasi-nonexpansive mapping of C into itself with sequence $\{k_n\} \subset [1,\infty)$ such that $F(S) \cap VI(C,A) \neq \emptyset$. Suppose $x_0 \in C$ and $\{x_n\}$ is given by*

$$\begin{cases} x_0 \in C, \quad \lambda > 0,\\ y_n = P_C(x_n-\lambda_n Ax_n),\\ z_n = \alpha_n x_n+(1-\alpha_n)S^n y_n,\\ H_n = \{v \in C : \|z_n-v\|^2 \leq \|x_n-v\|^2+\theta_n\},\\ W_n = \{z \in C : \langle z-x_n, x_0-x_n\rangle \leq 0\},\\ x_{n+1} = P_{H_n\cap W_n}(x_0), \quad n \geq 0,\end{cases} \tag{2.1}$$

where

$$\theta_n = (1-\alpha_n)(k_n^2-1)(diamC)^2 \to 0, \qquad as\ \ n \to \infty,$$

$\{\alpha_n\}$ is a sequence in $[0,1)$ and $\{\lambda_n\}$ is a sequence in $[0,2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\{\lambda_n\} \in [a,b]$ for some a, b with $0 < a < b < 2\alpha$, and $\lambda_n \to \lambda_0$, $\lim_{n\to\infty} \alpha_n = 0$. Assume that S is semi-compact. Then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,A)}(x_0)$.

Proof. First note that S has a fixed point in C by Lemma 1.1; that is, $F(S)$ is nonempty. Since C is a bounded set, therefore $\{x_n\}$,$\{Ax_n\}$ and $\{S^n x_n\}$ are also bounded.

Next observe that H_n is convex. Indeed, the defining inequality in H_n is equivalent to the inequality

$$2\langle(x_n-z_n), v\rangle \leq \|x_n\|^2-\|z_n\|^2+\theta_n,$$

which is affine (and hence convex) in v. Next observe that $F(S) \cap VI(C, A) \subset H_n$ for all n. Indeed, we have, for all $p \in F(S) \cap VI(C, A)$,

$$\begin{aligned} \|z_n - p\|^2 &= \|\alpha_n(x_n - p) + (1-\alpha_n)(S^n y_n - p)\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|S^n y_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)k_n^2\|x_n - p\|^2 \\ &= \|x_n - p\|^2 + [\alpha_n + (1-\alpha_n)k_n^2 - 1]\|x_n - p\|^2 \\ &\le \|x_n - p\|^2 + \theta_n. \end{aligned} \tag{2.2}$$

So $p \in H_n$ for all n. Next we show that

$$F(S) \cap VI(C, A) \subset H_n \cap W_n, \qquad for\ \ all\ \ n \ge 0. \tag{2.3}$$

It suffices to show that $F(S) \cap VI(C, A) \subset W_n$ for all $n \ge 0$. We prove this by induction. For $n = 0$, we have $F(S) \cap VI(C, A) \subset C = W_0$. Assume that $F(S) \cap VI(C, A) \subset W_n$. Since x_{n+1} is the projection of x_0 onto $H_n \cap W_n$, we have

$$\langle x_{n+1} - z, x_0 - x_{n+1}\rangle \ge 0, \qquad \forall\ \ z \in H_n \cap W_n. \tag{2.4}$$

As $F(S) \cap VI(C, A) \subset H_n \cap W_n$, the last inequality holds, in particular, for all $z \in F(S) \cap VI(C, A)$. This together with the definition of W_{n+1} implies that $F(S) \cap VI(C, A) \subset W_{n+1}$. Hence (2.4) holds for all $n \ge 0$.

Next we show that

$$\|x_{n+1} - x_n\| \to 0. \tag{2.5}$$

Indeed, by the definition of W_n, we have $x_n = P_{W_n}(x_0)$ which together with the fact that $x_{n+1} \in H_n \cap W_n \subset W_n$ implies that

$$\|x_0 - x_n\| \le \|x_0 - x_{n+1}\|.$$

This shows that the sequence $\{\|x_n - x_0\|\}$ is increasing. Since C is bounded, we obtain that the $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Noticing again that $x_n = P_{W_n}(x_0)$ and $x_{n+1} \in W_n$ which imply that $\langle x_{n+1} - x_n, x_n - x_0\rangle \ge 0$, and also noticing the identity

$$\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v\rangle, \qquad \forall u, v \in H,$$

we obtain

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\ &\le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \to 0, \qquad as\ \ n \to \infty. \end{aligned} \tag{2.6}$$

By the definition of y_n, we have

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|P_C(x_{n+1}-\lambda_{n+1}Ax_{n+1})-P_C(x_n-\lambda_nAx_n)\|\\ &\le \|x_{n+1}-\lambda_{n+1}Ax_{n+1}-x_n+\lambda_nAx_n\|\\ &\le \|(x_{n+1}-\lambda_{n+1}Ax_{n+1})-(x_n-\lambda_{n+1}Ax_n)\|\\ &\quad+|\lambda_n-\lambda_{n+1}|\|Ax_n\|\\ &\le \|x_{n+1}-x_n\|+|\lambda_n-\lambda_{n+1}|\|Ax_n\|.\end{aligned} \tag{2.7}$$

Since $\{Ax_n\}$ is bounded and $\|x_{n+1}-x_n\|\to 0$, we obtain $\|y_{n+1}-y_n\|\to 0$. From $x_{n+1}\in H_n$, we have

$$\begin{aligned}\|z_n-x_{n+1}\|^2 &\le \|x_n-x_{n+1}\|^2+\theta_n\to 0, \qquad as\ \ n\to\infty,\\ \|z_n-x_n\| &\le \|z_n-x_{n+1}\|+\|x_{n+1}-x_n\|\to 0.\end{aligned}$$

For $u\in F(S)\cap VI(C,A)$, from (1.3), we obtain

$$\begin{aligned}\|z_n-u\|^2 &= \|\alpha_nx_n+(1-\alpha_n)S^ny_n-u\|^2\\ &\le \alpha_n\|x_n-u\|^2+(1-\alpha_n)\|S^ny_n-u\|^2\\ &\le \alpha_n\|x_n-u\|^2+(1-\alpha_n)k_n^2\|y_n-u\|^2\\ &\le \alpha_n\|x_n-u\|^2+(1-\alpha_n)k_n^2\|x_n-u\|^2\\ &\quad+(1-\alpha_n)k_n^2a(b-2\alpha)\|Ax_n-Au\|^2\\ &\le \|x_n-u\|^2+(1-\alpha_n)(k_n^2-1)\|x_n-u\|^2\\ &\quad+(1-\alpha_n)k_n^2a(b-2\alpha)\|Ax_n-Au\|^2.\end{aligned} \quad .$$

Therefore, we have

$$\begin{aligned}&-(1-\alpha_n)k_n^2a(b-2\alpha)\|Ax_n-Au\|^2\\ &\le \|x_n-u\|^2-\|z_n-u\|^2+(1-\alpha_n)(k_n^2-1)\|x_n-u\|^2\\ &= (1-\alpha_n)(k_n^2-1)\|x_n-u\|^2+(\|x_n-u\|+\|z_n-u\|)\\ &\quad\times(\|x_n-u\|-\|z_n-u\|)\\ &\le (1-\alpha_n)(k_n^2-1)\|x_n-u\|^2+(\|x_n-u\|+\|z_n-u\|)\\ &\quad\times\|x_n-z_n\|.\end{aligned}$$

Since $k_n\to 1$ and $\|z_n-x_n\|\to 0$, we obtain $\|Ax_n-Au\|\to 0$. From (1.1), we have

$$\begin{aligned}\|y_n-u\|^2 &= \|P_C(x_n-\lambda_nAx_n)-P_C(u-\lambda_nAu)\|^2\\ &\le \langle(x_n-\lambda_nAx_n)-(u-\lambda_nAu),y_n-u\rangle\\ &= \frac{1}{2}\{\|(x_n-\lambda_nAx_n)-(u-\lambda_nAu)\|^2+\|y_n-u\|^2\\ &\quad-\|(x_n-\lambda_nAx_n)-(u-\lambda_nAu)-(y_n-u)\|^2\}\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{2}\{\|x_n-u\|^2+\|y_n-u\|^2-\|(x_n-y_n)-\lambda_n(Ax_n-Au)\|^2\}\\
&=\frac{1}{2}\{\|x_n-u\|^2+\|y_n-u\|^2-\|x_n-y_n\|^2\\
&\quad+2\lambda_n\langle x_n-y_n,Ax_n-Au\rangle-\lambda_n^2\|Ax_n-Au\|^2\}.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\|y_n-u\|^2\le\|x_n-u\|^2-\|x_n-y_n\|^2+2\lambda_n\langle x_n-y_n,Ax_n-Au\rangle\\
-\lambda_n^2\|Ax_n-Au\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
\|z_n-u\|^2&=\|\alpha_nx_n+(1-\alpha_n)S^ny_n-u\|^2\\
&\le\alpha_n\|x_n-u\|^2+(1-\alpha_n)\|S^ny_n-u\|^2\\
&\le\alpha_n\|x_n-u\|^2+(1-\alpha_n)k_n^2\|y_n-u\|^2\\
&\le(1-\alpha_n)(k_n^2-1)\|x_n-u\|^2+\|x_n-u\|^2-(1-\alpha_n)k_n^2\|x_n-y_n\|^2\\
&\quad+2\lambda_n(1-\alpha_n)k_n^2\langle x_n-y_n,Ax_n-Au\rangle-\lambda_n^2(1-\alpha_n)k_n^2\|Ax_n-Au\|^2.
\end{aligned}$$

Since $k_n\to1$, $\|z_n-x_n\|\to0$ and $\|Ax_n-Au\|\to0$, we obtain $\|x_n-y_n\|\to0$.

In virtue of

$$\begin{aligned}
\|z_n-S^ny_n\|=\alpha_n\|x_n-S^ny_n\|\to0,\\
\|z_n-y_n\|\le\|z_n-x_n\|+\|x_n-y_n\|\to0,
\end{aligned}$$

we have

$$\|S^ny_n-y_n\|\le\|S^ny_n-z_n\|+\|z_n-y_n\|\to0.$$

We deduce that

$$\begin{aligned}
\|Sy_n-y_n\|&\le\|Sy_n-S^{n+1}y_n\|+\|S^{n+1}y_n-S^{n+1}y_{n+1}\|\\
&\quad+\|S^{n+1}y_{n+1}-y_{n+1}\|+\|y_{n+1}-y_n\|\\
&\le L\|y_n-S^ny_n\|+\|S^{n+1}y_{n+1}-y_{n+1}\|+(1+L)\|y_n-y_{n+1}\|\to0
\end{aligned}$$

and

$$\begin{aligned}
\|S^nx_n-x_n\|&\le\|S^nx_n-S^ny_n\|+\|S^ny_n-y_n\|+\|y_n-x_n\|\\
&\le(L+1)\|y_n-x_n\|+\|S^ny_n-y_n\|\to0.
\end{aligned}$$

Similarity, we have $\|Sx_n-x_n\|\to0$.

By the assumption of Theorem 2.1, S is semi-compact, therefore it follows that there exists a subsequence $\{x_{n_i}\}\subset\{x_n\}$ such that $x_{n_i}\to w$. Hence we have that

$$\|Sw-w\|=\lim\|Sx_{n_i}-x_{n_i}\|=0,$$

i.e.,$w\in F(S)$.

We now prove that $w=P_{F(S)}(x_0)$ and $x_n\to w$. Put $w'=P_{F(S)}(x_0)$ and consider the sequence $\{x_0-x_{n_i}\}$. Then we have $x_0-x_{n_i}\to x_0-w$ and by the fact that $\|x_0 - x_{n+1}\| \leq \|x_0 - w'\|$ for all $n \geq 0$ which is implied by the fact that $x_{n+1} = P_{H_n \cap W_n}(x_0)$, we obtain

$$\|x_0 - w'\| \leq \|x_0 - w\| = \lim_{i\to\infty} \|x_0 - x_{n_i}\| = \lim_{i\to\infty} \|x_0 - x_{n_i+1}\| \leq \|x_0 - w'\|.$$

This implies that $\|x_0 - w'\| = \|x_0 - w\|$. (hence $w' = w$ by the uniqueness of the nearest point projection of x_0 onto $F(S)$.) It follows that $x_{n_i} \to w'$. Replacing $\{x_n\}$ with $\{x_{n_i}\}$, also there exists a convergence subsequence of $\{x_{n_i}\}$. Hence, we conclude that $x_n \to w' = w$.

Thus, $y_n \to w$. Next we show that $w \in VI(C, A)$.

Since $\{\lambda_n\} \subset [0, 2\alpha]$, $\lambda_n \to \lambda_0$, thus $y_n \to P_C(I - \lambda_0 A)w$. Indeed,

$$\begin{aligned}\|y_n - P_C(I - \lambda_0 A)w)\| &= \|P_C(I - \lambda_n A)x_n - P_C(I - \lambda_0 A)w\| \\ &\leq \|(I - \lambda_n A)x_n - (I - \lambda_0 A)w\| \\ &\leq \|x_n - w\| + \lambda_n \|Ax_n - Aw\| + |\lambda_n - \lambda_0| \|Aw\|.\end{aligned}$$

Since A is $1/\alpha$-Lipschitz continuous, hence,$\|Ax_n - Aw\| \to 0$. We have $y_n \to P_C(I - \lambda_0 A)w$. On the other hand, from $y_n \to w$ and the uniqueness of the limit, we have $w = P_C(I - \lambda_0 A)w$, i.e. $w \in VI(C, A)$. At the same time we also show that $\{x_n\}$ converges strongly to $w = P_{F(S)\cap VI(C,A)}(x_0)$. □

Remark 2.2. Theorem 2.1 is generalized Theorem 2.2 in [14]. The operator S extend from asymptotically nonexpansive mapping to asymptotically quasi-nonexpansive mapping. If $S = I$ is identical operator, then $\{x_n\}$ converges strongly to $P_{VI(C,A)}(x_0)$.

3. Applications

In this section, we prove two theorems in a real Hilbert space by using Theorem 2.1. A mapping $T : C \to C$ is called strictly pseudocontractive if there exists k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2$$

for all $x, y \in C$. If $k = 0$, then T is nonexpansive.

Put $A = I - T$, where $T : C \to C$ is a strictly pseudocontractive mapping with k. Then A is $(1 - k)/2$-inverse-strongly monotone (see [3]). Actually, we have, for all $x, y \in C$,

$$\|(I - A)x - (I - A)y\|^2 \leq \|x - y\|^2 + k\|Ax - Ay\|^2.$$

On the other hand, since H is a real Hilbert space, we have

$$\|(I - A)x - (I - A)y\|^2 = \|x - y\|^2 + \|Ax - Ay\|^2 - 2\langle x - y, Ax - Ay\rangle.$$

Hence we have

$$\langle x - y, Ax - Ay\rangle \geq \frac{1 - k}{2}\|Ax - Ay\|^2.$$

Using Theorem 2.1, we first prove a strong convergence theorem for finding a common fixed point of a asymptotically quasi-nonexpansive mapping and a strictly pseudocontractive mapping.

Theorem 3.1. *Let C be a bounded closed convex subset of a real Hilbert space H. let T be a k-strictly pseudocontractive mapping of C into itself such that $F(S)\cap F(T)\neq\emptyset$ and let S be a uniformly L-Lipschitzian, asymptotically quasi-nonexpansive mapping of C into itself with sequence $\{k_n\}\subset[1,\infty)$. Suppose $x_0\in C$ and $\{x_n\}$ is given by*

$$\begin{cases} x_0 \in C, \quad \lambda > 0, \\ y_n = (1-\lambda_n)x_n + \lambda_n T x_n, \\ z_n = \alpha_n x_n + (1-\alpha_n) S^n y_n, \\ H_n = \{v \in C : \|z_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n\}, \\ W_n = \{z \in C : \langle z - x_n, x_0 - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{H_n\cap W_n}(x_0), \quad n \geq 0, \end{cases}$$

where

$$\theta_n = (1-\alpha_n)(k_n^2-1)(diamC)^2 \to 0, \qquad as \ \ n\to\infty,$$

$\{\alpha_n\}$ is a sequence in $[0,1)$ and $\{\lambda_n\}$ is a sequence in $[0,1-k]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\{\lambda_n\}\in[a,b]$ for some a,b with $0<a<b<1-k$, and $\lambda_n\to\lambda_0$, $\lim_{n\to\infty}\alpha_n=0$. Assume that S is semi-compact.Then $\{x_n\}$ converges strongly to $P_{F(S)\cap F(T)}(x_0)$.

Proof. Put $A=I-T$. Then A is $(1-k)/2$-inverse-strongly monotone. We have $F(T)=VI(C,A)$ and $P_C(x_n-\lambda_n Ax_n)=(1-\lambda_n)x_n+\lambda_n Tx_n$(see[14]). So, by Theorem 2.1, we obtain the desired result. □

Using Theorem 2.1, we also have the following:

Theorem 3.2. *Let H be a real Hilbert space. Let A be an α-inverse-strongly monotone mapping of H into itself and let S be a uniformly L-Lipschitzian, asymptotically quasi-nonexpansive mapping of H into itself such that $F(S)\cap A^{-1}0\neq\emptyset$. Suppose*

$$\begin{cases} x_0 \in C, \quad \lambda > 0, \\ y_n = x_n - \lambda_n A x_n, \\ z_n = \alpha_n x_n + (1-\alpha_n) S^n y_n, \\ H_n = \{v \in C : \|z_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n\}, \\ W_n = \{z \in C : \langle z - x_n, x_0 - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{H_n\cap W_n}(x_0), \quad n \geq 0, \end{cases}$$

where

$$\theta_n = (1-\alpha_n)(k_n^2-1)M \to 0, \qquad as \ \ n\to\infty.$$

If we assume that $\{x_n\}$ is bounded sequence with bounds M,$\{\alpha_n\}$ is a sequence in $[0,1)$ and $\{\lambda_n\}$ is a sequence in $[0,2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\{\lambda_n\}\in[a,b]$ for some a,b with $0<a<b<2\alpha$, and $\lambda_n\to\lambda_0$, $\lim_{n\to\infty}\alpha_n=0$. Assume that S is semi-compact. Then $\{x_n\}$ converges strongly to $P_{F(S)\cap A^{-1}0}(x_0)$.

Proof. We have $A^{-1}0=VI(H,A)$. So, putting $P_H=I$, by Theorem 2.1, we obtain the desired result. □

References

[1] F. E. Browder, *Nonlinear monotone operators and convex sets in Banach spaces*, Bull. Amer. Math. Soc. **71** (1965), 780–785.

[2] F. E. Browder, *The fixed point theory of multi-valued mappings in topological vector spaces*, Math. Ann. **177** (1968), 283–301.

[3] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl. **20** (1967), 197–228.

[4] R. E. Bruck, *On the weak convergence of an ergodic iteration for the solution of variational inequalities for monotone operators in Hilbert space*, J. Math. Anal. Appl. **61** (1977), 159–164.

[5] H. Iiduka, W. Takahashi and M. Toyoda, *Approximation of solutions of variational inequalities for monotone mappings*, Pan Amer. Math. J. **14** (2004), 49–61.

[6] J. L. Lions and G. Stampacchia, *Variational inequalities*, Comm. Pure Appl. Math. **20** (1967), 493–517.

[7] F. Liu and M. Z. Nashed, *Regularization of nonlinear ill-posed variational inequalities and convergence rates*, Set-Valued Anal. **6** (1998), 313–344.

[8] K. Nakajo and W. Takahashi, *Strong and weak convergence theorems by an improved splitting method*, Comm. Appl. Nonlinear Anal. **9** (2002), 99–107.

[9] Z. Opial, *Weak convergence ofthe sequence of successive approximation for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.

[10] W. Takahashi, *Nonlinear complementarity problem and systems of convex inequalities*, J. Optim. Theory Appl. **24** (1978), 493–508.

[11] T. H. Kim and H. K. Xu, *Remarks on asymptotically nonexpansive mappings*, Nonlinear Anal. **41** (2000), 405–415.

[12] Hideaki Iiduka and Wataru Takahashi, *Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings*, Nonlinear Anal. **61** (2005), 341–350.

[13] Tae-Hwa Kima and Hong-Kun Xu, *Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups*, Nonlinear Anal. **64** (2006), 1140–1152.

[14] K. Goebel and W. A. Kirk, *A Fixed Point Theorem for Asymptotically Nonexpansive Mappings*, Proc. Amer. Math. Soc. **35** (1972), 171–174.

[15] Jong Kyu Kim, Young Man Nam and Jae Yull Sim, *Convergence theorems of implicit iterative sequences for a finite family of asymptotically quasi-nonexpansive type mappings*, Nonlinear Analysis **71** (2009), e2839–e2848.

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