THE DIVISOR CLASS GROUP OF SURFACES
OVER FINITE FIELDS

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Abstract. We investigate the divisor class group of surfaces over finite fields. For some surfaces the divisor class group depends on the characteristic of the field. We calculate the determinant of a matrix which will provide an information about the divisor class group of the surfaces.

1. Introduction

Every ring in this paper is assumed to be commutative and noetherian.

Let \((R, m)\) be a local ring with a maximal ideal \(m\). An element \(f \in R\) is called a determinant in \(R\), if it is a determinant of an \(n \times n\) matrix \((n \geq 2)\) with the entries in the maximal ideal \(m\).

It is due to Eisenbud that the unique factorization of a surface of embedding dimension 3 can be computed by determinants.

Theorem 1.1 ([5, p. 124]). Let \((R, m)\) be a 3-dimensional regular local ring and \(S = R/(f)\) with \(f \in R\). Then \(S\) is factorial if and only if \(f\) is not a determinant in \(R\).

Suppose that \(f\) is a determinant of an \(n \times n\) matrix \(A\) with entries in \(m\) and \(n \geq 2\). Let \(B\) be the \((n - 1) \times n\) matrix obtained from \(A\) by deleting the first row, then the ideal \(I\) of \((n - 1) \times (n - 1)\) minors of \(B\) is unmixed of height 2. Thus \(I/(f)\) is an unmixed ideal of height 1 in \(S\) that is not principal. So \(S\) is not factorial.

Therefore, for any noetherian local ring \((R, m)\) and \(f \in R\) if \(S = R/(f)\) is factorial, then \(f\) is not a determinant in \(R\). The converse holds only for a regular local ring with \(\dim R \leq 3\). If \(\dim R \geq 4\), then the converse is not true.
Theorem 1.2 ([4, Theorem 2.4]). Let $k$ be a real closed field and $R$ be the 4-dimensional regular local and $R = k[x, y, z, w]_{(x,y,z,w)}$. Then $f = x^2 + y^2 + z^2 + w^2$ is not a determinant in $R$.

The divisor class group of $k[x, y, z, w]_{(x,y,z,w)}/(x^2 + y^2 + z^2 + w^2)$ is infinite cyclic [6].

It is due to Gauss that if $R$ is factorial, then so is $R[x]$. But this is not true for the formal power series ring $R[[x]]$. Gauss's method cannot be applied to $R[[x]]$ since the units of $R[x]$ and those of $R[[x]]$ are different. Also the content of a formal power series is not defined. Counterexamples have been found by Salmon ([10]), Zinn-Justine and Danilov.

Example 1.3 ([14, 15]). Let $R = F(u)[[x, y, z]]/(x^2 + y^4 + uz^2)$, where $F$ is a field and $x, y, z$ and $u$ are variables. Then $R$ is factorial for all odd $(i, j) \neq (3, 3)$, but $R[[t]]$ is not factorial.

Surfaces of embedding dimension 3 were focused on due to following results by Samuel and Scheja.

Theorem 1.4 ([11]). Let $S$ be a locally Cohen-Macaulay factorial domain. If $S_p[[x]]$ is factorial for any height 2 prime ideal $p$ in $S$, $S[[x]]$ is factorial.

Theorem 1.5 ([13]). Let $(R, m)$ be a complete local factorial domain with depth $R \geq 3$, then $R[[x]]$ is factorial.

If $(S, n)$ is a 2-dimensional local domain of embedding dimension 3, then the divisor class group $Cl(S)$ of $S$ is generated by the classes of the height 1 prime ideals that is not contained in $n^e(R)$ where $e(R)$ is the multiplicity of $R$ [3]. Note that for the surfaces $S = R/(f)$ of a 3-dimensional regular local ring $(R, m)$, $e(R) = o(f)$ where $o(f) = e$ if $f \in m^e$, $f \not\in m^{e+1}$. Hence $Cl(S)$ is generated by the classes of the height 1 prime ideals that is not contained in $n^{o(f)}$. We calculate the divisor class group of some of the surfaces of embedding dimension 3.

This paper focuses on the surfaces of embedding dimension 3 over a finite field. Let $F$ be a finite field and $R = F[[x, y, z]]$. We study surfaces satisfying one of the following equations: $x^2 + y^3 + az^4 = 0$, $x^2 + y^3 + az^3 = 0$, $x^2 + y^2 + az^n = 0$ ($n \geq 2$, $a \in F$).

2. The Divisor Class Group over Finite Fields

For a 3-dimensional regular local ring $(R, m)$ with $m = (x, y, z)$, $R/(x^2 + y^3 + z^5)$
is factorial (cf. [9], [13], [3]). Also this is essentially the only nonregular factorial ring when $R/m$ is algebraically closed.

**Theorem 2.1** ([7]). Let $(S, \mathfrak{n})$ be a 2-dimensional nonregular local ring such that $S/\mathfrak{n}$ is algebraically closed field of characteristic $\neq 2, 3, 5$. Then the completion $\hat{S}$ of $S$ is factorial if and only if $S \cong R/(x^2 + y^3 + z^n)$ for some 3-dimensional regular local ring $R$ with a regular system of parameters $x, y, z$ of $R$.

Let $(S, \mathfrak{n})$ be a 2-dimensional non-regular local ring with $\mathfrak{n} = (x, y, z)$ and $S/\mathfrak{n}$ is real closed. Then Lipman further showed that $S$ has a rational singularity and is factorial if and only if one of the following equations holds in $S$: $x^2 + y^3 + z^n = 0$, $x^2 + y^3 + z = 0$, $x^2 + y^2 + z^n = 0$ ($n \geq 2$).

**Theorem 2.2** ([3, Theorem 2.2]). Let $F$ be a field and $a \in F$. Then $f = x^2 + y^3 + az^4$ is not a determinant in $R = F[[x, y, z]]/(x^2 + y^3 + az^4)$ if and only if $\sqrt{-a} \notin F$. Equivalently, $S = F[[x, y, z]]/(x^2 + y^3 + az^4)$ is factorial.

If $\alpha = \sqrt{-a} \in F$, then $x^2 + y^3 + az^4$ is a determinant in $R$. Note that

$$x^2 + y^3 + az^4 = \begin{pmatrix} x + \alpha z^2 & -y \\ y^2 & x - \alpha z^2 \end{pmatrix}$$

In this case the divisor class group $Cl(S)$ of $S$ is generated by the prime ideal $(x + \alpha z^2, y)S$ ([3, Theorem 3.2]).

Now consider the surface $S$ over a finite field $F$. Then the divisor class group $Cl(S)$ of $S$ depends on the characteristic of $F$.

**Theorem 2.3.** Let $F$ be a finite field of characteristic $p$ and $S = F[[x, y, z]]/(x^2 + y^3 + az^4)$ for $a \in F$.

1. If $p = 2$, then $S$ is not factorial for any $a$.
2. If $p$ is an odd prime, then $S$ is factorial only for $|F^*|/2$ elements $a \in F^*$.

**Proof.** Consider the group-homomorphism, $\epsilon_2 : F^* \longrightarrow F^*, \epsilon_2(a) = a^2$.

1. If $p = 2$, then $|F^*|$ is odd. So $\epsilon_2$ is an automorphism. So for any $a \in F^*$, $\alpha = \sqrt{-a} \in F$. Thus $x^2 + y^3 + az^4$ is a determinant in $F[[x, y, z]]$ and $S$ is not factorial by Theorem 2.2.
2. If $p$ is odd, then $|F^*|$ is even. So $\epsilon_2$ is not surjective. Also only for a half of elements in $F^*$, $a \notin \text{im}(\epsilon_2)$. For those $a$, $\sqrt{-a} \notin F$ and $S = F[[x, y, z]]/(x^2 + y^3 + az^4)$ is factorial by Theorem 2.2.
Now consider the surface $S = \mathbb{Z}_p[[x, y, z]]/(x^2 + y^3 + z^4)$ over the prime field $\mathbb{Z}_p$.

Note that the equation $T^2 + 1 = 0$ has no roots in $\mathbb{Z}_p$ if and only if $p$ is a prime integer of the form $4n + 3$.

If $p = 2$, then $\alpha = 1$ is a solution.

If $p$ is a prime integer of the form $4n + 1$, then $\alpha = (\frac{p-1}{2})!$ is a solution.

On the other hand, if $p$ is a prime integer of the form $4n + 3$, then $p$ is irreducible in the ring of Gaussian integers $\mathbb{Z}[i]$. So the equation $T^2 + 1 = 0$ has no roots in $\mathbb{Z}_p$.

**Corollary 2.4.** Let $p$ be a prime integer and $S = \mathbb{Z}_p[[x, y, z]]/(x^2 + y^3 + z^4)$. Then $S$ is factorial if and only if $p$ is a prime integer of the form $4n + 3$.

**Theorem 2.5** ([3, Theorem 2.2]). Let $F$ be a field and $a \in F$. Then $f = x^2 + y^3 + az^3$ is not a determinant in $R = F[[x, y, z]]$ if and only if $\sqrt{a} \notin F$. Equivalently, $S = F[[x, y, z]]/(x^2 + y^3 + az^3)$ is factorial.

If $\beta = \sqrt[3]{a} \in F$, then $x^2 + y^3 + az^3$ is a determinant in $R$. Note that

$$x^2 + y^3 + az^3 = \left( \begin{array}{c} x \\ y^2 - \beta yz + \beta^2 z^2 \\ x \end{array} \right)$$

Also if $\beta = \sqrt[3]{a} \in F$, then the divisor class group $Cl(S)$ of $S$ is generated by the prime ideal $(x, y + \beta z)S$ ([3, Theorem 3.2]).

**Theorem 2.6.** Let $F$ be a finite field of characteristic $p$ and $S = F[[x, y, z]]/(x^2 + y^3 + az^3)$ for $a \in F$.

1. If $p = 3$, then $S$ is not factorial for any $a$.
2. If $p$ is a prime integer of the form $3n + 1$, then $S$ is factorial only for $\frac{2}{3}|F^*|$ elements $a \in F^*$.
3. For a prime integer $p$ of the form $3n + 2$, put $m = [F : \mathbb{Z}_p]$. If $m$ is odd, then $S$ is not factorial for any $a$. If $m$ is even, then $S$ is factorial only for $\frac{2}{3}|F^*|$ elements $a \in F^*$.

**Proof.** Consider the group-homomorphism, $\epsilon_3 : F^* \longrightarrow F^*$, $\epsilon_3(a) = a^3$.

1. If $p = 3$, then gcd($|F^*|, 3) = 1$. So $\epsilon_3$ is an automorphism. So for any $a \in F^*$, $\beta = \sqrt[3]{a} \in F$. Thus $x^2 + y^3 + az^3$ is a determinant in $F[[x, y, z]]$ and $S$ is not factorial by Theorem 2.5.
2. If $p \equiv 1 \pmod{3}$, then $3||F^*|$. So $\epsilon_3$ is not surjective. Also only for $\frac{2}{3}F$ of elements $a$ in $F^*$, $a \notin im(\epsilon_3)$. For those $a$, $\sqrt[3]{a} \notin F$ and $S = F[[x, y, z]]/(x^2 + y^3 + az^3)$ is factorial by Theorem 2.2.
(3) If \( p \equiv 2 \pmod{3} \) and \( m \) is odd, \( \epsilon_3 \) is surjective. Also if \( p \equiv 2 \pmod{3} \) and \( m \) is even, \( \epsilon_3 \) is not surjective. Hence the conclusion follows.

For the surface \( S = \mathbb{Z}_p[[x, y, z]]/(x^2 + y^3 + az^3) \) over the prime field \( \mathbb{Z}_p \), Theorem 2.6 can be restated as follows.

**Corollary 2.7.** Let \( p \) be a prime integer and \( S = \mathbb{Z}_p[[x, y, z]]/(x^2 + y^3 + az^3) \) for \( a \in \mathbb{Z}_p \). Then \( S \) is factorial if and only if \( p \equiv 1 \pmod{3} \) and \( a \not\in \text{im}(\epsilon_3) \) where \( \epsilon_3 : \mathbb{Z}_p \to \mathbb{Z}_p, \epsilon_3(a) = a^3 \).

Let \( F \) be a field and \( m \geq 2 \). If \( i = \sqrt{-1} \in F \), then \( x^2 + y^2 + az^m \) is a determinant in \( F[[x, y, z]] \) for any \( a \in F \). Note that

\[
x^2 + y^2 + az^m = \begin{pmatrix} x + iy & -z \\ az^{m-1} & x - iy \end{pmatrix}.
\]

If \( m \) is even, \( \sqrt{-1} \notin F \) and \( \sqrt{-a} \notin F \); then \( x^2 + y^2 + az^m \) is not a determinant in \( F[[x, y, z]] \) ([3, Theorem 2.4]). So \( S = F[[x, y, z]]/(x^2 + y^2 + az^m) \) is factorial.

If \( m \) is odd, \( \sqrt{-1} \notin F \), \( \sqrt{-a} \notin F \) and \( F \) is real, then \( S = F[[x, y, z]]/(x^2 + y^2 + az^m) \) is factorial ([3, Theorem 2.4]).

If a field \( F \) is of characteristic \( p = 2 \) or \( p \equiv 1 \pmod{4} \), then \( \sqrt{-1} \in F \). So we can formulate the above result as follows.

**Theorem 2.8.** Let \( F \) be a field of characteristic \( p = 2 \) or \( p \equiv 1 \pmod{4} \). Then \( S = F[[x, y, z]]/(x^2 + y^2 + az^m) \) is not factorial for any \( a \in F \) and \( m \geq 2 \).

If a prime integer \( p \) is of the form \( 4n + 3 \), then \( \sqrt{-1} \notin \mathbb{Z}_p \) and \( |\mathbb{Z}_p^*| \) is even. So we obtain the following theorem.

**Theorem 2.9.** Let \( p \) be a prime integer of the form \( 4n + 3 \), and \( m \) even. Then \( S = \mathbb{Z}_p[[x, y, z]]/(x^2 + y^2 + az^m) \) is factorial only for \( |\mathbb{Z}_p^*|/2 \) elements \( a \in \mathbb{Z}_p^* \).

**References**


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