

ON AN EQUATION CONNECTED WITH THE THEORY FOR SPREADING OF ACOUSTIC WAVE

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ABSTRACT. In the paper, we study questions on classical solvability of nonlocal problems for a third-order linear hyperbolic equation in a rectangular domain. The Riemann method is applied to the Goursat problem and solution is obtained in the integral form. Investigated problems are reduced to the uniquely solvable Volterra-type equation of second kind. Influence effects of coefficients at lowest derivatives on correctness of studied problems are detected.

1. Introduction

At present, great attention is spared to problems of mathematical physics arising from study of questions of liquid filtration in porous surroundings [1], unsteady motion of ground waters with a free surface [5] and other wave processes in various surroundings (see, for example, [13], [14]).

This interest is connected with both great applied significance of such problems and mathematical originality expressed in nonclassical character of obtaining equations. Study of acoustic waves in surroundings, where spreading of a wave breaks state of thermodynamical and mechanical equilibrium, relates to this circle of questions. If we assume that perturbations are small and relaxation takes place by an exponential law, then changes of the density in such surrounding will be described with the help of the equation [14]

$$\tau \frac{\partial}{\partial t} \left(\frac{\partial^2 \rho(x, t)}{\partial t^2} - c_\infty^2 \Delta \rho(x, t) \right) + \frac{\partial^2 \rho(x, t)}{\partial t^2} - c_0^2 \Delta \rho(x, t) = 0. \quad (1.1)$$

Here τ , c_0 , c_∞ are characteristics of a material having the sense of time, relaxation, and limit phase velocities of sound, respectively.

If $\tau \ll T$ (T is the oscillation period), then sound spreads with the velocity c_0 , which is the same as for a surrounding without relaxation. For $\tau \gg T$ relaxation processes in the surrounding are "frozen" and sonic oscillations spreading with the velocity c_∞ , moreover $c_0 < c_\infty$ according to [14].

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For convenience of study, introduce dimensionless variables $x \sim x/(c_\infty\tau)$, $t \sim t/\tau$, remaining former notations, and suppose $u(x, t) \sim \rho(x, t)/c_\infty^2$, $\alpha = c_0^2/c_\infty^2$. Then equation (1.1) for an one-dimensional isotropic surrounding in new variables becomes the form

$$\frac{\partial}{\partial t} \left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} \right) + \frac{\partial^2 u}{\partial t^2} - \alpha \frac{\partial^2 u}{\partial x^2} = 0, \quad (1.2)$$

moreover $0 < \alpha < 1$ for real surroundings [14]–[18].

Equations of the form (1.1) or (1.2) arise also in investigating processes of spreading perturbations in viscoelastic and visco-plastic pivots [13].

Equations of the form (1.1) were considered in the work [18] where the solution of the Cauchy problem was constructed and its asymptotical estimate was obtained by a small parameter characterizing the dispersion of the sound velocity.

In the present work, we consider nonlocal boundary value problems for a third-order partial equation with a hyperbolic operator in the main part.

It is known, the Riemann function [2] plays the fundamental role in the theory of hyperbolic equations at a plane. Recently interest increases to construction of the Riemann function for equations of high, in particular, third order and to study of initial-boundary value problems for such equations [7]–[8].

Consider in the plane of variables x, y the third-order partial equation

$$\left(\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} \right) u_{xy} + Lu = f(x, y) \quad (1.3)$$

where α, β are given constants, moreover $\alpha^2 + \beta^2 \neq 0$, and L is a linear differential expression of the form

$$\begin{aligned} Lu \equiv & a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} + a_1(x, y)u_x \\ & + b_1(x, y)u_y + c_1(x, y)u. \end{aligned} \quad (1.4)$$

Coefficients $a(x, y), b(x, y), c(x, y), a_1(x, y), b_1(x, y), c_1(x, y)$ and the right side $f(x, y)$ of (1.3) are given real functions, but $u(x, y)$ is a real function to be found.

Without any loss of generality, suppose $\alpha > 0$ and $\beta > 0$. In fact, if $\alpha < 0, \beta > 0$ or $\alpha > 0, \beta < 0$, then changing the independent variable $x = 1 - \xi$ or $y = 1 - \eta$ these cases are reduced to the case of $\alpha > 0$ and $\beta > 0$.

It should be noted that equation (1.3) at $\alpha = 1$ and $\beta = 0$ is called pseudoparabolic, and both local and nonlocal boundary value problems for it were studied by many authors in various domains with the help of the Riemann method (see, for example [6]–[9]).

In the work [16] local and nonlocal boundary value problems are investigated for equation (1.3). However, in proofs of solvability for investigated problems in mentioned work, conditions are imposed upon coefficients of equation (1.3),

which are to be connected with equalities

$$2b(x, y) = \frac{\beta}{\alpha} a(x, y) + \frac{\alpha}{\beta} c(x, y), \quad a_1(x, y) = \frac{\alpha}{\beta} b_1(x, y),$$

every where in the considered domain.

In the present paper, we study questions on classical solvability of non-local boundary-value problems for the third-order hyperbolic equation with conditions of the Bitsadze-Samarsky-type and Samarsky-Ionkin-type conditions.

Boundary-value problems for partial differential equations of the second order with nonlocal conditions has been investigated in [15]–[17] and see also references therein.

Investigation of boundary-value problems are interesting on theoretical point of view. Also there are a number of the non-local boundary conditions for evolution problems that have various applications in chemical engineering, thermoelasticity, underground water flow and population dynamics and etc.: see, for example [11].

Boundary-value problems for equations of third-order with local and non-local boundary conditions are investigated by A. Bouziani [3], A. Bouziani and M. S. Temsi [4], V. I. Jegalov and A. N. Mironov [9], A. I. Kozhanov [10], A. M. Nakhushhev [11], O. S. Zikirov [20].

2. Formulation of the problems

Consider in the domain $D = \{(x, y) : 0 < x < l, 0 < y < h\}$ nonlocal problems for a third-order partial differential equation in the form

$$Mu \equiv \left(\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} \right) u_{xy} + Lu = f(x, y) \quad (2.1)$$

where α, β are given constants, moreover $\alpha^2 + \beta^2 \neq 0$, and L is a linear differential expression of the form (1.4). Using the methods of [16] and [19], we solve following problems:

Problem 1. *To find a regular in D solution $u(x, y)$ of equation (2.1) satisfying to the initial conditions*

$$u(x, 0) = \psi_1(x), \quad u_y(x, 0) = \psi_2(x), \quad 0 \leq x \leq l, \quad (2.2)$$

and the following boundary conditions:

$$u(0, y) = \lambda u(l, y) + \varphi_1(y), \quad 0 \leq y \leq h, \quad (2.3)$$

$$u_x(0, y) = \varphi_2(y), \quad 0 \leq y \leq h, \quad (2.4)$$

where $\lambda = \text{const}$, $\psi_i(x)$, $\varphi_i(y)$, ($i = 1, 2$) are given functions, such that

$$\psi_1'(0) = \varphi_2(0), \quad \psi_1(0) = \lambda \psi_1(l) + \varphi_1(0), \quad \psi_2(0) = \lambda \psi_2(0) + \varphi_1'(0).$$

Problem 2. To find a regular in D solution $u(x, y)$ of equation (2.1) satisfying to initial conditions (2.2) and the boundary conditions:

$$u(0, y) = \varphi_1(y), \quad 0 \leq y \leq h, \quad (2.5)$$

$$u_x(0, y) = \lambda u_x(l, y) + \varphi_2(y), \quad 0 \leq y \leq h, \quad (2.6)$$

where $\psi_i(x)$, $\varphi_i(y)$, ($i = 1, 2$) are given functions, moreover the following agreement conditions

$$\psi_1(0) = \varphi_1(0), \quad \psi_2(0) = \varphi_1(0), \quad \psi_1'(0) = \lambda \psi_1'(l) + \varphi_2(0).$$

Problem 3. To find a regular in D solution $u(x, y)$ of equation (2.1) Satisfying to the initial conditions and nonlocal boundary conditions:

$$u(0, y) = \lambda_1 u(l, y) + \varphi_1(y), \quad 0 \leq y \leq h, \quad (2.7)$$

$$u_x(0, y) = \lambda_2 u_x(l, y) + \varphi_2(y), \quad 0 \leq y \leq h, \quad (2.8)$$

where $\lambda_i = \text{const}$, $\psi_i(x)$, $\varphi_i(y)$, ($i = 1, 2$) are given functions, moreover

$$\psi_1(0) = \lambda_1 \psi_1(l) + \varphi_1(0), \quad \psi_1'(0) = \lambda_2 \psi_1'(l) + \varphi_2(0), \quad \psi_2(0) = \lambda_1 \psi_2(l) + \varphi_1'(0).$$

According to definition of nonlocal problems [12], nonlocal conditions (2.3) and (2.6) relate to the Bitsadze–Samarsky–type and Samarsky–Ionkin–type conditions, respectively.

Remark 1. Note that the hyperbolic properties of the problem (2.1)–(2.3) follow only from the derivatives u_{xxy} and u_{xyy} .

Remark 2. Since the family characteristics $x = \text{const}$ and $y = \text{const}$ is the double one for the hyperbolic equation (2.1), two conditions are given on the segments $[0, l]$ and $[0, h]$.

Remark 3. The family of characteristics of equation (2.1) contains real and different elements. It affects sufficiently both correctness of the problem formulation and its solvability. It should be noted that problem (2.1)–(2.3) represents natural development of known initial–boundary value and characteristic problems for pseudo–parabolic and linear hyperbolic equations of the third–order with two independent variables.

Definition 1. A function $u(x, y)$ continuous in D with its partial derivatives entering in equation (2.1) and satisfying to this equation in D is said to be a regular solution of equation (2.1).

Definition 2. The Riemann function is a solution $v(x, y) = v(x, y; \xi, \eta)$ of the following problem:

$$M^* v = 0, \quad (2.9)$$

$$v(\xi, y; \xi, \eta) = \omega_1(\xi, y), \quad v_x(\xi, y; \xi, \eta) = \exp \left(-\frac{1}{\alpha} \int_{\eta}^y a(\xi, t) dt \right), \quad (2.10)$$

$$v(x, \eta; \xi, \eta) = \omega_2(x, \eta), \quad v_y(x, \eta; \xi, \eta) = \exp\left(-\frac{1}{\beta} \int_{\xi}^x c(t, \eta) dt\right), \quad (2.11)$$

where (ξ, η) is arbitrary fixed point from the closed domain D ,

$$M^*v \equiv -\left(\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}\right) v_{xy} + (av)_{xx} + (2bv)_{xy} + (cv)_{yy} - (a_1v)_x - (b_1v)_y + c_1v,$$

$\omega_1(\xi, y)$ and $\omega_2(x, \eta)$ are respectively solutions of the following Cauchy problems

$$\begin{aligned} \beta\omega_{1yy}(\xi, y) - b(\xi, y)\omega_{1y}(\xi, y) + a_1(\xi, y)\omega_1(\xi, y) &= 0, \\ \omega_1(\xi, \eta) = 0, \quad \beta\omega_{1y}(\xi, \eta) &= 1; \end{aligned} \quad (2.12)$$

$$\begin{aligned} \alpha\omega_{2xx}(x, \eta) - b(x, \eta)\omega_{2x}(x, \eta) + b_1(x, \eta)\omega_2(x, \eta) &= 0, \\ \omega_2(\xi, \eta) = 0, \quad \alpha\omega_{2x}(\xi, \eta) &= 1. \end{aligned} \quad (2.13)$$

Obviously, problems (2.12) and (2.13) are uniquely solvable.

Assumption 1. For all $(x, y) \in D$, we assume that

$$\begin{aligned} a(x, y), \quad b(x, y), \quad c(x, y) &\in C^1(\overline{D}) \cap C^2(D); \\ a_1(x, y), \quad b_1(x, y) &\in C(\overline{D}) \cap C^1(D), \quad c_1(x, y) \in C(D). \end{aligned}$$

Assumption 2. We assume that

$$f(x, y) \in C^{(1, \delta)}(\overline{D}), \quad 0 < \delta < 1; \quad \psi_i(x) \in C^2[0, l], \quad \varphi_i(y) \in C^2[0, h]; \quad i = 1, 2.$$

In this paper, we show the existence and uniqueness of a classical solution of the boundary-value problems with the nonlocal conditions. For the proof of unique solvability, we use the methods of the Riemann's function and integral equations.

3. The Goursat problem and the Riemann function

In this section, we consider the subsidiary problem: *To find a function $u(x, y)$ being in D a solution of equation (2.1) and satisfying to conditions (2.2) and*

$$u(0, y) = \mu(y), \quad u_x(0, y) = \varphi_2(y), \quad 0 \leq y \leq h, \quad (3.1)$$

where $\mu(y)$ is an unknown function for now, moreover

$$\psi_1(0) = \mu(0), \quad \psi_2(0) = \mu'(0), \quad \psi_1'(0) = \varphi_2(0).$$

The straight lines $x = \text{const}$ and $y = \text{const}$ are characteristics of the equation (2.1), then problem $\{(2.1), (2.2), (3.1)\}$ is called *the Goursat's problem*. Following theorem is a solvability of the Goursat's problem.

Theorem 3.1. *Let Assumptions 1, 2 are fulfilled and $\mu(y) \in C^2[0, h]$. Then Goursat's problem is uniquely solvable.*

Proof. We shall prove Theorem 1 by the Riemann's method [19]. Let $u(x, y)$, $v(x, y) \in C^2(\bar{D}) \cap C^3(D)$. Then evident identities

$$\begin{aligned}
v \left(\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} \right) u_{xy} &= \frac{\partial}{\partial x} (\alpha v u_{xy} - \alpha v_{xy} u - \beta v_y u_y) \\
&\quad + \frac{\partial}{\partial y} (\beta v u_{xy} - \beta v_{xy} u - \alpha v_x u_x) \\
&\quad - u \left(\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} \right) v_{xy}; \\
v (a u_{xx} + 2b u_{xy} + c u_{yy}) &= \frac{\partial}{\partial x} [a v u_x - (a v)_x u + b v u_y - (b v)_y u] \\
&\quad + \frac{\partial}{\partial y} [b v u_x - (b v)_x u + c v u_y - (c v)_y u] \\
&\quad + u [(a v)_{xx} + (2b v)_{xy} + (c v)_{yy}]; \\
v (a_1 u_x + b_1 u_y + c_1 u) &= \frac{\partial}{\partial x} (a_1 v u) + \frac{\partial}{\partial y} (b_1 v u) - u [(a_1 v)_x + (b_1 v)_y - c_1 v],
\end{aligned}$$

imply validity of the equality

$$v M u - u M^* v = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}, \quad (3.2)$$

here

$$\begin{aligned}
P &= \alpha v u_{xy} - \alpha v_{xy} u - \beta v_y u_y + (a v) u_x - (a v)_x u + (b v) u_y - (b v)_y u + (a_1 v) u; \\
Q &= \beta v u_{xy} - \beta v_{xy} u - \alpha v_x u_x + (b v) u_x - (b v)_x u + (c v) u_y - (c v)_y u + (b_1 v) u; \\
M^* v &\equiv - \left(\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} \right) v_{xy} + (a v)_{xx} + (2b v)_{xy} + (c v)_{yy} - (a_1 v)_x - (b_1 v)_y \\
&\quad + c_1 v.
\end{aligned}$$

Suppose P, Q are continuous in the domain \bar{D} , and P_x, Q_y are continuous and bounded in D .

One can easily obtain with the help of the Riemann function $v(x, y; \xi, \eta)$ representation of the general solution $u(\xi, \eta)$ for equation (2.1) in the domain D . Indeed, integrating by parts equality (3.2) by the domain $D_0 = \{(x, y) : 0 < x < \xi, 0 < y < \eta\}$, we have

$$\begin{aligned}
u(\xi, \eta) &= \alpha v_x(0, \eta; \xi, \eta) u(0, \eta) + \beta v_y(\xi, 0; \xi, \eta) u(\xi, 0) \\
&\quad - \int_0^\xi [\beta v(x, 0; \xi, \eta) u_{xy}(x, 0) + c(x, 0) v(x, 0; \xi, \eta) u_y(x, 0) \\
&\quad + A(x; \xi, \eta) u_x(x, 0) + B(x; \xi, \eta) u(x, 0)] dx
\end{aligned}$$

$$\begin{aligned}
& - \int_0^{\eta} [\alpha v(0, y; \xi, \eta) u_{xy}(0, y) + a(0, y) v(0, y; \xi, \eta) u_x(0, y) \\
& \quad + A_1(y; \xi, \eta) u_y(0, y) + B_1(y; \xi, \eta) u(0, y)] dy \\
& + \int_0^{\xi} \int_0^{\eta} v(x, y; \xi, \eta) f(x, y) dx dy
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
A(x, \xi, \eta) &= -\alpha v_x(x, 0; \xi, \eta) + b(x, 0) v(x, 0; \xi, \eta); \\
B(x; \xi, \eta) &= -\beta v_{xy}(x, 0; \xi, \eta) - b(x, 0) v_x(x, 0; \xi, \eta) - \\
& \quad - c(x, 0) v_y(x, 0; \xi, \eta) - [b_x(x, 0) + c(x, 0) - d(x, 0)] v(x, 0; \xi, \eta); \\
A_1(y; \xi, \eta) &= -\beta v_y(0, y; \xi, \eta) + b(0, y) v(0, y; \xi, \eta); \\
B_1(y; \xi, \eta) &= -\alpha v_{xy}(0, y; \xi, \eta) - a(0, y) v_x(0, y; \xi, \eta) - \\
& \quad - b(0, y) v_y(0, y; \xi, \eta) - [a_x(0, y) + b_y(0, y) - e(0, y)] v(0, y; \xi, \eta).
\end{aligned}$$

Formula (3.3) can be considered as a representation of the general solution for equation (2.1) if we consider $u(0, y)$, $u_x(0, y)$, $u(x, 0)$ and $u_y(x, 0)$ as arbitrary continuously differentiable functions.

By virtue of boundary conditions (2.2) and (3.1), we obtain from (3.3) a representation for the solution of the Goursat problem for equation (2.1) in the form

$$\begin{aligned}
u(\xi, \eta) &= \alpha v_x(0, \eta; \xi, \eta) \mu(\eta) + \beta v_y(\xi, 0; \xi, \eta) \psi_1(\xi) \\
& - \int_0^{\xi} [\beta v(x, 0; \xi, \eta) \psi_2'(x) + c(x, 0) v(x, 0; \xi, \eta) \psi_2(x) \\
& \quad + A(x; \xi, \eta) \psi_1'(x) + B(x; \xi, \eta) \psi_1(x)] dx \\
& - \int_0^{\eta} [\alpha v(0, y; \xi, \eta) \varphi_2'(y) + a(0, y) v(0, y; \xi, \eta) \varphi_2(y) \\
& \quad + A_1(y; \xi, \eta) \mu'(y) + B_1(y; \xi, \eta) \mu(y)] dy \\
& + \int_0^{\xi} \int_0^{\eta} v(x, y; \xi, \eta) f(x, y) dx dy.
\end{aligned} \tag{3.4}$$

Thus, the solution of Goursat's problem for equation (2.1) is presentable in explicit form (3.4) if the Riemann's function $v(x, y; \xi, \eta)$ is known. \square

By the method, based on reduction to integral-differential equations of Volterra [16] and [19], one can prove the existence and uniqueness of the Riemann's function, which was determined by formulas (2.9)–(2.13).

Theorem 3.2. *Let Assumptions 1, 2 are fulfilled. Then the Riemann's function $v(x, y) = v(x, y; \xi, \eta)$ for the operator M exists and unique.*

Proof. Integrating equation (2.9) in x from ξ to x and in y from y to η and using conditions (2.10)–(2.13), as in [16], for determining the function $v(x, y; \xi, \eta)$ we get the integral equation

$$v(x, y) = \frac{1}{2(\alpha^2 + \beta^2)} \int_{\beta x - \alpha y}^{\alpha x + \beta y} K_0 v(\bar{x}(s), \bar{y}(s)) ds + \gamma(x, y), \quad (3.5)$$

here

$$\begin{aligned} K_0 v(\bar{x}(s), \bar{y}(s)) &= 2b(\bar{x}(s), \bar{y}(s))v(\bar{x}(s), \bar{y}(s)) \\ &+ \int_{\xi}^{\bar{x}(s)} [c_y(t, \bar{y}(s)) - e(t, \bar{y}(s))] v(t, \bar{y}(s)) ds \\ &+ \int_{\eta}^{\bar{x}(s)} [a_x(\bar{x}(s), \tau) - d((\bar{x}(s), \tau))] v(\bar{x}(s), \tau) d\tau \\ &+ \int_{\xi}^{\bar{x}(s)} \int_{\eta}^{\bar{y}(s)} f(t, \tau) v(t, \tau) d\tau dt; \end{aligned}$$

$$\bar{x}(s) = \frac{1}{\alpha^2 + \beta^2} (\beta^2 x - \alpha \beta y + \alpha s), \quad \bar{y}(s) = \frac{1}{\alpha^2 + \beta^2} (-\alpha \beta x + \alpha^2 y + \beta s);$$

$\gamma(x, y)$ is known function.

The generalized theorem on the stationary point implies that integral equation (3.5) has a unique solution. It follows from the Theorem 2 in [19].

In the case the solution of Goursat's problem for the equation (2.1) exists and we obtain representation (3.4). \square

For the Riemann function $v(x, y; \xi, \eta)$, the following statements immediately follow:

Lemma 3.3. *If $d(x, y) < 0$, $e(x, y) < 0$, $\forall (x, y) \in D$, then the inequalities*

$$v(x, \eta; l, \eta) < 0, \quad \forall x \in [0, l], \quad \alpha v_x(0, \eta; l, \eta) > 1, \quad (3.6)$$

$$v(\xi, y; \xi, h) < 0, \quad \forall y \in [0, h], \quad \beta v_y(\xi, 0; \xi, h) > 1, \quad (3.7)$$

hold for the function $v(x, y; \xi, \eta)$.

Proof. Consider the problem

$$\alpha v_{xx}(x, \eta; l, \eta) - b(x, \eta) v_x(x, \eta; l, \eta) + d(x, \eta) v(x, \eta; l, \eta) = 0, \quad (3.8)$$

$$v(x, \eta; l, \eta) |_{x=l} = 0, \quad \alpha v_x(x, \eta; l, \eta) |_{x=l} = 1; \quad (3.9)$$

Rewrite equation (3.5) in the form

$$\frac{\partial}{\partial x} \left[\alpha p(x; l, \eta) \frac{\partial v_x(x, \eta; l, \eta)}{\partial x} \right] + q(x, \eta) v(x, \eta; l, \eta) = 0, \quad (3.10)$$

where

$$p(x; l, \eta) = \exp \left[\int_x^l b(t, \eta) dt \right], \quad q(x, \eta) = p(x; l, \eta) d(x, \eta).$$

Let $v = v(x, \eta; l, \eta)$, $0 \leq x < l$ be the solution of equation (3.7) determinate by conditions (3.6). Then by virtue of the maximum principle and the Zaremba–Jiro principle we obtain [16] from (3.7) $v(x, \eta; l, \eta) < 0$, $\forall x \in [0, l)$.

Integrating equation (3.7) in limits from 0 up to l taking into account conditions (3.6) we obtain

$$\alpha p(x; l, \eta) v_x(0, \eta; l, \eta) = 1 + \int_0^l q(t, \eta) v(t, \eta; l, \eta) dt.$$

Since $v(x, \eta; l, \eta) < 0$, $d(x, \eta) < 0$, we obtain from the last equality $\alpha v_x(0, \eta; l, \eta) > 1$. Inequality (2.20) can be proved similarly to (2.19). \square

We obtain presentation (3.4) in the case of the solution of Goursat problem (2.1)–(2.2) and (3.1) exists.

Note that it is sufficient to establish the existence of the solution of Eq. (2.1) for the homogeneous conditions $\psi_i(x) = 0$, $i = 1, 2$; $\mu(y) = 0$, $\varphi_2(y) = 0$.

Indeed, this can be done introducing a new unknown function $z(x, y)$ by the formula

$$\begin{aligned} z(\xi, \eta) = & u(\xi, \eta) - \{\varphi_1(\eta) + \xi[\varphi(\eta) - \psi_1'(0)] \\ & + \psi_1(\xi) + \eta[\psi(\xi) - \psi(0)] - \psi'(0)\xi\eta - \psi_1(0)\}, \end{aligned}$$

which satisfies equation (2.1) with another right part and homogeneous conditions

$$z(0, \eta) = z_\xi(0, \eta) = z(\xi, 0) = z_\eta(\xi, 0) = 0. \quad (3.12)$$

Using properties of the Riemann's function [19] one can see that the function determinate by formula (3.8) satisfies equation (2.1) and homogeneous conditions (3.12).

Thus we have proved the unique solvability of the Goursat problem. Formula (3.4) allows to study different boundary–value problems for equations of the form (2.1). \square

4. Reduction of problems to the integral equations.

In this section we study questions on existence and uniqueness for the solutions of considered nonlocal problems.

Theorem 4.1. *Let Assumptions 1, 2 are fulfilled. Then non-local problem 1 is solvable, moreover uniquely at $\lambda \in \bar{\Xi}(0, 1)$.*

Proof. We showed above that if functions $\psi_i(x) \in C^2[0, l]$, $\mu(y)$, $\varphi_2(y) \in C^2[0, h]$, $i = 1, 2$, then the solution of the characteristic problem (2.1), (2.2), and (3.1) exists, it is unique and is representable in the form of (3.4). Rewrite representation (3.4) after some transformations in the form of

$$\begin{aligned}
u(\xi, \eta) = & [\alpha v_x(0, \eta; \xi, \eta) - A_1(\eta; \xi, \eta)]\mu(\eta) - \alpha v(0, \eta; \xi, \eta)\varphi_2(\eta) \\
& + [\beta v_y(\xi, 0; \xi, \eta) - A(\xi; \xi, \eta)]\psi_1(\xi) - \beta v_x(\xi, 0; \xi, \eta)\psi_2(\xi) \\
& + \beta v(0, 0; \xi, \eta)\psi_2(0) + \alpha v(0, 0; \xi, \eta)\psi_1'(0) \\
& + [A_1(0; \xi, \eta) + A(0; \xi, \eta)]\psi_1(0) \\
& + \int_0^\xi [A_x(x; \xi, \eta) - B(x; \xi, \eta)]\psi_1(x)dx \\
& + \int_0^\xi [\beta v_x(x, 0; \xi, \eta) - c(x, 0)v(x, 0; \xi, \eta)]\psi_2(x)dx \\
& + \int_0^\eta [\alpha v_y(0, y; \xi, \eta) - a(0, y)v(0, y; \xi, \eta)]\varphi_2(y)dy \\
& + \int_0^\eta [A_{1y}(y; \xi, \eta) - B_1(y; \xi, \eta)]\mu(y)dy \\
& + \int_0^\xi \int_0^\eta v(x, y; \xi, \eta)f(x, y)dx dy.
\end{aligned} \tag{4.1}$$

Thus, arbitrary solution of nonlocal problem 1 can be represented in the form of (4.1) if the continuously-differentiable function $\mu(y)$ is found.

For simplification of further calculations, denote

$$\begin{aligned}
F(\xi, \eta) = & [\alpha v_x(0, \eta; \xi, \eta) - A_1(\eta; \xi, \eta)]\mu(\eta) - \alpha v(0, \eta; \xi, \eta)\varphi_2(\eta) \\
& + [\beta v_y(\xi, 0; \xi, \eta) - A(\xi; \xi, \eta)]\psi_1(\xi) - \beta v_x(\xi, 0; \xi, \eta)\psi_2(\xi) \\
& + \beta v(0, 0; \xi, \eta)\psi_2(0) + \alpha v(0, 0; \xi, \eta)\psi_1'(0) \\
& + [A_1(0; \xi, \eta) + A(0; \xi, \eta)]\psi_1(0) + \int_0^\xi [A_x(x; \xi, \eta) - B(x; \xi, \eta)]\psi_1(x)dx
\end{aligned}$$

$$\begin{aligned}
& + \int_0^{\xi} [\beta v_x(x, 0; \xi, \eta) - c(x, 0)v(x, 0; \xi, \eta)]\psi_2(x)dx \\
& + \int_0^{\eta} [\alpha v_y(0, y; \xi, \eta) - a(0, y)v(0, y; \xi, \eta)]\varphi_2(y)dy \\
& + \int_0^{\xi} \int_0^{\eta} v(x, y; \xi, \eta)f(x, y)dx dy.
\end{aligned}$$

By virtue of nonlocal condition (2.3), we obtain unknown functions $\mu(y)$ satisfying to the condition

$$\mu(y) = \lambda u(l, y) + \varphi_1(y), \quad 0 \leq y \leq h. \quad (4.2)$$

Solving the nonlocal problem is reduced on the whole to finding functions $\mu(y)$.

We obtain from representation (4.1) at $\xi = l$:

$$\begin{aligned}
u(l, \eta) & = F(l, \eta) + [\alpha v_x(0, \eta; l, \eta) - A_1(\eta; l, \eta)]\mu(\eta) \\
& + \int_0^{\eta} [A_{1y}(y; l, \eta) - B_1(y; l, \eta)]\mu_1(y)dy.
\end{aligned}$$

If we multiply the last expression by λ , we obtain by virtue of condition (4.2) the Volterra-type equation with respect to $\mu(\eta)$:

$$\sigma(\eta)\mu(\eta) = \int_0^{\eta} k(y, \eta)\mu(y)dy + g(\eta), \quad (4.3)$$

here

$$\sigma(\eta) = 1 - \lambda[\alpha v_x(0, \eta; l, \eta) - A_1(\eta; l, \eta)], \quad k(y, \eta) = \lambda[A_{1y}(y; l, \eta) - B_1(y; l, \eta)],$$

$g(\eta) = \varphi_1(\eta) + \lambda F(l, \eta)$ is a known function.

By virtue of properties of the Riemann's function $v(x, y; \xi, \eta)$, one can easily make sure, equation (4.3) will be the second-order Volterra integral equation for all values of $\lambda \in (0, 1)$. For indicated values of λ , we find from equation (4.3) $\mu(\eta)$ in the form of

$$\mu(\eta) = g_1(\eta) + \int_0^{\eta} R(y, \eta)g_1(y)dy \quad (4.4)$$

where $g_1(\eta) = g(\eta)/\sigma(\eta)$, $R(y, \eta)$ is the resolvent of the kernel $k(y, \eta)/\sigma(\eta)$.

After definition of the function $\mu(\eta)$, investigated nonlocal problem 1 is reduced to the characteristic Goursat problem for equation (2.1), uniquely solvability for that is proved above.

Values of λ , at which a nonlocal problem is stated correctly, will be said to be regular values of this problem.

It is known [16], any value of $\lambda \in (0, 1)$ cannot be regular for nonlocal problem 1.

Now the following question arises naturally: what happens when conditions on coefficients of equation (2.1) are broken separately or simultaneously? If conditions on coefficients of the equation and boundary conditions are broken, then, as a simple example shows, nonlocal problem 1 can be stated incorrectly.

In fact, the function $u(x, y) = y^2(k^x + k^{-x})$, $k = (1 + \sqrt{1 - \lambda^2})/\lambda$ satisfies in the domain $D = \{(x, y) : 0 < x < 1, 0 < y < h\}$ to the equation

$$u_{xxy} + u_{xyy} - \frac{2}{y^2} u_x - \ln^2 k u_y = 0,$$

and the boundary conditions

$$u(0, y) = \lambda u(1, y), \quad u_x(0, y) = 0, \quad u(x, 0) = 0, \quad u_y(x, 0) = 0.$$

Similar result takes place also for nonlocal problem 2. □

Theorem 4.2. *Let Assumption 1 and Assumption 2 are fulfilled. Then the classical solution of the problem 3 exists and unique at $\lambda_i \in (0, 1)$, ($i = 1, 2$).*

Proof. To prove existence and uniqueness of the solution for nonlocal problem 3, we study the subsidiary Goursat problem for equation (2.1) with initial conditions (2.2) and boundary conditions

$$u(0, y) = \mu_1(y), \quad u_x(0, y) = \mu_2(y), \quad 0 \leq y \leq h, \quad (4.5)$$

where $\mu_i(y)$, ($i = 1, 2$) are for the present unknown functions, moreover the following equalities

$$\psi_1(0) = \mu_1(0), \quad \psi_2(0) = \mu_1'(0), \quad \psi_1'(0) = \mu_2(0)$$

hold.

By virtue of nonlocal conditions (2.7) and (2.8), we find unknown functions $\mu_1(y)$, $\mu_2(y)$ satisfying to the conditions

$$\mu_1(y) = \lambda_1 u(l, y) + \varphi_1(y), \quad 0 \leq y \leq h, \quad (4.6)$$

$$\mu_2(y) = \lambda_2 u_x(l, y) + \varphi_2(y), \quad 0 \leq y \leq h. \quad (4.7)$$

Solving the nonlocal problem is reduced on the whole to finding functions $\mu_1(y)$, $\mu_2(y)$.

We find from representation (4.1) at $\xi = l$:

$$\begin{aligned} u(l, \eta) &= F(l, \eta) + [\alpha v_x(0, \eta; l, \eta) - A_1(\eta; l, \eta)]\mu_1(\eta) - \alpha v(0, \eta; l, \eta)\mu_2(\eta) \\ &+ \int_0^\eta [\alpha v_y(0, y; l, \eta) - a(0, y)v(0, y; l, \eta)]\mu_2(y)dy \\ &+ \int_0^\eta [A_{1y}(y; l, \eta) - B_1(y; l, \eta)]\mu_1(y)dy. \end{aligned}$$

If we multiply the last expression by λ_1 , then by virtue of condition (4.6) we obtain the relation between functions $\mu_1(\eta)$ and $\mu_2(\eta)$:

$$\begin{aligned} A_{11}(\eta)\mu_1(\eta) + A_{12}(\eta)\mu_2(\eta) &= \int_0^\eta [k_{11}(y, \eta)\mu_1(y) + k_{12}(y, \eta)\mu_2(y)]dy \\ &+ f_1(\eta), \end{aligned} \quad (4.8)$$

here

$$\begin{aligned} A_{11}(\eta) &= 1 - \lambda_1[\alpha v_x(0, \eta; l, \eta) - A_1(\eta; l, \eta)], \\ A_{12}(\eta) &= \lambda_1\alpha v(0, \eta; l, \eta), \\ k_{11}(y, \eta) &= \lambda_1[A_{1y}(y; l, \eta) - B_1(y; l, \eta)], \\ k_{12}(y, \eta) &= \lambda_1[\alpha v_y(0, y; l, \eta) - a(0, y)v(0, y; l, \eta)], \end{aligned}$$

$f_1(\eta)$ is a known function.

Calculating the derivative of $u(\xi, \eta)$ from (4.1) by ξ and setting $\xi = l$, taking into account condition (4.7), we find after some transformations

$$\begin{aligned} A_{21}(\eta)\mu_1(\eta) + A_{22}(\eta)\mu_2(\eta) &= \int_0^\eta [k_{21}(y, \eta)\mu_1(y) + k_{22}(y, \eta)\mu_2(y)]dy \\ &+ f_2(\eta), \end{aligned} \quad (4.9)$$

here

$$\begin{aligned} A_{21}(\eta) &= -\lambda_2[\alpha v_{x\xi}(0, \eta; l, \eta) - A_{1\xi}(\eta; l, \eta)], \\ A_{22}(\eta) &= 1 - \lambda_2\alpha v_\xi(0, \eta; l, \eta), \\ k_{21}(y, \eta) &= \lambda_2[A_{1y\xi}(y; l, \eta) - B_{1\xi}(y; l, \eta)], \\ k_{22}(y, \eta) &= \lambda_2[\alpha v_{y\xi}(0, y; l, \eta) - a(0, y)v_\xi(0, y; l, \eta)], \end{aligned}$$

$f_2(\eta)$ is a known function.

Thus, to define functions $\mu_1(\eta)$ and $\mu_2(\eta)$, we obtain the system of integral equations. Hence, the question on solvability of studying nonlocal problem is reduced to the question on solvability of the system of equations (4.8) and (4.9).

Rewrite (4.8)–(4.9) in the form of

$$\mathcal{A}_1(\eta) \begin{pmatrix} \mu_1(\eta) \\ \mu_2(\eta) \end{pmatrix} = \int_0^\eta \mathcal{K}_1(y, \eta) \begin{pmatrix} \mu_1(y) \\ \mu_2(y) \end{pmatrix} dy + \begin{pmatrix} f_1(\eta) \\ f_2(\eta) \end{pmatrix}, \quad (4.10)$$

here

$$\mathcal{A}_1(\eta) = \begin{pmatrix} A_{11}(\eta) & A_{12}(\eta) \\ A_{21}(\eta) & A_{22}(\eta) \end{pmatrix}, \quad \mathcal{K}_1(y, \eta) = \begin{pmatrix} k_{11}(y, \eta) & k_{12}(y, \eta) \\ k_{21}(y, \eta) & k_{22}(y, \eta) \end{pmatrix},$$

On the base of Lemma 3.1, we have $\det|\mathcal{A}_1(\eta)| \neq 0$, $\forall \eta \in [0, h]$. Therefore the system of equations (4.10) is the system of integral Volterra equations of the second kind [12]. Finding from equation (4.10) $\mu_i(\eta)$, ($i = 1, 2$), we reduce nonlocal problem 3 to the Goursat problem for equation (2.1), correctness of which was established in 3. \square

References

- [1] G. N. Barenblatt, Yu. P. Jeltov and I. N. Kochina, *Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks*, J. Appl. Math. Mech. **24(5)** (1960), 1286–1303.
- [2] A. V. Bitsadze, *Some classes of partial differential equations*, Nauka, Moscow, 1981 (in Russian).
- [3] A. Bouziani, *On the solvability of a nonlocal problem arising in dynamics of moisture transfer*, Georgian Mathematical Journal **10(4)** (2003), 607–627.
- [4] A. Bouziani and M. S. Temsi, *On a Quasilinear Pseudohyperbolic Equations with a nonlocal Boundary Conditions*, Internat. J. Math. Analysis **3(3)** (2009), 109–120.
- [5] E. S. Dzekter, *Equation of the motion of underground waters with free surface in multilayer medis*, Dokl. Akad. Nauk SSSR **220(3)**, 540–543, (1975) (in Russian).
- [6] O. M. Dzhokhadze, *A General Darboux-Type Boundary Value Problem in Curvilinear Domains with Corners for a Third-Order Equation with Dominated Lower-Order Terms*, Siberian Mathematical Journal **43(2)** (2002), 235–250.
- [7] T. D. Dzhuraev, *Boundary-value problems for the mixed and mixed-composite type equations*, Fan, Tashkent, 1979 (in Russian).
- [8] T. D. Dzhuraev and O. S. Zikirov, *The Goursat problem and a non-local problem for a class of third-order equations*, In Non-classical equations of mathematical Physics, pp. 98–109, Novosibirsk, 2005. Institute of Mathematics (in Russian).
- [9] V. I. Jegalov and A. N. Mironov, *Differential Equations with older partial derivatives*, Kazan, 2001 (in Russian).
- [10] A. I. Kozhanov, *On a nonlocal boundary value problem with variable coefficients for the heat equation and the Aller equation*, Diffensialniye uravneniya **40(6)**, 763–774, 2004 (in Russian).
- [11] A. M. Nakhushev, *Problems with Displacements for Partial Differential Equations*, Nauka, Moscow, 2006 (in Russian).
- [12] A. M. Nakhushev, *Equations of Mathematical Biology*, Vysshaya shkola, Moscow, 1995 (in Russian).
- [13] Kh. A. Rakhmatulin and Yu. A. Dem'yanov, *Strength at intense short-time loading*, Logos, Moscow, 2009 (in Russian).
- [14] O. V. Rudenko and S. N. Soluyan, *Technical Bases of the Non-linear Acoustics*, Nauka, Moscow, 1975 (in Russian).
- [15] M. Sapagovas, *On the stability of a finite-difference scheme for nonlocal parabolic equations boundary-value problem*, Lietuvos Matematikos Rinkiny **48(3)** (2008), 339–356.

- [16] M. Kh. Shkhanukov, *On some boundary value problems for a third-order equation arising when modeling fluid filtration in porous media*, *Differentsialniye uravneniya* **18(4)**, 689–699, 1982(in Russian).
- [17] A. Stikonas A and O. Stikoniene, *Characteristic functions for Sturm–Liouville problems with nonlocal boundary–value problem*, *Mathematical Modelling and Analysis* **14(2)** (2009), 229–246.
- [18] V. V. Varlamov, *Time estimates for the Cauchy problem for a third-order hyperbolic equation*, *Internat. J. Math. and Math. Sci.* **33(17)** (2003), 1073–1981.
- [19] O. S. Zikirov, *On boundary–value problem for hyperbolic–type equation of the third order*, *Lietuvos Matematikos Rinkiny*s **47(4)** (2007), 591–603.
- [20] O. S. Zikirov, *A non–local Boundary Value Problems for Third–Order Linear Partial Differential Equation of Composite Type*, *Mathematical Modelling and Analysis* **14(3)** (2009), 407–421.

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