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CONVERGENCE THEOREMS OF A FINITE FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE TYPE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we study multi-step iterative algorithm with errors and give the necessary and sufficient condition to converge to common fixed points for a finite family of asymptotically quasi-nonexpansive type mappings in Banach spaces. Also we have proved a strong convergence theorem to converge to common fixed points for a finite family of said mappings on a nonempty compact convex subset of a uniformly convex Banach spaces. Our results extend and improve the corresponding results of [2, 4, 7, 8, 9, 10, 12, 15, 20].

1. Introduction

Let K be a subset of normed space E and $T\colon K \to K$ be a mapping. Then

(1) T is said to be an asymptotically nonexpansive mapping [5], if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \quad \forall x, y \in K. \tag{1}$$

(2) If for each $n \in \mathbb{N}$, there are constants $L > 0$ and $\alpha > 0$ such that

$$\|T^n x - T^n y\| \le L \|x - y\|^{\alpha}, \quad \forall x, y \in K, \tag{2}$$

then T is called a uniformly (L, α)-Lipschitz mapping. Every asymptotically nonexpansive mapping is a uniformly $(L, 1)$-Lipschitz mapping.

(3) T is said to be an asymptotically quasi-nonexpansive mapping, if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - p\| \le k_n \|x - p\|, \quad \forall x \in K \quad \text{and} \quad p \in F(T). \tag{3}$$

(4) T is said to be an asymptotically quasi-nonexpansive type mapping [13] if

$$\limsup_{n\to\infty} \left\{ \sup_{x\in K, p\in F(T)} \left(\|T^n x - p\|^2 - \|x - p\|^2 \right) \right\} \le 0. \tag{4}$$

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From the above definitions, it follows that if $F(T)$ is nonempty, then asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings are all special cases of asymptotically quasi-nonexpansive type mappings. But the converse does not hold in general.

In 1973, Petryshyn and Williamson [12] gave the necessary and sufficient conditions for Mann iterative sequence (cf.[11]) to converge to fixed points of quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [4] extended the results of Petryshyn and Williamson [12] and gave the necessary and sufficient conditions for Ishikawa iterative sequence to converge to fixed points for quasi-nonexpansive mappings.

Liu [9] extended the results of [4, 12] and gave the necessary and sufficient conditions for Ishikawa iterative sequence with errors to converge to fixed points of asymptotically quasi-nonexpansive mappings.

Iterative techniques for approximating fixed points of asymptotically nonexpansive and asymptotically quasi nonexpansive mappings in Banach spaces have been studied by many authors; See, [5, 8, 9, 15, 16, 17, 18] and the references therein. Related work can be found in [2, 7, 13, 20] and many others.

Recently, Tang and Peng [19] study the following iteration scheme in Banach space:

Let $\{T_i : i = 1, 2, \dots, k\}\colon K \to K$, where K is a nonempty subset of a Banach space E, be a finite family of uniformly quasi-Lipschitzian mappings. Let $x_1 \in K$, then the sequence $\{x_n\}$ is defined by

$$\begin{aligned} x_{n+1} &= a_{kn}x_n + b_{kn}T_k^n y_{(k-1)n} + c_{kn}u_{kn}, \\ y_{(k-1)n} &= a_{(k-1)n}x_n + b_{(k-1)n}T_{k-1}^n y_{(k-2)n} + c_{(k-1)n}u_{(k-1)n}, \\ y_{(k-2)n} &= a_{(k-2)n}x_n + b_{(k-2)n}T_{k-2}^n y_{(k-3)n} + c_{(k-2)n}u_{(k-2)n}, \\ &\vdots \\ y_{2n} &= a_{2n}x_n + b_{2n}T_2^n y_{1n} + c_{2n}u_{2n} \\ y_{1n} &= a_{1n}x_n + b_{1n}T_1^n x_n + c_{1n}u_{1n}, \qquad n \ge 1, \end{aligned} \tag{5}$$

where $\{a_{in}\}$, $\{b_{in}\}$, $\{c_{in}\}$ are sequences in $[0, 1]$ with $a_{in} + b_{in} + c_{in} = 1$ for all $i = 1, 2, \dots, k$ and $n \ge 1$, $\{u_{in},\ i = 1, 2, \dots, k,\ n \ge 1\}$ are bounded sequences in K. Also, they gave the necessary and sufficient condition to converge to common fixed points for a finite family of said mappings.

Remark 1. The iterative algorithm (5) is called multi-step iterative algorithm with errors. It contains well known iterations as special case. Such as, the modified Mann iteration (see, [16]), the modified Ishikawa iteration (see, [18]), the three-step iteration (see, [20]), the multi-step iteration (see, [7]).

The purpose of this paper is to study the multi-step iterative algorithm with bounded errors (5) for a finite family of asymptotically quasi-nonexpansive type mappings to converge to common fixed points in Banach spaces. The results obtained in this paper extend and improve the corresponding results of [2, 4, 7, 8, 9, 10, 12, 15, 20] and many others.

2. Preliminaries

The following lemmas will be used to prove the main results of this paper:

Lemma 2.1. ([17]) *Let $\{a_n\}$, $\{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le a_n + b_n, \qquad n \ge 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$. Then

(a) $\lim_{n\to\infty} a_n$ *exists.*

(b) *If $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.2. (Schu [16]) *Let E be a uniformly convex Banach space and $0 < a \le t_n \le b < 1$ for all $n \ge 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in E satisfying*

$$\limsup_{n\to\infty} \|x_n\| \le r, \quad \limsup_{n\to\infty} \|y_n\| \le r,$$
$$\lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r,$$

for some $r \ge 0$. Then

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

3. Main results

In this section, we prove strong convergence theorems of multi-step iterative algorithm with bounded errors for a finite family of asymptotically quasi-nonexpansive type mappings in a real Banach space.

Theorem 3.1. *Let E be a real arbitrary Banach space, K be a nonempty closed convex subset of E. Let $\{T_i : i = 1, 2, \dots, k\} \colon K \to K$ be a finite family of asymptotically quasi-nonexpansive type mappings. Let $\{x_n\}$ be the sequence defined by (5) with $\sum_{n=1}^{\infty} b_{in} < \infty$ and $\sum_{n=1}^{\infty} c_{in} < \infty$ for all $i = 1, 2, \dots, k$. If $\mathcal{F} = \cap_{i=1}^{k} F(T_i) \ne \emptyset$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i = 1, 2, \dots, k\}$ if and only if $\liminf\limits_{n\to\infty} d(x_n, \mathcal{F}) = 0$, where $d(x, \mathcal{F})$ denotes the distance between x and the set $\mathcal{F}$.*

Proof. The necessity is obvious and it is omitted. Now we prove the sufficiency. Since $\{u_{in},\ i = 1, 2, \dots, k,\ n \ge 1\}$ are bounded sequences in K, therefore there exists a $M > 0$ such that

$$M = \max\left\{\sup_{n\ge1} \|u_{in} - p\|,\ i = 1, 2, \dots, k\right\}.$$

Let $p \in \mathcal{F}$, it follows from definition (4) and for $i = 1, 2, \dots, k$, we have

$$\begin{aligned} &\limsup_{n\to\infty} \Big\{ \sup_{x\in K, p\in\mathcal{F}} \Big[\Big(\|T_i^n x - p\| - \|x - p\| \Big) \times \Big(\|T_i^n x - p\| + \|x - p\| \Big) \Big] \Big\} \\ &= \limsup_{n\to\infty} \Big\{ \sup_{x\in K, p\in\mathcal{F}} \Big[\|T_i^n x - p\|^2 - \|x - p\|^2 \Big] \Big\} \\ &\le 0. \end{aligned} \tag{6}$$

Therefore for $i = 1, 2, \dots, k$, we have

$$\limsup_{n\to\infty} \Big\{ \sup_{x\in K, p\in\mathcal{F}} \Big(\|T_i^n x - p\| - \|x - p\| \Big) \Big\} \le 0. \tag{7}$$

This implies that for any given $\varepsilon > 0$, there exists a positive integer n_0 such that for all $n \ge n_0$ and for $i = 1, 2, \dots, k$, we have

$$\sup_{x\in K, p\in\mathcal{F}} \Big\{ \|T_i^n x - p\| - \|x - p\| \Big\} < \varepsilon. \tag{8}$$

Since $\{x_n\}, \{y_{1n}\}, \dots, \{y_{(k-1)n}\} \subset E$, we have

$$\begin{aligned} \|T_1^n x_n - p\| - \|x_n - p\| &< \varepsilon, \quad \forall p \in \mathcal{F}, \quad \forall n \ge n_0, \\ \|T_2^n y_{1n} - p\| - \|y_{1n} - p\| &< \varepsilon, \quad \forall p \in \mathcal{F}, \quad \forall n \ge n_0, \\ \|T_3^n y_{2n} - p\| - \|y_{2n} - p\| &< \varepsilon, \quad \forall p \in \mathcal{F}, \quad \forall n \ge n_0, \\ &\vdots \\ &\vdots \\ \big\|T_k^n y_{(k-1)n} - p\big\| - \big\|y_{(k-1)n} - p\big\| &< \varepsilon, \quad \forall p \in \mathcal{F}, \quad \forall n \ge n_0. \end{aligned} \tag{9}$$

Thus for each $n \ge 1$ and for any $p \in \mathcal{F}$, using (5) and (9), we note that

$$\begin{aligned} \|y_{1n} - p\| &= \|a_{1n} x_n + b_{1n} T_1^n x_n + c_{1n} u_{1n} - p\| \\ &= \|a_{1n}(x_n - p) + b_{1n}(T_1^n x_n - p) + c_{1n}(u_{1n} - p)\| \\ &\le a_{1n} \|x_n - p\| + b_{1n} \|T_1^n x_n - p\| + c_{1n} \|u_{1n} - p\| \\ &\le a_{1n} \|x_n - p\| + b_{1n} \Big[\|x_n - p\| + \varepsilon \Big] + c_{1n} \|u_{1n} - p\| \\ &\le \big(a_{1n} + b_{1n}\big) \|x_n - p\| + b_{1n}\varepsilon + c_{1n} M \\ &= \big(1 - c_{1n}\big) \|x_n - p\| + b_{1n}\varepsilon + c_{1n} M \\ &\le \|x_n - p\| + b_{1n}\varepsilon + c_{1n} M \\ &= \|x_n - p\| + A_{1n} \end{aligned} \tag{10}$$

where $A_{1n} = b_{1n}\varepsilon + c_{1n} M$, since by assumption $\sum_{n=1}^{\infty} b_{1n} < \infty$ and $\sum_{n=1}^{\infty} c_{1n} < \infty$, it follows that $\sum_{n=1}^{\infty} A_{1n} < \infty$.

Furthermore, by inequality (9) and (10), we obtain

$$\begin{aligned}\|y_{2n}-p\| &= \|a_{2n}x_n + b_{2n}T_2^n y_{1n} + c_{2n}u_{2n} - p\| \\ &= \|a_{2n}(x_n-p) + b_{2n}(T_2^n y_{1n}-p) + c_{2n}(u_{2n}-p)\| \\ &\le a_{2n}\|x_n-p\| + b_{2n}\|T_2^n y_{1n}-p\| + c_{2n}\|u_{2n}-p\| \\ &\le a_{2n}\|x_n-p\| + b_{2n}\Big[\|y_{1n}-p\|+\varepsilon\Big] + c_{2n}\|u_{2n}-p\| \\ &\le a_{2n}\|x_n-p\| + b_{2n}\|y_{1n}-p\| + b_{2n}\varepsilon + c_{2n}M \\ &\le a_{2n}\|x_n-p\| + b_{2n}\Big[\|x_n-p\| + A_{1n}\Big] + b_{2n}\varepsilon + c_{2n}M \\ &\le \big(a_{2n}+b_{2n}\big)\|x_n-p\| + b_{2n}A_{1n} + b_{2n}\varepsilon + c_{2n}M \\ &= \big(1-c_{2n}\big)\|x_n-p\| + b_{2n}A_{1n} + b_{2n}\varepsilon + c_{2n}M \\ &\le \|x_n-p\| + A_{1n} + b_{2n}\varepsilon + c_{2n}M \\ &= \|x_n-p\| + A_{2n}\end{aligned} \tag{11}$$

where $A_{2n} = A_{1n} + b_{2n}\varepsilon + c_{2n}M$, since by assumption $\sum_{n=1}^{\infty} b_{2n} < \infty$, $\sum_{n=1}^{\infty} c_{2n} < \infty$ and $\sum_{n=1}^{\infty} A_{1n} < \infty$, it follows that $\sum_{n=1}^{\infty} A_{2n} < \infty$. Similarly, using (9) and (11), we see that

$$\begin{aligned}\|y_{3n}-p\| &= \|a_{3n}(x_n-p) + b_{3n}(T_3^n y_{2n}-p) + c_{3n}(u_{3n}-p)\| \\ &\le a_{3n}\|x_n-p\| + b_{3n}\|T_3^n y_{2n}-p\| + c_{3n}\|u_{3n}-p\| \\ &\le a_{3n}\|x_n-p\| + b_{3n}\Big[\|y_{2n}-p\|+\varepsilon\Big] + c_{3n}\|u_{3n}-p\| \\ &\le a_{3n}\|x_n-p\| + b_{3n}\|y_{2n}-p\| + b_{3n}\varepsilon + c_{3n}M \\ &\le a_{3n}\|x_n-p\| + b_{3n}\Big[\|x_n-p\| + A_{2n}\Big] + b_{3n}\varepsilon + c_{3n}M \\ &\le \big(a_{3n}+b_{3n}\big)\|x_n-p\| + b_{3n}A_{2n} + b_{3n}\varepsilon + c_{3n}M \\ &= \big(1-c_{3n}\big)\|x_n-p\| + b_{3n}A_{2n} + b_{3n}\varepsilon + c_{3n}M \\ &\le \|x_n-p\| + A_{2n} + b_{3n}\varepsilon + c_{3n}M \\ &= \|x_n-p\| + A_{3n}\end{aligned} \tag{12}$$

where $A_{3n} = A_{2n} + b_{3n}\varepsilon + c_{3n}M$, since by assumption $\sum_{n=1}^{\infty} b_{3n} < \infty$, $\sum_{n=1}^{\infty} c_{3n} < \infty$ and $\sum_{n=1}^{\infty} A_{2n} < \infty$, it follows that $\sum_{n=1}^{\infty} A_{3n} < \infty$. Continuing the above process, using (5) and (9), we get

$$\begin{aligned}\|x_{n+1}-p\| &= \big\|a_{kn}(x_n-p) + b_{kn}(T_k^n y_{(k-1)n}-p) + c_{kn}(u_{kn}-p)\big\| \\ &\le a_{kn}\|x_n-p\| + b_{kn}\big\|T_k^n y_{(k-1)n}-p\big\| + c_{kn}\|u_{kn}-p\| \\ &\le a_{kn}\|x_n-p\| + b_{kn}\Big[\big\|y_{(k-1)n}-p\big\|+\varepsilon\Big] + c_{kn}\|u_{kn}-p\| \\ &\le a_{kn}\|x_n-p\| + b_{kn}\big\|y_{(k-1)n}-p\big\| + b_{kn}\varepsilon + c_{kn}M \\ &\le a_{kn}\|x_n-p\| + b_{kn}\Big[\|x_n-p\| + A_{(k-1)n}\Big] + b_{kn}\varepsilon + c_{kn}M\end{aligned}$$

$$\begin{aligned}
&\leq \big(a_{kn}+b_{kn}\big)\left\|x_n-p\right\|+b_{kn}A_{(k-1)n}+b_{kn}\varepsilon+c_{kn}M \\
&= \big(1-c_{kn}\big)\left\|x_n-p\right\|+b_{kn}A_{(k-1)n}+b_{kn}\varepsilon+c_{kn}M \\
&\leq \left\|x_n-p\right\|+A_{(k-1)n}+b_{kn}\varepsilon+c_{kn}M \\
&= \left\|x_n-p\right\|+A_{kn}
\end{aligned} \tag{13}$$

where $A_{kn} = A_{(k-1)n} + b_{kn}\varepsilon + c_{kn}M$, since by assumption $\sum_{n=1}^{\infty} b_{kn} < \infty$, $\sum_{n=1}^{\infty} c_{kn} < \infty$ and $\sum_{n=1}^{\infty} A_{(k-1)n} < \infty$, it follows that $\sum_{n=1}^{\infty} A_{kn} < \infty$. By Lemma 2.1, we know that $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$.

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. From (13) we have

$$\begin{aligned}
\left\|x_{n+m}-p\right\| &\leq \left\|x_{n+m-1}-p\right\|+A_{k(n+m-1)} \\
&\leq \Big[\left\|x_{n+m-2}-p\right\|+A_{k(n+m-2)}\Big]+A_{k(n+m-1)} \\
&\leq \left\|x_{n+m-2}-p\right\|+\Big[A_{k(n+m-1)}+A_{k(n+m-2)}\Big] \\
&\leq \left\|x_{n+m-3}-p\right\|+\Big[A_{k(n+m-1)}+A_{k(n+m-2)}+A_{k(n+m-3)}\Big] \\
&\leq \dots \\
&\leq \dots \\
&\leq \left\|x_{n+m-3}-p\right\|+\Big[A_{k(n+m-1)}+A_{k(n+m-2)}+\dots+A_{kn}\Big] \\
&\leq \left\|x_n-p\right\|+\sum_{i=n}^{n+m-1} A_{ki},
\end{aligned} \tag{14}$$

for all $p \in \mathcal{F}$ and $m, n \in \mathbb{N}$. Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, for each $\varepsilon > 0$, there exists a natural number n_1 such that for $n \geq n_1$,

$$d(x_n, \mathcal{F}) < \frac{\varepsilon}{8} \quad \text{and} \quad \sum_{i=n_1}^{n+m-1} A_{ki} < \frac{\varepsilon}{2}. \tag{15}$$

Hence, there exists a point $q \in \mathcal{F}$ such that

$$\left\|x_{n_1}-q\right\| < \frac{\varepsilon}{4}. \tag{16}$$

By (14), (15) and (16), for all $n \geq n_1$ and $m \geq 1$, we have

$$\begin{aligned}
\left\|x_{n+m}-x_n\right\| &\leq \left\|x_{n+m}-q\right\|+\left\|x_n-q\right\| \\
&\leq \left\|x_{n_1}-q\right\|+\sum_{i=n_1}^{n+m-1} A_{ki}+\left\|x_{n_1}-q\right\| \\
&\leq 2\left\|x_{n_1}-q\right\|+\sum_{i=n_1}^{n+m-1} A_{ki} \\
&< 2\cdot\frac{\varepsilon}{4}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned} \tag{17}$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since E is complete, there exists a $p_1 \in E$ such that $x_n \to p_1$ as $n \to \infty$.

Now we have to prove that p_1 is a common fixed point of $\{T_i : i = 1, 2, \ldots, k\}$, that is, $p_1 \in \mathcal{F}$.

By contradiction, we assume that p_1 is not in $\mathcal{F}$. Since $\mathcal{F} = \cap_{i=1}^{k} F(T_i)$ is closed in Banach spaces, $d(p_1, \mathcal{F}) > 0$. So for all $p_2 \in \mathcal{F}$, we have

$$\|p_1 - p_2\| \le \|p_1 - x_n\| + \|x_n - p_2\| \,. \tag{18}$$

By the arbitrary of $p_2 \in \mathcal{F}$, we know that

$$d(p_1, \mathcal{F}) \le \|p_1 - x_n\| + d(x_n, \mathcal{F}). \tag{19}$$

By $\lim\limits_{n\to\infty} d(x_n, \mathcal{F}) = 0$, above inequality and $x_n \to p_1$ as $n \to \infty$, we have

$$d(p_1, \mathcal{F}) = 0, \tag{20}$$

which contradicts $d(p_1, \mathcal{F}) > 0$. Thus p_1 is a common fixed point of the mappings $\{T_i : i = 1, 2, \ldots, k\}$. This completes the proof. □

Theorem 3.2. *Let K be a nonempty compact convex subset of a uniformly convex Banach space E and for $i = 1, 2, \ldots, k$, let $T_i \colon K \to K$ be a finite family of uniformly (L_i, α_i)-Lipschitz and asymptotically quasi-nonexpansive type mappings. Let $\{x_n\}$ be the sequence defined by (5) with $\sum_{n=1}^{\infty} b_{in} < \infty$, $\sum_{n=1}^{\infty} c_{in} < \infty$ and $0 < \bar{\beta} \le b_{in} \le \beta < 1$ for all $i = 1, 2, \ldots, k$. If $\mathcal{F} = \cap_{i=1}^{k} F(T_i) \neq \emptyset$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i = 1, 2, \ldots, k\}$.*

Proof. From (13), we have

$$\|x_{n+1} - p\| \le \|x_n - p\| + A_{kn},$$

where $A_{kn} = A_{(k-1)n} + b_{kn}\varepsilon + c_{kn}M$, since by assumption $\sum_{n=1}^{\infty} b_{kn} < \infty$, $\sum_{n=1}^{\infty} c_{kn} < \infty$ and $\sum_{n=1}^{\infty} A_{(k-1)n} < \infty$, it follows that $\sum_{n=1}^{\infty} A_{kn} < \infty$. By Lemma 2.1, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \mathcal{F}$. Let $\lim_{n\to\infty} \|x_n - p\| = c$ for some $c > 0$. Then, from (10), we note that

$$\begin{aligned} \limsup_{n\to\infty} \|y_{1n} - p\| &\le \limsup_{n\to\infty} \Big(\|x_n - p\| + A_{1n} \Big) \\ &\le \limsup_{n\to\infty} \|x_n - p\| = c, \end{aligned} \tag{21}$$

and

$$\begin{aligned} \limsup_{n\to\infty} \|T_1^n x_n - p\| &\le \limsup_{n\to\infty} \Big(\|x_n - p\| + \varepsilon \Big) \\ &\le c + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary given, so we have

$$\limsup_{n\to\infty} \|T_1^n x_n - p\| \le c, \tag{22}$$

and

$$\begin{aligned}
\lim_{n\to\infty}\|y_{1n}-p\| &= \lim_{n\to\infty}\|a_{1n}x_n+b_{1n}T_1^nx_n+c_{1n}u_{1n}-p\| \\
&= \lim_{n\to\infty}\|(1-b_{1n}-c_{1n})x_n+b_{1n}T_1^nx_n+c_{1n}u_{1n}-p\| \\
&= \lim_{n\to\infty}\|(1-b_{1n})(x_n-p+c_{1n}(u_{1n}-x_n)) \\
&\quad + b_{1n}(T_1^nx_n-p+c_{1n}(u_{1n}-x_n))\| \\
&= c.
\end{aligned} \tag{23}$$

Again since $\lim_{n\to\infty}\|x_n-p\|$ exists, so $\{x_n\}$ is a bounded sequence in K. By virtue of condition $\sum_{n=1}^{\infty}c_{in}<\infty$ for all $i=1,2,\dots,k$ and the boundedness of the sequence $\{x_n\}$ and $\{u_{1n}\}$, we have

$$\begin{aligned}
\limsup_{n\to\infty}\|x_n-p+c_{1n}(u_{1n}-x_n)\| &\le \limsup_{n\to\infty}\|x_n-p\| \\
&\quad + \limsup_{n\to\infty}\Big(c_{1n}\|u_{1n}-x_n\|\Big) \\
&\le c,\ \ p\in\mathcal{F}.
\end{aligned} \tag{24}$$

It follows from (22) that

$$\begin{aligned}
\limsup_{n\to\infty}\|T_1^nx_n-p+c_{1n}(u_{1n}-x_n)\| &\le \limsup_{n\to\infty}\|T_1^nx_n-p\| \\
&\quad + \limsup_{n\to\infty}\Big(c_{1n}\|u_{1n}-x_n\|\Big) \\
&\le \limsup_{n\to\infty}\Big(\|x_n-p\|+\varepsilon\Big) \\
&\quad + \limsup_{n\to\infty}\Big(c_{1n}\|u_{1n}-x_n\|\Big) \\
&\le c+\varepsilon,\ \ p\in\mathcal{F}.
\end{aligned}$$

Since $\varepsilon>0$ is arbitrary given, so we have

$$\limsup_{n\to\infty}\|T_1^nx_n-p+c_{1n}(u_{1n}-x_n)\|\le c. \tag{25}$$

Therefore, from (23)-(25) and Lemma 2.2 we know that

$$\lim_{n\to\infty}\|T_1^nx_n-x_n\|=0. \tag{26}$$

Again from (11), we note that

$$\begin{aligned}
\limsup_{n\to\infty}\|y_{2n}-p\| &\le \limsup_{n\to\infty}\Big(\|x_n-p\|+A_{2n}\Big) \\
&\le \limsup_{n\to\infty}\|x_n-p\|=c,
\end{aligned} \tag{27}$$

and from (21), we note that

$$\begin{aligned}
\limsup_{n\to\infty}\|T_2^ny_{1n}-p\| &\le \limsup_{n\to\infty}\Big(\|y_{1n}-p\|+\varepsilon\Big) \\
&\le c+\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary given, so we have

$$\limsup_{n\to\infty} \|T_2^n y_{1n} - p\| \le c. \tag{28}$$

Next, consider

$$\begin{aligned}\limsup_{n\to\infty} \|T_2^n y_{1n} - p + c_{2n}(u_{2n} - x_n)\| &\le \limsup_{n\to\infty} \|T_2^n y_{1n} - p\| \\ &\quad + \limsup_{n\to\infty} \Big(c_{2n} \|u_{2n} - x_n\| \Big) \\ &\le \limsup_{n\to\infty} \Big(\|y_{1n} - p\| + \varepsilon \Big) \\ &\quad + \limsup_{n\to\infty} \Big(c_{2n} \|u_{2n} - x_n\| \Big) \\ &\le c + \varepsilon, \ \ p \in \mathcal{F}.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary given, so we have

$$\limsup_{n\to\infty} \|T_2^n y_{1n} - p + c_{2n}(u_{2n} - x_n)\| \le c. \tag{29}$$

Also,

$$\begin{aligned}\limsup_{n\to\infty} \|x_n - p + c_{2n}(u_{2n} - x_n)\| &\le \limsup_{n\to\infty} \|x_n - p\| \\ &\quad + \limsup_{n\to\infty} \Big(c_{2n} \|u_{2n} - x_n\| \Big) \\ &\le c, \ \ p \in \mathcal{F},\end{aligned} \tag{30}$$

and

$$\begin{aligned}\lim_{n\to\infty} \|y_{2n} - p\| &= \lim_{n\to\infty} \|a_{2n} x_n + b_{2n} T_2^n y_{1n} + c_{2n} u_{2n} - p\| \\ &= \lim_{n\to\infty} \|(1 - b_{2n} - c_{2n}) x_n + b_{2n} T_2^n y_{1n} + c_{2n} u_{2n} - p\| \\ &= \lim_{n\to\infty} \|(1 - b_{2n})(x_n - p + c_{2n}(u_{2n} - x_n)) \\ &\quad + b_{2n}(T_2^n y_{1n} - p + c_{2n}(u_{2n} - x_n))\| \\ &= c.\end{aligned} \tag{31}$$

Therefore, from (29)-(31) and Lemma 2.2 we know that

$$\lim_{n\to\infty} \|T_2^n y_{1n} - x_n\| = 0. \tag{32}$$

Now, we shall show that $\lim_{n\to\infty} \|T_3^n y_{2n} - x_n\| = 0$. For each $n \ge 1$,

$$\begin{aligned}\|x_n - p\| &\le \|T_2^n y_{1n} - x_n\| + \|T_2^n y_{1n} - p\| \\ &\le \|T_2^n y_{1n} - x_n\| + \Big(\|y_{1n} - p\| + \varepsilon \Big).\end{aligned} \tag{33}$$

Using (32), we have

$$\begin{aligned}c &= \lim_{n\to\infty} \|x_n - p\| \\ &\le \liminf_{n\to\infty} \|y_{1n} - p\| .\end{aligned}$$

It follows from (21) that

$$\begin{aligned} c &= \lim_{n\to\infty} \|x_n - p\| \\ &\le \liminf_{n\to\infty} \|y_{1n} - p\| \\ &\le \limsup_{n\to\infty} \|y_{1n} - p\| \le c. \end{aligned} \tag{34}$$

This implies that

$$\lim_{n\to\infty} \|y_{1n} - p\| = c. \tag{35}$$

On the other hand, we have

$$\|y_{2n} - p\| \le \Big(\|x_n - p\| + A_{2n} \Big), \ \forall n \ge 1,$$

where $\sum_{n=1}^{\infty} A_{2n} < \infty$. Therefore

$$\begin{aligned} \limsup_{n\to\infty} \|y_{2n} - p\| &\le \limsup_{n\to\infty} \Big(\|x_n - p\| + A_{2n} \Big), \\ &\le c, \end{aligned} \tag{36}$$

and hence

$$\begin{aligned} \limsup_{n\to\infty} \|T_3^n y_{2n} - p\| &\le \limsup_{n\to\infty} \Big(\|y_{2n} - p\| + \varepsilon \Big) \\ &\le c + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary given, so we have

$$\limsup_{n\to\infty} \|T_3^n y_{2n} - p\| \le c. \tag{37}$$

Next, consider

$$\begin{aligned} \limsup_{n\to\infty} \|T_3^n y_{2n} - p + c_{3n}(u_{3n} - x_n)\| &\le \limsup_{n\to\infty} \|T_3^n y_{2n} - p\| \\ &\quad + \limsup_{n\to\infty} \Big(c_{3n} \|u_{3n} - x_n\| \Big) \\ &\le \limsup_{n\to\infty} \Big(\|y_{2n} - p\| + \varepsilon \Big) \\ &\quad + \limsup_{n\to\infty} \Big(c_{3n} \|u_{3n} - x_n\| \Big) \\ &\le c + \varepsilon, \ p \in \mathcal{F}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary given, so we have

$$\limsup_{n\to\infty} \|T_3^n y_{2n} - p + c_{3n}(u_{3n} - x_n)\| \le c. \tag{38}$$

Also,

$$\begin{aligned} \limsup_{n\to\infty} \|x_n - p + c_{3n}(u_{3n} - x_n)\| &\le \limsup_{n\to\infty} \|x_n - p\| \\ &\quad + \limsup_{n\to\infty} \Big(c_{3n} \|u_{3n} - x_n\| \Big) \\ &\le c, \ p \in \mathcal{F}, \end{aligned} \tag{39}$$

and

$$\begin{aligned}\lim_{n\to\infty}\|y_{3n}-p\| &= \lim_{n\to\infty}\|a_{3n}x_n+b_{3n}T_3^n y_{2n}+c_{3n}u_{3n}-p\| \\ &= \lim_{n\to\infty}\|(1-b_{3n}-c_{3n})x_n+b_{3n}T_3^n y_{2n}+c_{3n}u_{3n}-p\| \\ &= \lim_{n\to\infty}\|(1-b_{3n})(x_n-p+c_{3n}(u_{3n}-x_n)) \\ &\quad + b_{3n}(T_3^n y_{2n}-p+c_{3n}(u_{3n}-x_n))\| \\ &= c.\end{aligned} \tag{40}$$

Therefore, from (38)-(40) and Lemma 2.2 we know that

$$\lim_{n\to\infty}\|T_3^n y_{2n}-x_n\| = 0. \tag{41}$$

Similarly, by using the same argument as in the proof above, we have

$$\lim_{n\to\infty}\left\|T_i^n y_{(i-1)n}-x_n\right\| = 0, \tag{42}$$

for all $i = 2, 3, \ldots, k$.

Since K is compact, $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{x_{n_j}\}_{j=1}^{\infty}$. Let

$$\lim_{j\to\infty} x_{n_j} = p. \tag{43}$$

Then from (5) and (42), we have

$$\begin{aligned}\left\|x_{n_j+1}-x_{n_j}\right\| &\le b_{k_{n_j}}\left\|T_k^{n_j} y_{(k-1)n_j}-x_{n_j}\right\| + c_{k_{n_j}}\left\|u_{k_{n_j}}-x_{n_j}\right\| \\ &\to 0, \quad \text{as} \quad j\to\infty.\end{aligned} \tag{44}$$

From (5) and (26), we have

$$\begin{aligned}\|y_{1n}-x_n\| &\le b_{1n}\|T_1^n x_n-x_n\| + c_{1n}\|u_{1n}-x_n\| \\ &\to 0, \quad \text{as} \quad n\to\infty.\end{aligned} \tag{45}$$

Again from (26) and (43), we have

$$\lim_{j\to\infty} T_1^{n_j} x_{n_j} = p. \tag{46}$$

Since $\lim_{j\to\infty} x_{n_j+1} = p$, we have

$$\lim_{j\to\infty} T_1^{n_j+1} x_{n_j+1} = p. \tag{47}$$

From (44), (46) and (47), we have

$$\begin{aligned}0 &\le \|p-T_1p\| \\ &\le \left\|p-T_1^{n_j+1}x_{n_j+1}\right\| + \left\|T_1^{n_j+1}x_{n_j+1}-T_1^{n_j+1}x_{n_j}\right\| \\ &\quad + \left\|T_1^{n_j+1}x_{n_j}-T_1p\right\| \\ &\le \left\|p-T_1^{n_j+1}x_{n_j+1}\right\| + L_1\left\|x_{n_j+1}-x_{n_j+1}\right\|^{\alpha_1} + L_1\left\|T_1^{n_j}x_{n_j}-p\right\|^{\alpha_1} \\ &\to 0 \quad \text{as} \quad j\to\infty.\end{aligned} \tag{48}$$

From (32) and (43), we have

$$\lim_{j\to\infty} T_2^{n_j} y_{1n_j} = p. \tag{49}$$

Since $\lim_{j\to\infty} x_{n_j+1} = p$, we have

$$\lim_{j\to\infty} T_2^{n_j+1} y_{1n_j+1} = p. \tag{50}$$

From (44), (45), (49) and (50), we have

$$\begin{aligned}
0 &\leq \|p - T_2 p\| \\
&\leq \left\|p - T_2^{n_j+1} y_{1n_j+1}\right\| + \left\|T_2^{n_j+1} y_{1n_j+1} - T_2^{n_j+1} x_{n_j+1}\right\| \\
&\quad + \left\|T_2^{n_j+1} x_{n_j+1} - T_2^{n_j+1} x_{n_j}\right\| + \left\|T_2^{n_j+1} x_{n_j} - T_2^{n_j+1} y_{1n_j}\right\| \\
&\quad + \left\|T_2^{n_j+1} y_{1n_j} - T_2 p\right\| \\
&\leq \left\|p - T_2^{n_j+1} y_{1n_j+1}\right\| + L_2 \left\|y_{1n_j+1} - x_{n_j+1}\right\|^{\alpha_2} \\
&\quad + L_2 \left\|x_{n_j+1} - x_{n_j}\right\|^{\alpha_2} + L_2 \left\|x_{n_j} - y_{1n_j}\right\|^{\alpha_2} \\
&\quad + L_2 \left\|T_2^{n_j} y_{1n_j} - p\right\|^{\alpha_2} \\
&\to 0 \quad \text{as} \quad j \to \infty.
\end{aligned} \tag{51}$$

Now, from (5) and (32), we have

$$\begin{aligned}
\|y_{2n} - x_n\| &\leq b_{2n} \|T_2^n y_{1n} - x_n\| + c_{2n} \|u_{2n} - x_n\| \\
&\to 0, \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{52}$$

Again from (41) and (43), we have

$$\lim_{j\to\infty} T_3^{n_j} y_{2n_j} = p. \tag{53}$$

Since $\lim_{j\to\infty} x_{n_j+1} = p$, we have

$$\lim_{j\to\infty} T_3^{n_j+1} y_{2n_j+1} = p. \tag{54}$$

From (44), (52), (53) and (54), we have

$$\begin{aligned}
0 &\le \|p - T_3 p\| \\
&\le \left\|p - T_3^{n_j+1} y_{2n_j+1}\right\| + \left\|T_3^{n_j+1} y_{2n_j+1} - T_3^{n_j+1} x_{n_j+1}\right\| \\
&\quad + \left\|T_3^{n_j+1} x_{n_j+1} - T_3^{n_j+1} x_{n_j}\right\| + \left\|T_3^{n_j+1} x_{n_j} - T_3^{n_j+1} y_{2n_j}\right\| \\
&\quad + \left\|T_3^{n_j+1} y_{2n_j} - T_3 p\right\| \\
&\le \left\|p - T_3^{n_j+1} y_{2n_j+1}\right\| + L_3 \left\|y_{2n_j+1} - x_{n_j+1}\right\|^{\alpha_3} \\
&\quad + L_3 \left\|x_{n_j+1} - x_{n_j}\right\|^{\alpha_3} + L_3 \left\|x_{n_j} - y_{2n_j}\right\|^{\alpha_3} \\
&\quad + L_3 \left\|T_3^{n_j} y_{2n_j} - p\right\|^{\alpha_3} \\
&\to 0 \quad \text{as} \quad j \to \infty.
\end{aligned} \tag{55}$$

Similarly, from (5) and (42), we have

$$\begin{aligned}
\left\|y_{(k-1)n} - x_n\right\| &\le b_{(k-1)n} \left\|T_{k-1}^n y_{(k-2)n} - x_n\right\| + c_{(k-1)n} \left\|u_{(k-1)n} - x_n\right\| \\
&\to 0, \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{56}$$

Again from (42) and (43), we have

$$\lim_{j\to\infty} T_k^{n_j} y_{(k-1)n_j} = p. \tag{57}$$

Since $\lim_{j\to\infty} x_{n_j+1} = p$, we have

$$\lim_{j\to\infty} T_k^{n_j+1} y_{(k-1)n_j+1} = p. \tag{58}$$

From (44), (56), (57) and (58), we have

$$\begin{aligned}
0 &\le \|p - T_k p\| \\
&\le \left\|p - T_k^{n_j+1} y_{(k-1)n_j+1}\right\| + \left\|T_k^{n_j+1} y_{(k-1)n_j+1} - T_k^{n_j+1} x_{n_j+1}\right\| \\
&\quad + \left\|T_k^{n_j+1} x_{n_j+1} - T_k^{n_j+1} x_{n_j}\right\| + \left\|T_k^{n_j+1} x_{n_j} - T_k^{n_j+1} y_{(k-1)n_j}\right\| \\
&\quad + \left\|T_k^{n_j+1} y_{(k-1)n_j} - T_k p\right\| \\
&\le \left\|p - T_k^{n_j+1} y_{(k-1)n_j+1}\right\| + L_k \left\|y_{(k-1)n_j+1} - x_{n_j+1}\right\|^{\alpha_k} \\
&\quad + L_k \left\|x_{n_j+1} - x_{n_j}\right\|^{\alpha_k} + L_k \left\|x_{n_j} - y_{(k-1)n_j}\right\|^{\alpha_k} \\
&\quad + L_k \left\|T_k^{n_j} y_{(k-1)n_j} - p\right\|^{\alpha_k} \\
&\to 0 \quad \text{as} \quad j \to \infty.
\end{aligned} \tag{59}$$

Hence

$$\lim_{n\to\infty} \|p - T_i p\| = 0, \quad \forall i = 1, 2, \ldots, k. \tag{60}$$

Thus p is a common fixed point of the mappings $\{T_i : i = 1, 2, \dots, k\}$. Since the subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ converges to p and $\lim_{n\to\infty} \|x_n - p\|$ exists, we conclude that $\lim_{n\to\infty} x_n = p$. This completes the proof. □

Remark 2. Theorem 3.1 extends and improves the corresponding result of Khan et al. [7] and Tang and Peng [19] to the case of more general class of asymptotically quasi-nonexpansive or uniformly quasi-Lipschitzian mappings considered in this paper.

Remark 3. Theorem 3.1 also extend and improve the corresponding results of [2, 4, 8, 9, 12, 15]. Especially Theorem 3.1 extends and improves Theorem 1 and 2 in [9], Theorem 1 in [8] and Theorem 3.2 in [15] in the following ways:

(1) The asymptotically quasi-nonexpansive mapping in [8], [9] and [15] is replaced by finite family of asymptotically quasi-nonexpansive type mappings.

(2) The usual Ishikawa iteration scheme in [8], the usual modified Ishikawa iteration scheme with errors in [9] and the usual modified Ishikawa iteration scheme with errors for two mappings in [15] are extended to the multi-step iteration scheme with errors for a finite family of mappings.

Remark 4. Theorem 3.2 extends and improves the corresponding result of [10] in the following aspect:

(1) The asymptotically quasi-nonexpansive mapping in [10] is replaced by finite family of asymptotically quasi-nonexpansive type mappings.

(2) The usual modified Ishikawa iteration scheme with errors in [10] is extended to the multi-step iteration scheme with errors for a finite family of mappings.

Remark 5. Theorem 3.1 also extends the corresponding result of [20] to the case of more general class of asymptotically nonexpansive mappings and multi-step iteration scheme with errors for a finite family of mappings considered in this paper.

Remark 6. Our results also extend the corresponding results of Chidume and Ofoedu [3] to the case of more general class of total asymptotically nonexpansive mappings considered in this paper.

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