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ON SIGNED SPACES

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Abstract. We denote by $\mathcal{Q}(A)$ the set of all matrices with the same sign pattern as A. A matrix A has *signed null-space* provided there exists a set $\mathcal{S}$ of sign patterns such that the set of sign patterns of vectors in the null-space of $\widetilde{A}$ is $\mathcal{S}$, for each $\widetilde{A} \in \mathcal{Q}(A)$. In this paper, we show that the number of sign patterns of elements in the row space of S^*-matrix is $3^{m+1} - 2^{m+2} + 2$. Also the number of sign patterns of vectors in the null-space of a totally L-matrix is obtained.

1. Introduction

The *sign* of a real number a is defined by

$$\operatorname{sign}(a) = \begin{cases} -1 & \text{if } a < 0, \\ 0 & \text{if } a = 0, \text{ and} \\ 1 & \text{if } a > 0. \end{cases}$$

A *sign pattern* is a $(0, 1, -1)$-matrix. The *sign pattern of a matrix* A is the matrix obtained from A by replacing each entry by its sign. We denote by $\mathcal{Q}(A)$ the set of all matrices with the same sign pattern as A. The *zero pattern* of a matrix A is the $(0, 1)$ matrix obtained from A by replacing each nonzero entry by 1.

Let A be an m by n matrix and b an m by 1 vector. The linear system $Ax = b$ has *signed solutions* provided there exists a collection $\mathcal{S}$ of n by 1 sign patterns such that the set of sign patterns of the solutions to $\widetilde{A}x = \tilde{b}$ is $\mathcal{S}$, for each $\widetilde{A} \in \mathcal{Q}(A)$ and $\tilde{b} \in \mathcal{Q}(b)$. This notion generalizes that of a sign-solvable linear system (see [1] and references therein). The linear system, $Ax = b$, is *sign-solvable* provided each linear system $\widetilde{A}x = \tilde{b}$ ($\widetilde{A} \in \mathcal{Q}(A)$, $\tilde{b} \in \mathcal{Q}(b)$) has a solution and all solutions have the same sign pattern. Thus, $Ax = b$ is sign-solvable if and only if $Ax = b$ has signed solutions and the set $\mathcal{S}$ is singleton.

A matrix A has *signed null-space* provided $Ax = 0$ has signed solutions. Thus, A has signed null-space if and only if there exists a set $\mathcal{S}$ of sign patterns

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such that the set of sign patterns of vectors in the null-space of $\widetilde{A}$ is $\mathcal{S}$, for each $\widetilde{A} \in \mathcal{Q}(A)$. An *L-matrix* is a matrix, A, with the property that each matrix in $\mathcal{Q}(A)$ has linearly independent rows. A square L-matrix is a *sign-nonsingular*, or *SNS*-matrix for short. A *totally L-matrix* is an $m \times n$ matrix such that each $m \times m$ submatrix is an SNS-matrix. An $m \times n$ totally L-matrix with $n = m+1$ is called *S^*-matrix.* An S^*-matrix is called *S-matrix* if it is row-mixed. It is known that totally L-matrices are matrices with signed null-spaces. Hence the set of matrices with signed null-spaces generalizes the set of totally L-matrices. Matrices with signed null-spaces are characterized in [4, 5, 6, 7, 8, 9].

In this paper, we consider matrices with signed null-space and signed row-space and we show that the number of sign patterns of elements in the row space of S^*-matrix is $3^{m+1} - 2^{m+2} + 2$. Also we obtain that

$$|\text{full sign patterns of } NS(A)| + |\text{full sign patterns of } RS(A)| = 2^n$$

if A be an m by n non-degenerate matrix. Using this property, we obtain the number of sign patterns of vectors in the null-space of a totally L-matrix.

For a given m by n matrix A, we denote the row-space and null-space of A by RS(A) and NS(A), respectively. We denote diag$(d_1, d_2, \ldots, d_n)$ for the n by n diagonal matrix whose (i,i)-entry is d_i. Also, we denote a zero matrix of an appropriate size by O. For a set S of matrices, the set of all sign patterns of matrices in S is denoted by $\mathcal{SP}(S)$.

2. Signed spaces

The matrix A has *signed row-space* provided there exists a set $\mathcal{S}$ of sign patterns such that the set of sign patterns of vectors in the row-space of $\widetilde{A}$ is $\mathcal{S}$, for each $\widetilde{A} \in \mathcal{Q}(A)$. As the row-space of a matrix is the orthogonal complement of its null-space, it is natural to conjecture that A has signed null-space if and only if A has signed row-space. The next theorem shows that this is indeed the case.

Theorem 2.1. ([4]) *Let A be an m by n matrix. Then A has signed row-space if and only if A has signed null-space.*

Let B be an m by n $(0,1,-1)$-matrix. The matrix A is *conformally contractible* to B provided there exists an index k such that the rows and columns of A can be permuted so that A has the form

$$\left[\begin{array}{c|c|c} B[\langle m\rangle, \langle n\rangle \setminus \{k\} & x & y \\ \hline 0 \ \cdots \ 0 & 1 & -1 \end{array}\right], \tag{1}$$

where $x = [x_1, \ldots, x_m]^T$ and $y = [y_1, \ldots, y_m]^T$ are $(0,1,-1)$ vectors such that $x_i y_i \geq 0$ for $i = 1, 2, \ldots, m$, and the sign pattern of $x + y$ is the kth column of B.

Corollary 2.2. *Let A be an m by n matrix and let B be a matrix obtained from A by a conformal contraction. Then A has signed row-space if and only if B has signed row-space.*

Proof. It is known that if A is an m by n matrix and B is a matrix obtained from A by a conformal contraction, then A has signed nullspace if and only if B has signed nullspace. Since A has signed row-space if and only if A has signed null-space by Theorem A, we have the result. $\square$

A vector is *mixed* if it has a positive entry and a negative entry. A matrix is *row-mixed* if each of its rows is mixed. A vector is *balanced* if it is the zero vector or is mixed. The notion of a *row-balanced* matrix is defined analogously. A *signing* is a nonzero, diagonal $(0, 1, -1)$-matrix. A signing is *strict* if each of its diagonal entries is nonzero. A matrix B is *strictly row-mixable* provided there exists a strict signing D such that BD is row-mixed. Let S be a set of sign patterns. A nonzero sign pattern x in S is *minimal* if a sign pattern $x^{'}$ obtained from x by replacing any nonzero entry with 0 is not in S.

For a given m by n row-mixed matrix A, let $\mathcal{M}_A$ be $\{D :$ minimal signing such that AD is balanced $\}$ and let $\mathcal{D}_A$ be $\{d = (d_1, \ldots, d_n) | \text{diag}(d_1, \ldots, d_n) \in \mathcal{M}_A\}$. And let $\mathcal{V}_A$ be $\{v = (v_1, \ldots, v_n) \in \mathbb{R}^n | v_i d_i = 0$ for all i or there exist i, j with $v_i d_i > 0$ and $v_j d_j < 0$ for all $d \in \mathcal{D}_A\}$. Then we have the following question:

Problem. *Is $\mathcal{S}P(RS(A))$ equal to $\mathcal{S}P(\mathcal{V}_A)$ if A has signed null-space?.*

We can derive one direction of the result easily as seen in the following.

Proposition 2.3. *If A be an m by n mixed matrix which A has signed null-space, then $\mathcal{S}P(RS(A)) \subseteq \mathcal{S}P(\mathcal{V}_A)$.*

Proof. Let $v = (v_1, \ldots, v_n) \in \text{RS}(A)$ and let D be a minimal signing such that AD is balanced. Without loss of generality, we may assume that $D = \text{diag}(d_1, \ldots, d_k, 0, \cdots, 0)$. Then there exists $\mathbf{x} \in \text{NS}(AD)$ such that

$$\mathbf{x} = (x_1, \ldots, x_k, 0, \ldots, 0),\ x_i > 0,\ i = 1, \ldots, k.$$

Since $(v_1 d_1, \ldots, v_n d_n) \in \text{RS}(AD) = \text{NS}(AD)^{\perp}$,

$$(v_1 d_1, \ldots, v_n d_n) \cdot (x_1, \ldots, x_k, 0, \ldots, 0) = 0$$

and hence $\sum_{i=1}^{k} v_i d_i x_i = 0$. Thus $v_i d_i = 0$ for all i or there exist i, j with $v_i d_i > 0$ and $v_j d_j < 0$. This implies $\mathcal{S}P(\text{RS}(A)) \subseteq \mathcal{S}P(\mathcal{V}_A)$. $\square$

We can show that the Problem is true if A is an m by $m+1$ S-matrix. Let A be m by $m+1$ S-matrix. Then its nullspace is spanned by a vector $\mathbf{a}$ each of whose entries is positive. Hence $\mathcal{M}_A$ consists of the $m+1$-square identity matrix I_{m+1}. Let $\mathbf{v} \in \mathcal{V}_A$. If $\mathbf{v} \neq \mathbf{0}$, then there exists a vector

$\mathbf{v}'$ such that $\mathcal{SP}(\mathbf{v}) = \mathcal{SP}(\mathbf{v}')$ and $\mathbf{v}' \cdot \mathbf{a} = 0$. Hence $\mathbf{v}' \in \mathrm{R}S(A)$. Thus $\mathcal{SP}(\mathrm{RS}(A)) = \mathcal{SP}(\mathcal{V}_A)$.

From this fact, we have $\mathcal{SP}(\mathrm{RS}(A)) = \mathcal{SP}(\mathrm{RS}(B))$ for any two m by $m+1$ S-matrices A, B. Moreover, $|\mathcal{SP}(\mathrm{RS}(A))| = |\mathcal{SP}(\mathrm{RS}(B))|$ for any two m by $m+1$ S^*-matrices A, B.

Proposition 2.4. *$\mathcal{SP}(RS(A)) = \mathcal{SP}(\mathcal{V}_A)$ if A has an m by n $(m < n)$ totally L matrix.*

Proof. Let A be an m by n totally L matrix. Since we have indicated the equality for $n = m+1$, we will prove it for $n = m+2$. Without loss of generality, we may assume that A is row mixed. Then every vector in $\mathcal{D}_A$ has at most one zero entry. Choose two vectors $\mathbf{v}, \mathbf{w}$ in $\mathcal{D}_A$ such that each of them has exactly one zero entry. Then there exist $\mathbf{a}, \mathbf{b}$ in $\mathbf{R}^{m+2}$ such that $\mathcal{SP}(\mathbf{a}) = \mathcal{SP}(\mathbf{v}), \mathcal{SP}(\mathbf{b}) = \mathcal{SP}(\mathbf{w})$ and $\{\mathbf{a}, \mathbf{b}\}$ is a basis of $\mathrm{NS}(A)$. Let $\mathbf{v} \in \mathcal{V}_A$. If $\mathbf{v} \neq 0$, then there exists a vector $\mathbf{v}'$ such that $\mathcal{SP}(\mathbf{v}) = \mathcal{SP}(\mathbf{v}')$ and $\mathbf{v}' \cdot \mathbf{a} = 0, \mathbf{v}' \cdot \mathbf{b} = 0$. Hence $\mathbf{v}' \in \mathrm{R}S(A)$. Thus $\mathcal{SP}(\mathcal{V}_A) \subseteq \mathcal{SP}(\mathrm{RS}(A))$. □

Let A be an m by $m+2$ totally L matrix. Notice that there exists a vector $\mathbf{d}_i = (d_1, \ldots, d_{m+2})$ in $\mathcal{D}_A$ such that $d_i = 0$ for each $i = 1, 2, \ldots, m+2$. Since $\mathcal{SP}(\mathcal{D}_A) \subseteq \mathcal{SP}(\mathrm{NS}(A))$, any nonzero vector in the row space of A should have at least 3 nonzero entries. In fact, any element of $\mathcal{SP}(\mathrm{RS}(A))$ with exactly 3 non-zero entries in the same positions is unique as shown in the following. Let $\mathbf{e}_{ijk}$ denote the vector all of whose entries are 0 except for the i-th, j-th, k-th entries which are 1 of suitable size.

Corollary 2.5. *Let A be an m by $m+2$ totally L-matrix, and let v be in $\mathcal{SP}(RS(A))$ such that the zero pattern of v is $\mathbf{e}_{ijk}$ for some i, j and k with $1 \leq i < j < k \leq m+2\}$. There is no sign pattern in $\mathcal{SP}(RS(A))$ different from $\pm v$ whose zero pattern is $\mathbf{e}_{ijk}$.*

Proof. Let $\mathbf{v} = (v_1, v_2, \cdots, v_{m+2}) \in \mathcal{SP}(\mathrm{RS}(A)$ have 3 non-zero entries. Without loss of generality, we may assume that the zero pattern of $\mathbf{v}$ is $\mathbf{e}_{123}$. Suppose that $\mathbf{w} = (w_1, w_2, \ldots, w_{m+2}) \in \mathcal{SP}(\mathrm{RS}(A)$ such that $w \neq \pm v$ and the zero pattern of $\mathbf{w}$ is $\mathbf{e}_{123}$. Then there are i, j in $\{1, 2, 3\}$ such that $v_i v_j w_i w_j = -1$. Let k be the integer which is neither i nor j in $\{1, 2, 3\}$. Let $\mathbf{d}_k = (d_1, d_2, \ldots, d_{m+2}) \in \mathcal{D}$ such that $d_k = 0$. Since $v_i d_i v_j d_j < 0$ and $w_i d_i w_j d_j < 0$ by Proposition 3, $v_i v_j w_i w_j = 1$ which is impossible. □

Notice that a matrix which is not an SNS-matrix but has signed null-space has at least 3 sign patterns of vectors in its null-space. The following proposition characterizes the matrices whose signed null-space has exactly 3 sign patterns.

Proposition 2.6. *Let A be an m by $m+1$ matrix. Then the following are equivalent.*

(a) *A has signed null-space,*
(b) $|\mathcal{S}P(NS(A))| = 3$,
(c) *A is permutation equivalent to a matrix of the form*

$$\begin{bmatrix} B & C \\ O & D \end{bmatrix}$$

where B is an S^*-matrix and D is an (vacuously) SNS-matrix.

Proof. $(c) \Rightarrow (b)$. There is nothing to prove.

$(b) \Rightarrow (a)$. Without loss of generality, we may assume that the sign patterns of null-space of A are $\pm(1, 1, \ldots, 1)$ and 0. Let $\mathbf{a}, \mathbf{b}$ be non-zero vectors in null-space of A. Let the sign patterns of $\mathbf{a}$ be $(1, 1, \ldots, 1)$. If $\mathbf{a}$ is not scalar multiple of $\mathbf{b}$, then we have a sign pattern of the row space of A which is not $\pm(1, 1, \ldots, 1)$ and 0. This is impossible. Hence null-space of A is generated by a positive vector. This means that A is an S-matrix.

$(a) \Rightarrow (c)$. A is permutation equivalent to a matrix of the form

$$\begin{bmatrix} B & C \\ O & D \end{bmatrix}$$

where B is a row-mixable matrix and columns of D are linearly independent. If the number of rows of D is more than that of columns of D, the matrix B has a square sub-matrix which is row-mixed. This means B has no signed null-space and hence A has no signed null-space. Hence D is an SNS-matrix. B is an m by $m+1$ row-mixable matrix which has signed null-space. Hence B is an S^*-matrix. □

Let S_m be the m by $m+1$ matrix such that

$$S_m = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}$$

where the unspecified entries are zero. Notice that the matrix S_m is an S-matrix.

Let P_m be the set of $\mathbf{v} \in \mathcal{S}P(\mathrm{RS}(S_m))$ whose first nonzero entry is positive. Let a_m be the number of elements in P_m. For $m \geq 2$, P_m has exactly a_{m-1} sign patterns whose last entry is 0. Hence we have the following result.

Lemma 2.7. $a_m = 5a_{m-1} - 6a_{m-2} + 1,\ (m \geq 3)$. *Here, $a_1 = 1$ and $a_2 = 6$.*

Proof. We will prove it by induction on m. There is nothing to prove for $m = 1, 2$. Let $m \geq 3$. It is easy to show that P_m has exactly a_{m-1} sign patterns $\mathbf{v} = (v_1, v_2, \ldots, v_{m+1})$ such that $v_1, v_2, \ldots, v_m$ are fixed and v_{m+1} is one of among 1, 0, -1 . Notice that the other sign patterns $\mathbf{w}$ of P_m have nonzero in the last entry. Let $\mathbf{v}^{'}$ and $\mathbf{w}^{'}$ be the vectors obtained from $\mathbf{v}$ and

$\mathbf{w}$ by adding the last component which is 0 respectively. Hence we can get the sign patterns of the row-space of P_{m+1} by acting the last row of S_{m+1} to $\mathbf{v}'$ and $\mathbf{w}'$. Thus we obtain 5 distinct sign patterns from each $\mathbf{w}'$ and 9 distinct sign patterns from three patterns $\mathbf{v}'$. Hence we have $a_{m+1} = 5a_m - 6a_{m-1} + 1$. □

Proposition 2.8. *The number of sign patterns of elements in the row-space of S_m is $3^{m+1} - 2^{m+2} + 2$.*

Proof. From $a_m = 5a_{m-1} - 6a_{m-2} + 1$, we have the characteristic equation $x^2 - 5x + 6 = 0$. Hence we can put $a_m = \alpha 2^m + \beta 3^m + \gamma$. Then we have

$$\begin{cases} \alpha + \beta + \gamma &= 0, \\ 2\alpha + 3\beta + \gamma &= 1, \\ 4\alpha + 9\beta + \gamma &= 6. \end{cases}$$

Thus

$$\begin{aligned} a_m &= -2 \cdot 2^m + \frac{3}{2} \cdot 3^m + \frac{1}{2} \\ &= -2^{m+1} + \frac{1}{2}(3^{m+1} + 1). \end{aligned}$$

The number of sign patterns of elements of row-space of S_m is $2a_m + 1 = 3^{m+1} - 2^{m+2} + 2$. □

Corollary 2.9. *The number of sign patterns of elements in the row-space of an S^*-matrix is $3^{m+1} - 2^{m+2} + 2$.*

An m by n matrix is *non-degenerate* if its m by m sub-matrices are invertible. A sign pattern v of a vector is *full* sign pattern if v has no zero entry. Let $\mathcal{FSP}(\mathrm{RS}(A))$ and $\mathcal{FSP}(\mathrm{NS}(A))$ denote the set of all full sign patterns of vectors in the row-space of A and null-space of A respectively.

Proposition 2.10. *For any m by n non-degenerate matrix A,*

$$|\mathcal{FSP}(NS(A))| + |\mathcal{FSP}(RS(A))| = 2^n.$$

Proof. Let s be a full sign pattern such that $\mathcal{Q}(s) \cap \mathrm{RS}(A) = \emptyset$. Without loss of generality, we may assume that $s = (+, \ldots, +)^T$. By the separation theorem for convex sets, there exists a nonzero vector $\mathbf{x} \in \mathrm{NS}(A)$ such that $\mathbf{x} \geq 0$. We may assume $x = (x_1, \ldots, x_k, 0, \ldots, 0)$ where $x_i > 0$, $i = 1, 2, \ldots, k$. Since A is non-degenerate, $k > m$. We also have an element $\mathbf{y}_i = (y_{i1}, \ldots, y_{im}, 0, \ldots, 0, y_{ii}, 0, \ldots, 0) \in \mathrm{NS}(A)$ where $y_{ij} \neq 0$, $i = k+1, \ldots, n$ and $j = 1, \ldots, m, i$ since every $m+1$ columns of A are linearly dependent. Then $\mathbf{x} + \epsilon_{k+1}\mathbf{y}_{k+1} + \ldots + \epsilon_n \mathbf{y}_n \in \mathrm{NS}(A)$ for any real ϵ_i, $i = k+1, \ldots, n$. Hence there exists a vector $\mathbf{x}'$ such that $\mathbf{x}' \in \mathcal{Q}(s) \cap \mathrm{NS}(A)$. Since $\mathrm{RS}(A)$ is orthogonal complement of $\mathrm{NS}(A)$, $\mathcal{SP}(\mathrm{NS}(A)) \cap \mathcal{SP}(\mathrm{RS}(A)) = \mathbf{0}$. Thus we have the result. □

To show that $|\mathcal{SP}(\mathrm{NS}(A))| = 4m + 9$ if A is an m by $m+2$ totally L-matrix, we need a lemma which owes to P. Delsarte and Y. Kamp.

Lemma 2.11. ([2]) *For a non-degenerate m by n matrix A,*

$$|\mathcal{F}SP(RS(A))| = 2\sum_{i=0}^{m-1}\binom{n-1}{i}.$$

Proposition 2.12. *Let A be an m by $m+2$ totally L-matrix. Then*

$$|\mathcal{S}P(NS(A))| = 4m+9.$$

Proof. By lemma 11, we have

$$|\mathcal{F}SP(\mathrm{RS}(A))| = 2\sum_{i=0}^{m-1}\binom{m+1}{i} = 2^{m+2} - 2m - 4.$$

Hence $|\mathcal{F}SP(\mathrm{NS}(A))| = 2^{m+2} - (2^{m+2} - 2m - 4) = 2m + 4$. Every nonzero sign pattern of $\mathrm{NS}(A)$ which is not full sign pattern of $\mathrm{NS}(A)$ must have exactly one zero entry. Such a sign pattern is also unique. Hence total number of nonzero sign patterns of $\mathrm{NS}(A)$ which are not full sign patterns of $\mathrm{NS}(A)$ is $2(m+2)$. Since $\mathbf{0}$ is a sign pattern of $\mathrm{RS}(A)$, we have $|\mathcal{S}P(\mathrm{NS}(A))| = 2m + 4 + 2(m+2) + 1 = 4m + 9$. □

References

[1] R. A. Brualdi and B. L. Shader, *The matrices of sign-solvable linear systems*, Cambridge Univ. Press, 1995.

[2] P. Delsarte and Y. Kamp, *Low rank matrices with a given sign pattern*, SIAM. J. Disc. MATH. **2-1** (1989), 51–63.

[3] K. G. Fisher, W. Morris and J. Shapiro, *Mixed Dominating Matrices*, Linear Algebra and its Applications **270** (1998), 191–214.

[4] S. J. Kim and B. L. Shader, *Linear systems with signed solutions*, Linear Algebra and its Applications **313** (2000), 21–40.

[5] S. J. Kim and B. L. Shader, *Sign solvable cone system*, Linear and multilinear algebra **vol.50**, no.1 (2002), 23–32.

[6] S. J. Kim and B. L. Shader, *On Matrices which has signed nullspaces*, Linear Algebra Appl. **353** (2002), 245–255.

[7] S. J. Kim, B. L. Shader and S. G. Hwang, *On Matrices with signed nullspaces*, SIAM J. on Matrix Analysis and Appl. **vol.24**, no.2 (2002), 570–780.

[8] S. J. Kim and T. Y. Choi, *A Characterization of an SN-matrix related with L-matrix*, Honam Mathematical Journal **28** (2006), 333–342.

[9] S. J. Kim and T. Y. Choi, *A Note ON SN-matrices*, Honam Mathematical Journal **vol.30**, no.4 (2008), 659–670.

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