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A Comparison of Group Steiner Tree Formulations

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그룹-스타이너-트리 문제의 수학적 모형에 대한 연구

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The group Steiner tree problem is a generalization of the Steiner tree problem that is defined as follows. Given a weighted graph with a family of subsets of nodes, called groups, the problem is to find a minimum weighted tree that contains at least one node in each group. We present some existing and some new formulations for the problem and compare the relaxations of such formulations.

Keywords: Group Steiner tree, Combinatorial Optimization, Formulation.

1. Introduction

In the group Steiner tree problem (GSTP), we are given an undirected graph $G=(V,E)$ with a nonnegative cost function $c: E \rightarrow \mathrm{R}_+$ and subsets of nodes $V_1, V_2, \cdots, V_m$, called groups. Let $K=\{1, 2, \cdots, m\}$ be the index set of the groups. There may exist nodes that do not belong to any of the groups. A group Steiner tree is defined as a tree that contains at least one node in each group. Then the GSTP is to find a group Steiner tree of minimum cost, where the cost of a tree is the sum of costs of its edges. We assume that the groups are pairwise disjoint but this assumption does not restrict the generality of the problem since we can easily transform a problem not satisfying this assumption to one satisfying it (Garg *et al*., 2000). The GSTP is NP-hard since it is a generalization of the Steiner tree problem (STP).

Reich and Widmayer (Reich and Widmayer 1990) have introduced the GSTP with its applications in VLSI design. Other applications can be found in (Dror *et al*., 2000; Garg *et al*., 2000; Myung *et al*., 1995). Due to its practical and theoretical significance, a lot of research attention has been given to this problem (Chekuri *et al*., 2006; Dror *et al*., 2000; Duin *et al*., 2004; Feremans *et al*., 2001; Ferreira and Filho 2006; Garg *et al*., 2000; Houari and Chaouachi, 2006; Ihler *et al*., 1999; Salazar, 2000; Yang and Gillard, 2000). If $V=V_1 \cup \cdots \cup V_m$ and we are restricted to select exactly one node per group, the resulting problem becomes the generalized minimum spanning tree problem (GMSTP). Myung *et al*. (1995) introduced the GMSTP and many research considered the problem (Feremans *et al*., 2002; Golden *et al*., 2005; Pop *et al*., 2006; Wang *et al*., 2006). The GMSTP is also NP-hard. As many of the works on the GSTP and the GMSTP have been done independently, in some papers the name of GMSTP is used to indicate the GSTP and a couple of different names other than the above two were also used. However, both problems can not be trivially transformed to each other.

In this paper, our objective is to describe various integer programming formulations for the GSTP and compare the linear programming (LP) relaxations of them. We consider the formulations already presented in the literature and also introduce some new ones. A comparison of different formulations is an interesting and meaningful subject in combinatorial

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optimization and similar studies have been done on the STP (Goemans and Myung 1993; Polzin and Daneshmand 2001) and GMSTP (Feremans *et al*., 2002).

2. Notation

Throughout the paper, we frequently use the following notation. We refer to undirected graphs as graphs and to directed graphs as digraphs. In a graph $G=(V, E)$, the elements of E are called edges and the edge e between the node i and j is denoted by $\{i, j\}$ or $\{j, i\}$. In a digraph $D=(V, A)$, the elements of A are called arcs and the arc a from node i to node j is denoted by (i, j). (i, j) and (j, i) do not represent the same arc. Bidirecting an edge $e=\{i, j\}$ means replacing the edge by two arcs in opposite direction, (i, j) and (j, i). Given a graph $G=(V, E)$ and a family of subsets of V, $S_1, \cdots, S_p$, $\delta_G(S_1, \cdots, S_p)$ represents the set of edges with end nodes in different subsets. For a set of $S\subseteq V$, we also use $\delta_G(S)$ to mean $\delta_G(S, V\backslash \mathrm{S})$ and use $E_G(S)$ to denote the set of edges in E with both end nodes in S. When we consider a digraph $D=(V, A)$ we use the following notation. For a set $S \subseteq V$, we use $\delta^-_D(S)=\{(i, j)\in A | i \not\in S, j\in S\}$, $\delta^+_D(S)=\delta^-_D(V\backslash S)$ and $A_D(S)=\{(i, j) | i\in S, j\in S\}$. For exposition brevity, we skip the subscripts G and D, when the underlying graph or digraph is clear in the context and we write $\delta^-_D(i)$ (resp. $\delta^+_D(i)$ or $\delta_G(i)$) instead of $\delta^-_D(\{i\})$ (resp. $\delta^+_D(\{i\})$ or $\delta_G(\{i\})$).

If x is defined on the elements of a set M (typically M is an edge set E, an arc set A or a vertex set V) then we denote $\Sigma_{i\in N} x_i$ for $N\subseteq M$ by $x(N)$. The only exceptions are $\delta(.)$ $\delta^-(.)$, $\delta^+(.)$, $E(.)$ and $A(.)$ which were defined previously.

3. Existing Formulations

Because of the similarities among the STP, the GMSTP and the GSTP, we may expect to formulate the GSTP by directly using the formulations for the STP and the GMSTP. Actually, some formulations of the GSTP can be obtained by slightly modifying the ones for the other problems. However, in order to use a certain formulation for the STP and the GMSTP, we have to either know in a priori at least one node in a selected tree or to assume that exactly one node is selected per group. For this reason, some researchers transformed the problem into a degree constrained STP (Dror *et al*., 2000; Houari and Chaouachi, 2006). In this section, we introduce several formulations for the GSTP that have already appeared in the literature. To describe a selected graph, we define an incidence vector x such that $x_e=1$ if edge e is included in the selected subgraph and 0 otherwise.

3.1 Multicut Based Formulation

Given a graph $G=(V, E)$ and a partition $S_1, \cdots, S_p$ of V, that is defined as a set of disjoint subsets of V whose union is V, we call $\delta(S_1, \cdots, S_p)$ a multicut. Multicut is a generalization of a simple cut that is defined as $\delta(S)$ for a nonempty set $S\subseteq V$. We will say that a partition $S_1, \cdots, S_p$ of V is a group-partition, if for every component S_i of the partition, $V\backslash S_i$ contains at least one group V_k for some $k\in K$, i.e., $S_i\cap V_k=\varnothing$. Ferreira and de Oliveria Filho (Ferreira and de Oliveria Filho, 2006) proposed the following formulation for the GSTP using a class of multicut constraints defined on group-partitions.

$$
\begin{aligned}
\text{(mcut)}\quad \min\ & \sum_{e\in E} c_e x_e && (1)\\
s.t.\ & x(\delta(S_1, \cdots, S_p)) \geq 1, &&\\
& \text{for all group-partitions } S_1, \cdots, S_p \text{ of } V && (2)\\
& x_e \geq 0,\ e\in E && (3)\\
& x : \text{integer} && (4)
\end{aligned}
$$

(mcut) is based on the fact that a minimal subgraph having a path between any pair of groups becomes a group Steiner tree. Ferreira and de Oliveria Filho (Ferreira and de Oliveria Filho, 2006) proved that the separation problem for the constraints (2) is NP-complete. Therefore, we can not expect to solve the LP relaxation of (mcut) in polynomial time and furthermore, we will show later that the LP relaxation of (mcut) is not tight compared with other formulations presented in this paper. However, (mcut) is a unique formulation found in the literature that uses only edge variables, that is, it is a natural formulation for the GSTP.

3.2 Node Variable Based Formulations

Although there are no node weights, we can use node variables to describe which nodes are included in the selected subgraph. For this purpose, we define an incidence vector y such that $y_i=1$ if node i is in the selected subgraph and 0 otherwise. Feremans *et al*. (Feremans *et al*., 2001) proposed the following formulation for the GSTP.

$$
\begin{aligned}
\text{(sub1)}\quad \min\ & \sum_{e\in E} c_e x_e &&\\
s.t.\ & \text{(3), (4) and} &&\\
& y(V_k) \geq 1, \qquad k\in K && (5)\\
& x(E) = y(V) - 1, && (6)
\end{aligned}
$$

$$x(E(S)) \leq y(S) - y_i, \quad i \in S \subseteq V \tag{7}$$

$$x_e \leq 1, \quad e \in E \tag{8}$$

$$y_i \leq 1, \quad i \in V \tag{9}$$

$$y_i \geq 0, \quad i \in V \tag{10}$$

$$y: \text{integer} \tag{11}$$

The constraints (5) ensure that at least one node is selected per group and the equation (6) implies that the number of edges in a tree equals one less than the number of spanned nodes. The constraints (7) are well known generalized subtour elimination constraints that prevent the solution from containing cycles.

Formulations similar to (sub1) are also considered for the STP by Goemans (1994) and for the GMSTP by Myung *et al.* (1995). Salazar (2000) have shown that if $V_k \subseteq S$ for some $k \in K$, generalized subtour elimination constraints (7) for S can be replaced by $x(E(S)) \leq y(S) - 1$. Using this observation, Salazar proposed the following formulation.

$$\text{(sub2)} \quad \min \sum_{e \in E} c_e x_e$$

$$s.t. \quad \text{(3), (4), (5), (6), (8), (9), (10), (11) and}$$

$$x(E(S)) \leq y(S) - y_i, \quad i \in S \in \Theta_1 \tag{12}$$

$$x(E(S)) \leq y(S) - 1, \quad i \in S \in \Theta_2 \tag{13}$$

where $\Theta_1 = \{S \subseteq V \mid S \not\supseteq V_k, \forall k \in K\}$ and
$\Theta_2 = \{S \subseteq V \mid S \supseteq V_k, \text{ for } some\ k \in K\}$.

Both formulations (sub1) and (sub2) are equivalent but their LP relaxations, in which the integer constraints (4) and (11) are omitted, provide different objective values. We will analyze them in Section 5.

3.3 Degree Constrained STP Formulations

The GSTP can be transformed to a degree constrained STP. Suppose that $G=(V,E)$ with a set of groups $V_1, \cdots, V_m$ and a nonnegative cost function c are given as an instance of the GSTP. We add one dummy node per group and place edges between a dummy node and the nodes in the group. We assume 0 edge weight for all newly added edges. Let $T=\{n+1, \cdots, n+m\}$ be the set of dummy nodes where node $n+k$ is assigned to group V_k and let E_k for $k \in K$ be the set of edges between node $n+k$ and the nodes in V_k. Let $\widetilde{G}=(\widetilde{V}, \widetilde{E})$ be the augmented graph where $\widetilde{V}=V \cup T$ and $\widetilde{E}=E \cup E_1 \cup \cdots \cup E_m$. Consider the following degree constrained STP, in which our objective is to select a minimum cost tree connecting dummy nodes such that edges adjacent to each dummy node is at most one. In this problem, all nodes in V are Steiner nodes that may or may not be contained in the selected tree. It is not difficult to know that there exists a one-to-one match between a degree constrained Steiner tree of $\widetilde{G}$ and a group Steiner tree of G and both trees have the same objective value.

Duin *et al.*(2004) considered the degree constrained STP to develop a heuristic of solving the GSTP. We can obtain various formulations for the degree constrained STP from those for the STP by simply adding the constraints forcing each dummy node to have a single degree. In the literature, a myriad of formulations for the STP have been proposed and a comparison of their LP relaxations was also well studied (Goemans and Myung, 1993; Polzin and Daneshmand, 2001). Houari and Chaouachi (Houari and Chaouachi, 2006) presented three formulations for the GSTP in such a way. Here, we introduce a flow based formulation that appeared in (Houari and Chaouachi, 2006). To construct a tight formulation, they describe it on a digraph that is obtained by bidirecting each edge of a given graph. Let A be the set of arcs obtained by bidirecting the edges of E, i.e., $A=\{(i,j) \mid \{i,j\} \in E\}$. For the edges in $E_k, k \in K$, we replace each edge by one arc because we don't need the incoming arcs to the root dummy node and the outgoing arcs from the remaining dummy nodes due to the degree constaints. We set node $n+1$ as the root node and define $A_1=\{(n+1,j) \mid j \in V_1\}$ and $A_k=\{(j,n+k) \mid j \in V_k\}$ for $k \in K \setminus \{1\}$ Let $D=(\widetilde{V}, \widetilde{A})$ denote the resulting digraph where $\widetilde{A}=A \cup A_1 \cup \cdots \cup A_m$ and let $K_1=K \setminus \{1\}$. In a directed graph, each arc replacing the edge e has the cost c_e.

Our objective is to select a minimal digraph having a directed path from node $n+1$ to node $n+k$, for each $k \in K_1$. We additionally define an incidence vector w such that $w_a=1$ if arc a is included in the subgraph and 0 otherwise. The flow formulation considers w_a as the capacity of arc a and determines the variables such that one unit of flow can be sent from node $n+1$ to node $n+k$ for each $k \in K_1$, which we will call the flow of commodity k. So, we additionally need flow variables f_a^k representing the flow of commodity k in arc a. Then the following flow based formulation describes the GSTP.

$$\text{(dcs)} \quad \min \sum_{e \in E} c_e x_e$$

$$s.t. \quad w(\delta^+(n+1))=1, \tag{14}$$

$$w(\delta^-(n+k))=1, \quad k \in K_1 \tag{15}$$

$$f^k(\delta^+(j)) - f^k(\delta^-(j)) = \left.\begin{cases} 1 & \text{if } j=n+1 \\ -1 & \text{if } j=n+k \\ 0 & \text{if } j \notin \{n+1, n+k\} \end{cases}\right\}, \quad k \in K_1 \tag{16}$$

$$f_a^k \le w_a, \quad a \in \widetilde{A}, \ k \in K_1 \tag{17}$$

$$x_e = w_{ij} + w_{ji}, \ e = \{i, j\} \in E \tag{18}$$

$$f_a^k \ge 0, \ a \in \widetilde{A}, \ k \in K_1 \tag{19}$$

$$w_a \ge 0, \ a \in \widetilde{A} \tag{20}$$

$$w : \text{integer} \tag{21}$$

The constraints (16) ensure the existence of a unit flow from the root dummy node to the other dummy nodes. Although we don't need x variables, we insert it for later use when we compare the LP relaxations of the formulations.

4. New Compact Formulations

In this section, we present three new formulations, one using node variables and the other two using a dummy node and a degree constraint. The first formulation comes from the observation that the constraints (8), (9) and (12) can be omitted in (sub2). In other words, the following formulation is also valid for the GSTP.

$$\text{(sub3)} \ \min \ \sum_{e \in E} c_e x_e$$
$$s.t. \ \text{(3), (4), (5), (6), (10), (11), (13)}$$

It is not trivial to show that (sub3) is a valid formulation and we will show it in Section 5.

Our next two formulations are motivated by the observation that we can use only one dummy node to describe various formulations that can be obtained via the degree constrained STP. As defined in Section 3.3, node $n+1$ denotes a dummy node associated with the group V_1 and E_1 denotes the set of edges with 0 weight between the dummy node and each node in V_1. Our two formulations are also based on a digraph. Let A and A_1 be the set of arcs as defined in the previous section. Then our formulations are defined on a digraph, $D = (\widetilde{V}, \widetilde{A})$ where $\widetilde{V} = V \cup \{n+1\}$ and $\widetilde{A} = A \cup A_1$. Notice that D is a subgraph of the digraph we considered in Section 3.3. If we restrict the degree of node $n+1$ to 1, a subgraph having a path from node $n+1$ to at least one node in each group corresponds to a group Steiner tree. We describe two formulations, one using flow variables and the other using the cut constraint. We use an incidence vector w to identify which arcs to be included in a selected subgraph and for a flow based formulation, we use flow variables f_a^k representing the flow destined for group $k \in K_1$ in arc a. The following flow based formulation describes the GSTP.

$$\text{(flow)} \ \min \ \sum_{e \in E} c_e x_e$$
$$s.t. \ \text{(14), (17), (18), (19), (20), (21) and}$$

$$f^k(\delta^+(j)) - f^k(\delta^-(j)) = \begin{cases} 1 & \text{if } \ j = n+1 \\ 0 & \text{if } j \not\in \{n+1\} \cup V_k \end{cases}, \quad k \in K_1 \tag{22}$$

The constraints (22) ensure the existence of a unit flow from the root node to each group, which implies that there exist a path from node $n+1$ to each group in a selected graph.

Our last formulation is the following cut based formulation.

$$\text{(cut)} \ \min \ \sum_{e \in E} c_e x_e$$
$$s.t. \ \text{(14), (18), (20), (21), and}$$

$$w(\delta^-(S)) \ge 1, \ S \in \Theta_2 \tag{23}$$

It is well known that the constraints (22) and (17) can be replaced by the so-called cut constraints (23) by the max-flow min-cut theorem.

Formulation (sub3) has less constraints than both (sub1) and (sub2) and formulations (flow) and (cut) also have less constraints than similar formulations appeared in Section 3.3. So our new formulations are more compact than the corresponding ones presented in Section 3. Nevertheless, the LP relaxations of our new formulations provide lower bounds as good as any other existing formulations. We will show it in the next section. Ferreira and de Oliveria Filho (Ferreira and de Oliveria Filho, 2006) also proposed a formulation using flow variables and another one using the cut constraints but both of them were proved to be invalid (Myung, 2007).

5. A Comparison of LP Relaxations

In this section, we compare the LP relaxations of the formulations considered in Section 3 and 4. The optimal objective value of the LP relaxation of each formulation becomes a lower bound for the GSTP. If the computing time for solving the LP relaxations of the formulations we consider are same, the one giving higher lower bound is most preferable. We will compare the formulations in terms of the lower bounds they can provide. If(·) is an integer programming formulation presented in Section 3 and 4, we let F(·) denote a feasible region of its LP relaxation where the integrality restriction on the variable, (4), (11) and (21) are removed and $F_x(\,\cdot\,)$ is the projection of $F(\,\cdot\,)$ in the space of x variables. We also use $v(\,\cdot\,)$ as the optimal objective value of the LP relaxation.

Among the formulations presented in this paper, the following relations hold.

Theorem 1 : For an arbitrary instance of the GSTP with a nonnegative cost function c, v(sub2) = v(sub3) = v(dcs) = v (cut) = v(flow), v(sub1) $\leq$ v(sub2), and v(mcut) $\leq$ v(sub2). And there exist an instance for which v(mcut) < v(sub1) and one for which v(mcut) > v(sub1).

Proof : We prove the first part of the theorem by proving the following four claims.

(i) v(sub2) = v(sub3)

Since F(sub2) $\subseteq$ F(sub3), v(sub2) $\geq$ v(sub3). To show that v(sub2) $\leq$ v(sub3), we prove that every minimal member of F(sub3) belongs to F(sub2). Consider a $(x, y) \in F$(sub3) such that $(\hat{x}, \hat{y}) \not\in F$(sub3) for all $(\hat{x}, \hat{y}) \leq (x, y)$, $(\hat{x}, \hat{y}) \neq (x, y)$.

First we show that (x, y) satisfies the constraints (12). We will show that if (x, y) violates any of (12), it is not a minimal member that contradicts our assumption. Suppose that (x, y) violates (12) for some $i \in S$ and $S \in \Theta_1$. We assume that we select an inequality such that $|S|$ is the minimum among the violated inequalities and $y_i = \max_{v \in S} y_v$. Notice that such selection gives the most violated inequality among (12) for S. It must be $x_e > 0$ for some $e \in E(S)$ and $y_j > 0$ for some $j \in \{i\} \cup (V \backslash S)$ by (6). If $y_i > 0$, set $j = i$, otherwise set $j = v$ for some $v \in V \backslash S$ with $y_v > 0$. If node j belongs to one of the groups, that is $j \in V_k$ for some $k \in K$, we claim that $y(V_k) > 1$. Otherwise, i.e., $y(V_k) = 1$, we reach the following contradicting result.

$$\begin{aligned} x(E(S \cup V_k)) &\geq x(E(S)) \\ &> y(S) - y_i \\ &\geq y(S \backslash V_k) \\ &= y(S \cup V_k) - 1 \end{aligned}$$

The second inequality is due to our assumption. The third inequality holds when $i \in V_k$ and it also holds when $i \not\in V_k$ since $i \not\in V_k$ implies $j \neq i$ in which $y_v = 0$ for all $v \in S$.

Now we show that we can decrease x_e and y_j without violating the constraints of (sub3). If we decrease x_e and y_j by the same small amount, the resulting variables never violate (3), (5) (by our previous claim), (6), and (10). They also satisfy (13) unless there exists $S' \in \Theta_2$ such that $e \not\in E(S')$, $j \in S'$ and $x(E(S') = y(S') - 1$. Suppose that such S' exists. Then the following relations hold.

$$\begin{aligned} x(E(S \cap S')) &\geq x(E(S)) + x(E(S')) - x(E(S \cup S')) \\ &> y(S) - y_i + y(S') - 1 - y(S \cup S') + 1 \\ &= y(S) - y_i + y(S') - y(S \cup S') \end{aligned}$$

The second inequality is due to our assumption that $x(E(S)) > y(S) - y_i$ and $x(E(S')) = y(S') - 1$ and the fact that (x, y) satisfies (13) for $S \cup S'$. When $j = i$, $y(S) - y_i + y(S') - y(S \cup S')$ becomes $y(S \cap S') - y_i$ and when $j \in V$\S, it becomes 0 since $y_v = 0$ for all $v \in S$. In the former case, (x, y) does not satisfy (12) for $S \cap S'$ and i and in the latter case not satisfy (12) for $S \cap S'$ and any $v \in S \cap S'$. Both cases contradict the minimality of S.

Secondly, we prove that (x, y) satisfies the constraints (9). Suppose that $y_i > 1$ for $i \in V$. We show that we can decrease y_i and x_e for some $e \in E$ with $x_e > 0$ without violating (3), (5), (6), and (10). We first show that there exists at most one constraint (13) such that $i \in S$ and (x, y) satisfies with equality. Suppose that there exist two distinct S, $S' \in \Theta_2$ such that $i \in S \cap S'$, $x(E(S)) = y(S) - 1$ and $x(E(S')) = y(S') - 1$. Then the following relations hold.

$$\begin{aligned} x(E(S \cap S')) &\geq x(E(S)) + x(E(S')) - x(E(S \cup S')) \\ &\geq y(S) - 1 + y(S') - 1 - y(S \cup S') + 1 \\ &= y(S \cap S') - 1 \\ &> y(S \cap S') - y_i \end{aligned}$$

This contradicts our previous claim that (x, y) satisfies the constraints (12). Notice that we can decrease x_e for some $e \in E$ with $x_e > 0$ and y_i by the same small amount without violating (3), (5), (6), and (10). Moreover, if there exists a constraint (13) for some $S \in \Theta_2$ such that $i \in S$ and (x, y) satisfies with equality, we can appropriately select $e \in E(S)$ with $x_e > 0$ and decrease y_i and x_e without violating the constraint.

Finally, we prove that (x, y) satisfies the constraints (8). As (x, y) satisfies (12) and (9), for every $e = \{i, j\} \in E$, $x_e \leq y_i$ by (12) for $S = \{i, j\}$ and $x_e \leq y_i \leq 1$ by (9) for $i \in V$.

(ii) v(sub2) = v(cut)

We first prove that v(sub2) $\geq$ v(cut) by showing that F_x (sub2) $\subseteq$ F_x(cut) and then prove that v(sub3) $\leq$ v(cut) by showing that F_x(cut) $\subseteq$ F_x(sub3). If our claim is true, it implies that v(sub2) = v(cut) by (i).

Consider an $(x, y) \in F$(sub2), we claim that there always exists $w \in R^{|A \cup A_1|}$ that satisfies (14), (18), (20), and

$$w(\delta^-(i)) = y_i, \ i \in V. \tag{24}$$

This can be viewed as a feasibility problem for a bipartite transportation network composed of a set of supply nodes and a set of demand nodes, as shown in <Figure 1>. One supply node is associated with node $n+1$ and the other supply nodes with the edges in E of G. Demand nodes are associated with the nodes in V of G. Arcs of the transportation network correspond to the variables w_{ij} such that $w_{n+1,j}$ denotes the flow on the arc from the supply node associated with node $n+1$ to the one associated with node $j \in V$ and w_{ij} for $e=\{i, j\} \in E$ denotes the flow from the node associated with edge e to the node associated with node $j \in V$. The amount of supplies and demands for supply and demand nodes are shown in <Figure 1> and the capacity of each arc is infinity. To check the feasibility of the transportation problem, let's construct a flow network by adding a source node s, a sink node t, arcs from s to each supply node, and arcs from each demand node to t. The capacity of an arc from s to a supply node is set equal to the supply of the supply node and that of one from a demand node to t is set equal to the demand of the demand node. Then the transportation problem is feasible if and only if the maximum flow from s to t is $x(E)+1$.

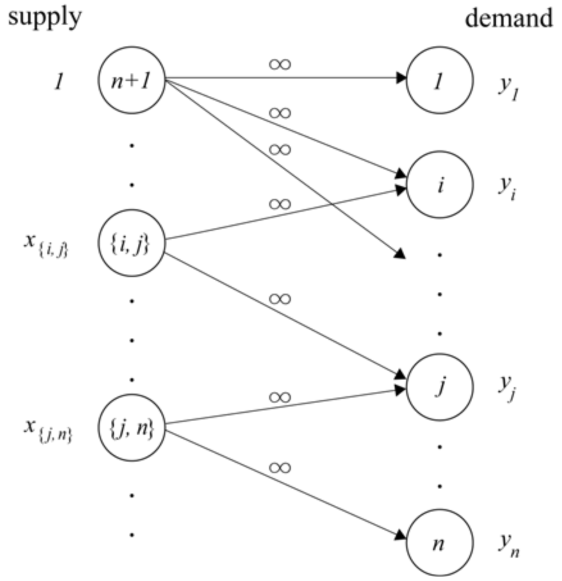

Figure 1. A bipartite transportation network

Now we prove that the maximum flow is $x(E)+1$ by showing that the minimum capacity of $s-t$ cut is $x(E)+1$. Consider a $s-t$ cut and let $S \subseteq V$ be a subset of nodes in G whose corresponding demand nodes in the transportation network are on the s side of the cut. The cut has finite capacity only when every supply node associated with an edge in $E \backslash E(S)$ is on the t side of the cut and if at least one demand node associated with a node in V_1 belong to $V \backslash S$, the supply node associated with node $n+1$ should be on the t side. So, if $V_1 \subseteq S$, the cut value is at least $x(E)-x(E(S))+y(S)$ and otherwise, the cut value is at least $x(E)-x(E(S))+y(S)+1$. In the former case, by (13) and in the latter case, by (12), the cut value is at least $x(E)+1$. Therefore, there exists w satisfying (14), (18), (20) and (24).

Finally, for any $S \in \Theta_2$,

$$\begin{aligned} w(\delta^-(S)) &= \sum_{i \in S} w(\delta^-(i)) - w(A(S)) \\ &= y(S) - x(E(S)) \\ &\geq 1, \qquad (\text{by } (13)) \end{aligned}$$

so $(x, w) \in F(\text{cut})$.

Next, we prove that $F_x(\text{cut}) \subseteq F_x(\text{sub3})$. Consider a $(x, w) \in F(\text{cut})$ and let $y_i = w(\delta^-(i))$ for all $i \in V$. We will show that $(x, y) \in F(\text{sub3})$. First, (x, y) satisfies the constraints (5) and (6) since $y(V_k) = \Sigma_{i \in V_k} w(\delta^-(i)) \geq w(\delta^-(V_k)) \geq 1$ and $x(E) = \Sigma_{i \in V} w(\delta^-(i)) - w(\delta^+(n+1)) = y(V) - 1.(x, y)$ also satisfies the constraints (13) since for each $S \in \Theta_2$,

$$\begin{aligned} x(E(S)) &= \sum_{i \in S} w(\delta^-(i)) - w(\delta^-(S)) \\ &= y(S) - w(\delta^-(S)) \\ &\leq y(S) - 1, \qquad (\text{by } (23)) \end{aligned}$$

Finally, (x, y) obviously satisfies (3) and (10).

(iii) v(cut) = v(flow) = v(dcs)

From (22), we can obtain the equations $\Sigma_{j \in V_k}\{f^k(\delta^+(j)) - f^k(\delta^-(j))\}$= -1 for each $k \in K_1$. Using the max-flow min-cut theorem, we can show that F(cut) is the projection of F(flow) onto the (x,w) space. Moreover, F(flow) is the projection of F(dcs) onto the space of the variables for (flow).

(iv) v(sub1) ≤ v(sub2) and v(mcut) ≤ v(sub2)

It is obvious that $F(\text{sub2}) \subseteq F(\text{sub1})$. Now we show that $F_x(\text{sub2}) \subseteq F(\text{mcut})$. Consider a $(x,y) \in F(\text{sub2})$ and let S_1 $\cdots$, S_p be a group-partition of V. If $p=2$, x satisfies (2) because

$$\begin{aligned} x(\delta(S_1, S_2)) = x(E) - x(E(S_1)) - x(E(S_2)) \geq y(V) - 1 \\ y(S_1) + 1 - y(S_2) + 1 = 1. \end{aligned}$$

The inequality comes from (6) and (13). Suppose that $p \geq 3$. Note that

$$\begin{aligned} &\sum_{j=1, j \neq i}^{p} x(E(S_j)) + x(\delta(S_1, \cdots, S_{i-1}, S_{i+1}, \cdots, S_p)) \\ &= x(E(V \backslash S_i)) \leq y(V \backslash S_i) - 1. \end{aligned}$$

If we sum the first and the last mathematical expression for all i's, we have the following inequality :

$$(p-1)\sum_{i=1}^{p} x(E(S_i)) + (p-2)x(\delta(S_1,\cdots,S_p)) \le \sum_{i=1}^{p} y(V\backslash S_i) - p.$$

Since $\Sigma_{i=1}^{p} x(E(S_i)) = x(E) - x(\delta(S_1,\cdots,S_p))$ and $\Sigma_{i=1}$ $y(V\backslash S_i) = (p-1)y(V)$, we can have $(p-1)x(E) - x(\delta(S_1,$ $\cdots, S_p)) \le (p-1)y(V) - p$.

Therefore, x also satisfies (2) by (6).

Finally, we present two instances of the GSTP, one for which v(mcut) < v(sub1) and the other for which v(mcut) > v(sub1). Both instances are defined on an undirected graph $G=(V,E)$ with $V=\{1,2,3\}$ and $E=\{\{1,2\},\{2,3\},\{1,3\}\}$. The first instance has three groups each of which contains one node and assumes unit weight on each edge. Then $x_{\{1,2\}} = x_{\{2,3\}} = x_{\{1,3\}} = 0.5$ is an optimal solution of the LP relaxation of (mcut) and v(mcut) = 1.5. But in the LP relaxation of (sub1), all y variables should equal to 1 that implies $x(E) = 2$ by (6). Therefore, v(sub1) = 2. The other instance has two groups $V_1 = \{1,2\}$ and $V_2 = \{3\}$ and assumes $c_{\{1,2\}} = 0$, $c_{\{2,3\}} = 1$, $c_{\{1,3\}} = \infty$. Notice that v(sub1) = 0.5 since an optimal solution of the LP relaxation of (sub1) is (x, y) with $y_1 = y_2 = 0.5$, $y_3 = 1$, $x_{\{1,2\}} = x_{\{2,3\}} = 0.5$, and $x_{\{1,3\}} = 1$. But in the LP relaxation of (mcut), v(mcut)≥ 1 since $x_{\{1,3\}} + x_{\{2,3\}} \ge 1$ by (2). □

The proof for (i) is also valid when (x, y) are restricted to have integer values, so the following statement holds.

Corollary 2 : (sub3) is a valid formulation for the GSTP.

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