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# VISCOSITY METHODS OF APPROXIMATION FOR A COMMON SOLUTION OF A FINITE FAMILY OF ACCRETIVE OPERATORS

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Abstract. In this paper, we try to extend the viscosity approximation technique to find a particular common zero of a finite family of accretive mappings in a Banach space which is strictly convex reflexive and has a weakly sequentially continuous duality mapping. The explicit viscosity approximation scheme is proposed and its strong convergence to a solution of a variational inequality is proved.

## 1. Introduction

Let $E$ be a Banach space with a dual space of $E^*$, $C$ a nonempty closed convex subset of $E$, and $T : C \to C$ a mapping. Recall that $T$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A point $x \in C$ is a fixed point of $T$ proved $Tx = x$. Denote by $Fix(T)$ the set of fixed points of $T$. $f : C \to C$ is a contraction on $C$ if there exists a constant $\beta \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \beta\|x - y\|$, $\forall x, y \in C$. The normalized duality mapping $J$ from $E$ to $2^{E^*}$ is given by $J(x) = \{g \in E^* : \langle x, g\rangle = \|x\|^2 = \|g\|^2\}$, $x \in E$ where $E^*$ denotes the dual space of $E$ and $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing.

Recall that an operator $A$ with $D(A)$ and $R(A)$ in $E$ is said to be accretive, if for each $x_i \in D(A)$ and $y_i \in A(x_i)$ $(i = 1, 2)$, there is a $j \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, j\rangle \geq 0.$$

An accretive operator $A$ is $m$-accretive if $R(I + \lambda A) = E$ for all $\lambda > 0$. Denote by $N(A)$ the zero set of $A$: i.e.,

$$N(A) := A^{-1}(0) = \{x \in D(A) : 0 \in Ax\}.$$

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If $A$ is accretive, then we can define, for each $r > 0$, a nonexpansive single-valued mapping $J_r : R(I+rA) \to D(A)$ by $J_r := (I+rA)^{-1}$, which is called the resolvent of $A$. we also know that for an accretive operator $A$, $N(A) = Fix(J_r)$.

Recently, Zegeye and Shahzad [13] have proved the strong convergence theorem for a finite family of accretive operators, let $l \geq 1$ be a positive integer, and define the set $\Lambda = 1, 2, \cdots, l$. We also can see [6], J. S. Jung also has proved the strong convergence of an iterative method for finding common zeros of a finite family of accretive operators.

**Theorem 1.1.** ([13]) *Let $E$ be a strictly convex and real reflexive Banach space $E$ which has a uniformly Gâteaux differentiable norm, and $K$ a nonempty closed convex subset of $E$. Let $A_i : i \in \Lambda : K \to E$ be a finite family of $m$-accretive operators with $\bigcap_{i=1}^{l} N(A_i) \neq \emptyset$. Assume that every nonempty closed bounded convex subset of $E$ has the fixed point property for nonexpansive mappings. For any given $u$, $x_0 \in C$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) S_r x_n, \quad n \geq 0, \tag{1}$$

*where $S_r = a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_l J_{A_l}$ with $J_{A_i} = (I + A_i)^{-1}$, for $i = 0, 1, 2, \cdots, l$, $a_i \in (0, 1), \sum_{i=0}^{l} a_i = 1$, and $\{\alpha_n\}$ a real sequence satisfying the conditions (C1) $\lim_{n\to\infty} \alpha_n = 0$; (C2) $\sum_{n=0}^{\infty} \alpha_n = +\infty$ and (C3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$ or(C3)* $\lim_{n\to\infty} \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_{n+1}} = 0$. Then the sequence $\{x_n\}$ converges strongly to a common zero of $\{A_i : i \in \Lambda\}$.*

And in [5], L. Hu, L. Liu generalized and extended the result of Zegeye and Shahzad [13], they proved the following theorem:

**Theorem 1.2.** ([5]) *Let $E$ be a strictly convex and real reflexive Banach space $E$ which has a uniformly Gâteaux differentiable norm, and $C$ a nonempty closed convex subset of $E$. Let $\{A_i : i \in \Lambda\} : C \to E$ be a finite family of accretive operators satisfying the following range conditions:*

$$cl(D(A_i)) \subseteq C \subset \bigcap_{r>0} R(I + rA_i), \quad i = 1, 2, \cdots, l.$$

*Assume that $\bigcap_{i=1}^{l} N(A_i) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $(0, 1)$ and $\{r_n\}$ is a sequence in $(0, +\infty)$, satisfying conditions:*

(i) *(C1)$\lim_{n\to\infty} \alpha_n = 0$; (C2)$\sum_{n=0}^{\infty} \alpha_n = +\infty$;*
(ii) *$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;*
(iii) *$\lim_{n\to\infty} r_n = r, r \in R^+$.*

*For any $u \in C$, $x_0 \in C$, the sequence $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S_{r_n} x_n, \quad n \geq 0, \tag{2}$$

*where $S_{r_n} = a_0 I + a_1 J_{r_n}^1 + a_2 J_{r_n}^2 + \cdots + a_l J_{r_n}^l$ with $J_{r_n}^i = (I + r_n A_i)^{-1}$, for $i = 0, 1, 2, \cdots, l$, $a_i \in (0, 1), \sum_{i=0}^{l} a_i = 1$. Then the sequence $\{x_n\}$ converges strongly to a common zero of $\{A_i : i \in \Lambda\}$.*

The viscosity iterative has been studied by many researchers (see, [7], [8], [3], [12]). In 2000, Moudafi [7] introduced viscosity approximation method and proved that if $E$ is a real Hilbert space, for given $x_0 \in C$, the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} := \alpha_n f(x_n) + (1-\alpha_n)Tx_n, n \geq 0, \tag{3}$$

where $f : C \to C$ is a contraction mapping with constant $\beta \in (0,1)$ and $\alpha_n \subseteq (0,1)$ satisfies certain conditions, converges strongly to a fixed point of $T$ in $C$ which is the unique solution to the following variational inequality:

$$\langle (I-f)x^*, x-x^* \rangle \geq 0, \quad \forall x \in Fix(T).$$

In 2004, Xu [12] studied further the viscosity approximation method for nonexpansive mappings in uniformly smooth Banach spaces. This result of Xu [12] extends Theorem 2.2 of Moudafi [7] to a Banach space setting.

In 2006, Paul-Emile Maing é [8] considered the general iterative method

$$x_{n+1} = \alpha_n T_n x_n + (1-\alpha_n) J^A_{r_n} x_n, \tag{4}$$

for calculating a particular zero of $A$, an $m$-accretive oprator in a Banach space $E$, $T_n$ being a sequence of nonexpansive self-mappings in $E$. Under suitable conditions on the parameters and $E$, they stated strong and weak convergence results of $\{x_n\}$.

Motivated and inspired by above works, in this paper, we introduce and study the following iterative algorithm in strictly convex reflexive Banach spaces $E$ with a weakly sequentially continuous duality mapping from $E$ to $E^*$: for given $x_0 \in C$, let the sequence $\{x_n\}$ be defined by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_{r_n} x_n, \quad \forall n \geq 0, \tag{5}$$

where $S_{r_n} = a_0 I + a_1 J^1_{r_n} + a_2 J^2_{r_n} + \cdots + a_l J^l_{r_n}$ with $J^i_{r_n} = (I + r_n A_i)^{-1}$ for $i = 1, 2, \cdots, l$, $a_i \in (0,1)$, $\sum_{i=0}^{l} a_i = 1$ and $\{r_n\} \subset (0, +\infty)$. $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$. The present results improve and extend many known results in the literature.

## 2. Preliminaries

Recall that a gauge function $\phi : R^+ \to R^+$ such that $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. The duality mapping $J_\phi : E \to E^*$ associated with a gauge function $\phi$ is defined by

$$J_\phi(x) = \{u^* \in E^* : \langle x, u^* \rangle = \|x\| \|u^*\|, \ \ \|u^*\| = \phi(\|x\|), \forall x \in E\}.$$

In the particular case $\phi(t) = t$, the duality map $J = J_\phi$ is called the normal duality map. We note that $J_\phi(x) = \frac{\phi(\|x\|)}{\|x\|} J(x)$, for $x \neq 0$. It is known that if $E$ is smooth then $J_\phi$ is single valued and norm-to-weak$^*$ continuous(see[2]).

Following Browder [1], we say that a Banach space $E$ has the weak continuous duality mapping if there exists a gauge function $\phi$ for which the duality map $J_\phi$ is single valued and weak to weak$^*$ sequentially continuous (i.e., if

$\{x_n\}$ is a sequence in $E$ weakly convergent to a point $x$, then the sequence $\{J_\phi(x_n)\}$ converges weak$^*$ to $J_\phi(x)$). If Banach space $E$ admits weakly sequentially continuous duality mapping $J$, then by ([4] Lemma 1), we get that duality mapping $J$ is single-valued. It is well known $l^p(1 < p < \infty)$ spaces have a weakly continuous duality mapping $J_\phi$ with a gauge function $\phi(t) = t^{p-1}$. Setting

$$\Phi(t) = \int_0^t \phi(\tau)\mathrm{d}\tau, \quad t \geq 0,$$

one can see that $\Phi(t)$ is a convex function and $J_\phi = \partial\Phi(\|x\|)$, for $x \in E$, where $\partial$ denotes the subdifferential in the sense of convex analysis.

Recall that a Banach space $E$ is said to be smooth if and only if the duality mapping $J$ is single-valued. A Banach space $E$ is called strictly convex if for $a_i \in (0, 1)$, $i \in \Lambda$, such that $\sum_{i=1}^{l} a_i = 1$, we have $\|a_1x_1 + a_2x_2 + \cdots + a_lx_l\| < 1$ for $x_i \in U$, $i \in \Lambda$ and $x_i \neq x_j$ for some $i \neq j$. For in a strictly convex Banach space we have that if $\|x_1\| = \|x_2\| = \cdots = \|x_l\| = \|a_1x_1 + a_2x_2 + \cdots + a_lx_l\|$, for $x_i \in E$, $a_i \in (0, 1)$, $i \in \Lambda$ and $\sum_{i=1}^{l} a_i = 1$, then $x_1 = x_2 = \cdots = x_l$.

Let $C$ a nonempty closed convex subset of $E$ and $Q$ a mapping of $E$ onto $C$. Then $Q$ is said to be sunny if $Q(Q(x) + t(x - Q(x))) = Q(x)$ for all $x \in E$ and $t \geq 0$. A mapping $Q$ of $E$ into $E$ is said to be a retraction if $Q^2 = Q$. If a mapping $Q$ is a retraction, then $Q(z) = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of $Q$. A subset $C$ of $E$ is said to be a sunny nonexpansive retract of $E$ if there exists a sunny nonexpansive retraction of $E$ onto $C$ and it is said to be a nonexpansive retract of $E$ if there exists a nonexpansive retraction of $E$ onto $C$. If $E = H$, the metric projection $P_C$ is a sunny nonexpansive retraction from $H$ to any closed convex subset of $H$.

**Lemma 2.1.** (see [10]) *Let $E$ be a smooth Banach space and $C$ a nonempty subset of $E$. Let $Q : E \to C$ be a retraction and $J$ the normalized duality mapping on $E$. Then the following are equivalent:*

(i) *$Q$ is sunny nonexpansive;*
(ii) *$\langle x - Q(x), J(y - Q(x))\rangle \leq 0$, for all $x \in E$ and $y \in K$.*

We note that Lemma 2.1 still holds if the normalized duality map $J$ is replaced with the general duality map $J_\phi$, where $\phi$ is a gauge function.

**Lemma 2.2.** (see [11]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\mu_n, \quad \forall n \geq 0,$$

*where (i) $0 < \alpha_n < 1$, (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose, either $\sigma_n = o(\alpha_n)$, or $\sum_{n=1}^{\infty} \sigma_n < \infty$, where $\sigma_n = \alpha_n\mu_n$ or (iii) $\limsup_{n\to\infty} \mu_n \leq 0$. Then $s_n \to 0$ as $n \to \infty$.*

**Lemma 2.3.** (see [2]) *Let $E$ be a real Banach space. Then for all $x, y \in E$ we get that*

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, j_\phi(x + y)\rangle, \quad j_\phi \in J_\phi. \tag{6}$$

**Lemma 2.4.** (The Resolvent Identity) *For $\lambda > 0$ and $\mu > 0$ and $x \in E$,*

$$J_\lambda x = J_\mu(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J_\lambda x).$$

**Lemma 2.5.** (see [9]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $E$ such that*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n, \quad \forall n \geq 0,$$

*where $\{\beta_n\}$ is a sequence in $(0, 1)$ such that $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Assume*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

*Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

**Lemma 2.6.** ([13]) *Let $C$ be a nonempty closed convex subset of a strictly convex Banach space $E$. Let $\{A_i : 1 < i < l\} : C \to E$ be a finite family of accretive operators such that $\bigcap_{i=1}^{l} N(A_i) \neq \emptyset$, satisfying the range conditions:*

$$cl(D(A_i)) \subseteq C \subset \bigcap_{r>0} R(I + rA_i), \quad i = 1, 2, \cdots, l.$$

*Let $a_0, a_1, \cdots, a_l$ be real numbers in $(0, 1)$ such that $\sum_{i=0}^{l} a_i = 1$ and $S_r = a_0 I + a_1 J_r^1 + a_2 J_r^2 + \cdots + a_l J_r^l$, where $J_r^i = (I + rA_i)^{-1}$ and $r > 0$. Then $S_r$ is nonexpansive and $Fix(S_r) = \bigcap_{i=1}^{l} N(A_i)$.*

**Lemma 2.7.** (Demiclosedness Principle) *If $K$ is closed convex subset of a real space $E$ satisfying Opial's condition and $T$ is a nonexpansive mapping, then $x_n \rightharpoonup x$ and $(I - T)x_n \to y$ implies that $(I - T)x = y$.*

## 3. Main results

Throughout this section, we assume:

(i) $E$ is a strictly convex reflexive Banach space with a weakly sequentially continuous duality mapping $J_\phi$ for some gauge $\phi$. $C$ is a nonempty closed convex subset of $E$ which is also a sunny nonexpansive retract of $E$.

(ii) The real sequence $\{\alpha_n\}$ satisfies the two conditions: (C1)$\lim_{n\to\infty} \alpha_n = 0$, and (C2)$\sum_{n=0}^{\infty} \alpha_n = +\infty$.

**Theorem 3.1.** *Let $\{A_i : 1 < i < l\} : C \to E$ be a finite family of accretive operators satisfying the following range conditions:*

$$cl(D(A_i)) \subseteq C \subset \bigcap_{r>0} R(I + rA_i), \quad i = 1, 2, \cdots, l.$$

*Assume that $\bigcap_{i=1}^{l} N(A_i) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are are three sequences in $(0, 1)$ and $\{r_n\}$ is a sequence in $(0, +\infty)$, satisfying conditions*

$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\lim_{n\to\infty} r_n = r$, $r \in R^+$. *For any* $x_0 \in C$, *the sequence* $\{x_n\}$*is given by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_{r_n} x_n, \quad n \geq 0, \tag{7}$$

*where* $f : C \to C$ *is a contraction with constant* $\beta$, *and* $S_{r_n} = a_0 I + a_1 J^1_{r_n} + a_2 J^2_{r_n} + \cdots + a_l J^l_{r_n}$ *with* $J^i_{r_n} = (I + r_n A_i)^{-1}$, *for* $i = 0, 1, 2, \cdots, l$, $a_i \in (0, 1)$, $\sum_{i=0}^{l} a_i = 1$. *Then the sequence* $\{x_n\}$ *converges strongly to* $x^* = Q(f(x^*))$, *which is a common zero of* $\{A_i : i \in \Lambda\}$ . *Moreover,* $x^*$ *is the solution of the variational inequality:*

$$\langle (I - f)x^*, J(x^* - x)\rangle \leq 0, \quad \forall x \in \bigcap_{i=1}^{l} N(A_i). \tag{$*$}$$

*Proof.* By Lemma 2.6, this implies that $F := Fix(S_{r_n}) = \bigcap_{i=1}^{l} N(A_i) \neq \emptyset$. Take $p \in F$, we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n (f(x_n) - p) + \beta_n (x_n - p) + \gamma_n (S_{r_n} - p)\| \\
&\leq \alpha_n \|f(x_n) - p\| + (1 - \alpha_n)\|x_n - p\| \\
&\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n)\|x_n - p\| \\
&\leq [1 - (1 - \beta)\alpha_n]\|x_n - p\| + \alpha_n \|f(p) - p\|.
\end{aligned}$$

By induction, we obtain for all $n \geq 0$,

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{1}{1 - \beta}\|f(p) - p\|\}.$$

Therefore, the sequences $\{x_n\}$, $\{f(x_n)\}$ and $\{S_{r_n} x_n\}$ are bounded. Rewrite the iterative process (7) as follow:

$$\begin{aligned}
x_{n+1} &= \beta_n x_n + (1 - \beta_n)\frac{\alpha_n f(x_n) + \gamma_n S_{r_n} x_n}{1 - \beta_n} \\
&= \beta_n x_n + (1 - \beta_n) y_n,
\end{aligned}$$

where $y_n = \frac{\alpha_n}{1-\beta_n} f(x_n) + \frac{\gamma_n}{1-\beta_n} S_{r_n} x_n$. We get that $\{y_n\}$ is also bounded. After some manipulation this yields

$$\begin{aligned}
y_{n+1} - y_n &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} f(x_{n+1}) - \frac{\alpha_n}{1 - \beta_n} f(x_n) \\
&\quad + \left(1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\right)(S_{r_{n+1}} x_{n+1} - S_{r_n} x_n) \\
&\quad + \left(\frac{\alpha_n}{1 - \beta_n} - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\right) S_{r_n} x_n.
\end{aligned}$$

By the resolvent identity, it follows that

$$\begin{aligned}\|J^i_{r_{n+1}}x_{n+1} - J^i_{r_n}x_n\| &= \|J^i_{r_n}(\frac{r_n}{r_{n+1}}x_{n+1} + (1-\frac{r_n}{r_{n+1}})J^i_{r_{n+1}}x_{n+1}) - J^i_{r_n}x_n\| \\ &\leq \|\frac{r_n}{r_{n+1}}(x_{n+1}-x_n) + (1-\frac{r_n}{r_{n+1}})(J^i_{r_{n+1}}x_{n+1} - J^i_{r_n}x_n)\| \\ &\leq \frac{r_n}{r_{n+1}}\|x_{n+1}-x_n\| + |1-\frac{r_n}{r_{n+1}}|M,\end{aligned}$$

where $M = \sup_{n\geq 1}\{x_n - J^i_{r_{n+1}}x_n\}$. Since $S_{r_n} = a_0 I + \sum_{n=1}^{l} a_i J^i_{r_n}$, we have

$$\begin{aligned}\|S_{r_{n+1}}x_{n+1} - S_{r_n}x_n\| &= \|a_0(x_{n+1}-x_n) + \sum_{n=1}^{l} a_i(J^i_{r_{n+1}}x_{n+1} - J^i_{r_n}x_n)\| \\ &\leq a_0\|x_{n+1}-x_n\| + \sum_{n=1}^{l} a_i\|J^i_{r_{n+1}}x_{n+1} - J^i_{r_n}x_n\| \\ &\leq \left[\frac{r_n}{r_{n+1}} + a_0(1-\frac{r_n}{r_{n+1}})\right]\|x_{n+1}-x_n\| \\ &\quad + (1-a_0)|1-\frac{r_n}{r_{n+1}}|M.\end{aligned}$$

It follows that

$$\begin{aligned}&\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\| \\ &\leq \|\frac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_{n+1}) - \frac{\alpha_n}{1-\beta_n}f(x_n)\| + (1-\frac{\alpha_{n+1}}{1-\beta_{n+1}})\|S_{r_{n+1}}x_{n+1} - S_{r_n}x_n\| \\ &\quad + |\frac{\alpha_n}{1-\beta_n} - \frac{\alpha_{n+1}}{1-\beta_{n+1}}|\|S_{r_n}x_n\| - \|x_{n+1}-x_n\| \\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - f(x_n)\| + |\frac{\alpha_n}{1-\beta_n} - \frac{\alpha_{n+1}}{1-\beta_{n+1}}|\|f(x_n)\| \\ &\quad + (1-\frac{\alpha_{n+1}}{1-\beta_{n+1}})[\frac{r_n}{r_{n+1}} + a_0(1-\frac{r_n}{r_{n+1}})]\|x_{n+1}-x_n\| \\ &\quad + (1-\frac{\alpha_{n+1}}{1-\beta_{n+1}})(1-a_0)|1-\frac{r_n}{r_{n+1}}|M \\ &\quad + |\frac{\alpha_n}{1-\beta_n} - \frac{\alpha_{n+1}}{1-\beta_{n+1}}|\|S_{r_n}x_n\| - \|x_{n+1}-x_n\| \\ &= \left\{\left(1-\frac{\alpha_{n+1}}{\beta_{n+1}}\right)\left[\frac{r_n}{r_{n+1}} + a_0(1-\frac{r_n}{r_{n+1}})\right] + \frac{\alpha_{n+1}}{1-\beta_{n+1}}\beta - 1\right\}\|x_{n+1}-x_n\| \\ &\quad + (1-\frac{\alpha_{n+1}}{1-\beta_{n+1}})(1-a_0)|1-\frac{r_n}{r_{n+1}}|M \\ &\quad + |\frac{\alpha_n}{1-\beta_n} - \frac{\alpha_{n+1}}{1-\beta_{n+1}}|(\|S_{r_n}x_n\| + f\|(x_n)\|),\end{aligned}$$

from $\{x_n\}$, $\{f(x_n)\}$ and $\{S_{r_n}x_n\}$ are bounded, $\lim_{n\to\infty} r_n = r$, $\lim_{n\to\infty}\alpha_n = 0$, we have

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \leq 0.$$

Consequently, by Lemma 2.5, we obtain $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

$$\begin{aligned}
&\|x_{n+1} - \frac{\beta_n x_n + \gamma_n S_{r_n} x_n}{1-\alpha_n}\| \\
&= \|\alpha_n f(x_n) + (1-\alpha_n)\frac{\beta_n x_n + \gamma_n S_{r_n} x_n}{1-\alpha_n} - \frac{\beta_n x_n + \gamma_n S_{r_n} x_n}{1-\alpha_n}\| \\
&\le \alpha_n \|f(x_n) - \frac{\beta_n x_n + \gamma_n S_{r_n} x_n}{1-\alpha_n}\| \\
&\to 0 \quad (n\to\infty),
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + \gamma_n x_n + \gamma_n S_{r_n} x_n - x_n\| \\
&= \|\alpha_n f(x_n) + \gamma_n S_{r_n} x_n - (\alpha_n + \gamma_n) x_n\| \\
&\le (1-\beta_n)\|y_n - x_n\| \\
&\to 0.
\end{aligned}$$

Obviously,

$$\|x_n - \frac{\beta_n x_n + \gamma_n S_{r_n} x_n}{1-\alpha_n}\| = \frac{\gamma_n}{1-\alpha_n}\|x_n - S_{r_n} x_n\|.$$

By the conditions $\lim_{n\to\infty}\alpha_n = 0$ and $\limsup_{n\to\infty}\beta_n < 1$, it follows that $\liminf_{n\to\infty}\gamma_n > 0$. Therefore, we obtain

$$\|x_n - S_{r_n} x_n\| \le \frac{1-\alpha_n}{\gamma_n}\|x_n - x_{n+1}\| \left( \|x_{n+1} - \frac{\beta_n x_n + \gamma_n S_{r_n} x_n}{1-\alpha_n}\| \right) \to 0 \ \ (n\to\infty).$$

By the resolvent identity and $S_{r_n} = a_0 I + \sum_{i=1}^{l} J^i_{r_n}$, this implies that

$$\begin{aligned}
\|S_{r_n} x_n - S_r x_n\| &= \|\sum_{i=1}^{l} a_i (J^i_{r_n} x_n - J_r x_n)\| \\
&\le \sum_{i=1}^{l} a_i \|J^l_r(\frac{r}{r_n} x_n + (1-\frac{r}{r_n}) J^i_{r_n} x_n) - J^i_r x_n)\| \\
&\le \sum_{i=1}^{l} a_i \|(\frac{r}{r_n} x_n + (1-\frac{r}{r_n}) J^i_{r_n} x_n) - x_n)\| \\
&\le \sum_{i=1}^{l} a_i |1-\frac{r}{r_n}| \|x_n - J^i_{r_n} x_n\| \to 0 \quad (n\to\infty).
\end{aligned}$$

Hence, we have

$$\|x_n - S_r x_n\| \le \|x_n - S_{r_n} x_n\| + \|S_{r_n} x_n - S_r x_n\| \to 0 \quad (n\to\infty).$$

Next we shall show that

$$\limsup_{n\to\infty}\langle x^* - f(x^*), J_\phi(x^* - x_{n+1})\rangle \le 0.$$

Since $E$ is reflexive and $\{x_n\}$ is bounded, we may assume $x_{n_k} \rightharpoonup \omega$ such that

$$\limsup_{n\to\infty}\langle x^* - f(x^*), J_\phi(x^* - x_{n+1})\rangle = \limsup_{k\to\infty}\langle x^* - f(x^*), J_\phi(x^* - x_{n_k})\rangle.$$

From the Dimclosedness Principle, we have $\omega \in F$. On the other hand, from the standard characterization of retraction onto $F$ and the assumption that the duality mapping $J_\phi$ is weakly sequentially continuous, Lemma 2.1 gives that

$$\begin{aligned}\limsup_{n\to\infty}\langle x^* - f(x^*), J_\phi(x^* - x_{n+1})\rangle &= \limsup_{k\to\infty}\langle x^* - f(x^*), J_\phi(x^* - x_{n_k})\rangle \\ &= \langle x^* - f(x^*), J_\phi(x^* - \omega) \leq 0.\end{aligned}$$

From Lemma 2.3, we get that

$$\begin{aligned}&\Phi(\|x_{n+1} - x^*\|) \\ &= \Phi(\|\alpha_n(f(x_n) - f(x^*)) + \beta_n(x_n - x^*) + \gamma_n(S_{r_n}x_n - x^*) \\ &\quad + \alpha_n(f(x^*) - x^*)\|) \\ &\leq \Phi(\|\alpha_n(f(x_n) - f(x^*)) + \beta_n(x_n - x^*) + \gamma_n(S_{r_n}x_n - x^*)\|) \\ &\quad + \alpha_n\langle f(x^*) - x^*, J_\phi(x_{n+1} - x^*)\rangle \\ &\leq \Phi((\alpha_n\beta + \beta_n + \gamma_n)\|x_n - x^*\|) + \alpha_n\langle f(x^*) - x^*, J_\phi(x_{n+1} - x^*)\rangle \\ &\leq (1 - \alpha_n(1-\beta))\Phi(\|x_n - x^*\|) + \alpha_n\langle f(x^*) - x^*, J_\phi(x_{n+1} - x^*)\rangle.\end{aligned}$$

By the Lemma 2.2, we have $x_n \to x^*$ as $n \to \infty$. Moreover, $x^*$ satisfying condition $(*)$ follows from the property of $Q$. To show that it is unique, let $y^* \in F$ be another solution of the variational inequality $(*)$ in $F$. Then adding $\langle f(x^*) - x^*, J_\phi(y^* - x^*)\rangle \leq 0$ and $\langle f(y^*) - y^*, J_\phi(x^* - y^*)\rangle \leq 0$, we have $(1-\beta)\phi(\|x^* - y^*\|)\|x^* - y^*\| \leq 0$. This implies that $x^* = y^*$ □

**Theorem 3.2.** *Let $\{A_i : i \in \Lambda\} : C \to E$ be a finite family of accretive operators satisfying the following range conditions:*

$$cl(D(A_i)) \subseteq C \subset \bigcap_{r>0} R(I + rA_i), \quad i = 1, 2, \cdots, l.$$

*And assume that $\bigcap_{i=1}^{l} N(A_i) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three sequences in $(0,1)$ and $r > 0$ a real number satisfying $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. For any $x_0 \in C$, the sequence $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_r x_n, \quad n \geq 0,$$

*where $S_r = a_0 I + a_1 J_r^1 + a_2 J_r^2 + \cdots + a_l J_r^l$, with $J_r^i = (I + rA_i)^{-1}$ for $0 < a_i < 1$, $i = 0, 1, 2, \cdots, l$, $\sum_{i=0}^{l} a_i = 1$. Then the sequence $\{x_n\}$ converges strongly to $x^*$, which is a common zero of $\{A_i : i \in \Lambda\}$ and the unique solution of the variational inequality*

$$\langle (I - f)x^*, J(x^* - x)\rangle \leq 0, \quad \forall x \in \bigcap_{i=1}^{l} N(A_i).$$

As direct consequences of Theorem 3.1 and Theorem 3.2, we obtain the two corollaries below:

**Corollary 3.3.** *Let $\{A_i : i \in \Lambda\} : C \to E$ be a finite family of m-accretive operators. Assume that $\bigcap_{i=1}^{l} N(A_i) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three sequences in $(0,1)$ and $r > 0$ a real number satisfying the condition $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. For any $x_0 \in C$, the sequence $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S x_n, \quad n \geq 0,$$

*where $S = a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_l J_{A_l}$, with $J_{A_i} = (I + A_i)^{-1}$ for $0 < a_i < 1$, $i = 0, 1, 2, \cdots, l$, $\sum_{i=0}^{l} a_i = 1$. Then the sequence $\{x_n\}$ converges strongly to $x^*$, which is a common zero of $\{A_i : i \in \Lambda\}$ and the unique solution of the variational inequality*

$$\langle (I - f)x^*, J(x^* - x)\rangle \leq 0, \quad \forall x \in \bigcap_{i=1}^{l} N(A_i).$$

**Corollary 3.4.** *Let $A : C \to E$ be an m-accretive operator such that $N(A) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three sequences in $(0,1)$ and $r_n \in (0, +\infty)$, satisfying the conditions $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, and $\lim_{n\to\infty} \frac{r_n}{r_{n+1}} = 1$. For any $x_0 \in C$, the sequence $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_{r_n} x_n, \quad n \geq 0,$$

*where $J_{r_n} = (I + r_n A)^{-1}$. Then the sequence $\{x_n\}$ converges strongly to $x^*$, which is a zero of $A$ and the unique solution of the variational inequality*

$$\langle (I - f)x^*, J(x^* - x)\rangle \leq 0, \quad \forall x \in N(A).$$

**Theorem 3.5.** *Let $\{A_i : 1 < i < l\} : C \to E$ be a finite family of accretive operators satisfying the following range conditions:*

$$cl(D(A_i)) \subseteq C \subset \bigcap_{r>0} R(I + rA_i), \quad i = 1, 2, \cdots, l.$$

*Assume that $\bigcap_{i=1}^{l} N(A_i) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$ and $\{r_n\}$ is a sequence in $(0, +\infty)$, satisfying conditions $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and $\lim_{n\to\infty} r_n = r$, $r \in (0, +\infty)$. For $x_0 \in C$, the sequence $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)(\lambda x_n + (1 - \lambda) S_{r_n} x_n), \quad \forall n \geq 0,$$

*where $f : C \to C$ is a contraction with constant $\beta$, and $S_r = a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_l J_{A_l}$ with $J_{A_i} = (I + A_i)^{-1}$, for $i = 0, 1, 2, \cdots, l$, $a_i \in (0,1)$, $\sum_{i=0}^{l} a_i = 1$. Then the sequence $\{x_n\}$ converges strongly to $x^*$, which is a common zero of $\{A_i : i \in \Lambda\}$ and the solution of the variational inequality:*

$$\langle (I - f)x^*, J(x^* - x)\rangle \leq 0, \quad \forall x \in \bigcap_{i=1}^{l} N(A_i). \qquad (*)$$

*Proof.* Taking $\beta_n = (1-\alpha_n)\lambda, \quad \forall n \in N$, we have
$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} (1-\alpha_n)\lambda = \lambda \in (0,1).$$
By theorem 3.1, we obtain the conclusion. □

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