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Some Conjectures for the Newsvendor Problem under Progressive Multiple Discounts

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■ Abstract ■

This paper investigates properties of the newsvendor problem under a schedule involving progressive multiple discounts compared with the standard newsvendor problem under a no-discounts schedule. Unlike most conventional approaches using the criticial fractile to analyze the retailer and/or supplier behavior(s) in the newsvendor problem, our approach uses riskless profit. From the properties revealed through a series of computational experiments, two conjectures regarding the relationship between the expected profits of both newsvendor problems as a generalization over Khouja's argument (1995) are raised. Those conjectures encourage newsvendors who may face budget or warehouse capacity restriction to use the extended model under a multiple-discounts schedule rather than the standard model with no-discounts schedule because they apply for every order quantity as well as the optimal order quantity. In addition to the conjectures, some insightful results are found to justify the implementation of a multiple-discounts schedule from the computational experiments and a new interpretation for implementation of a multiple-discounts schedule that has not been addressed in Khouja is provided.

Keyword : Newsvendor Problem, Progressive Multiple Discounts

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1. Introduction

The classical newsvendor problem involves identifying the optimal order quantity under a single-period probabilistic demand framework. Such a static or single-period inventory model has wide applicability because it is relevant to many real-world situations. For example, it is often used to aid decision-making in the fashion and sporting goods industries, both at the manufacturing and at the retail levels [8]. In this paper, two versions of the newsvendor problem are considered : the standard version under a no-discounts schedule and the extended alternative under a schedule of multiple progressive discounts.

There exists a significant body of literature regarding the newsvendor problem. Extensive reviews can be found in Silver et al. [27], Khouja [13], Qin et al. [24]. Quantity discount is one of the major extensions over the standard newsvendor problem. A quantity discount has been considered as an effective tool for achieving channel coordination for maximum joint profit to both the retailer and the supplier [7]. Some authors do not consider quantity discount as a tool for achieving channel coordination for maximum joint profit to both the retailer and the supplier [1, 29]. For comprehensive review of the quantity discounts, Benton and Park [2] and Munson and Rosenblatt [20] are referred.

Since Hadley and Whitin [9] provided the classical formulation of the newsvendor problem under the assumption that a single discount is used to sell excess inventory or that excess inventory is disposed of if the order quantity is greater than the realized demand, several extensions of the newsvendor problem have been identified and successfully addressed. Juker and Rosenblatt [10] considered three types of discounts : all-unit discounts, incremental discounts, and carload-lot discounts. Pantumsinchai and Knowles [21] offered a general algorithm for solving the extended newsvendor problem in which order quantity is made up of a number of containers with standard sizes. Khjoua [11] solved an extended newsvendor problem in which multiple discounts are used to sell excess inventory, and Khjoua [12] developed an algorithm for determining the optimal order quantity when both a supplier all-units discount and a retailer (newsvendor) multiple discounts are offered.

Another approach to extending the classical newsvendor problem seeks to determine the optimal order quantity when demand is price-dependent and/or quantity discounts are offered [6, 14, 17, 22, 23, 25, 30]). Zang [31] considers the multi-item newsvendor problem with budget constraint and quantity discounts.

Some authors addressed the newsvendor problem in the case of incomplete information regarding the applicable probability distribution. Scarf [26] formulated the newsvendor problem where only the mean and the variance of demand are known and the demand distribution is unknown; he derived a closed-form solution for optimal order quantity, thereby maximizing the expected profit. Gallego and Moon [8] and Moon and Silver [19] extended Scarf's model.

Khouja [11] argues that multiple discounts provide higher *optimal* expected profit than using a single discount but does not provide the rigorous proof. The purpose of the paper is to raise practically valuable conjectures for implementation of a multiple progressive discounts as a generalization over Khouja's argument. The question is, "Can the newsvendor offering a multiple progressive discounts to customers always expect greater profit for *every* order quantity as well as the optimal solution than the one offering no-discounts?" Our analysis and extended experiments strongly support the conjectures that the answer is yes. Those conjectures encourage newsvendor who may face budget or warehouse capacity restriction to safely use the extended model under a multiple-discount schedule rather than the standard model with no-discounts schedule because they apply for every order quantity as well as the optimal order quantity if they are proved.

Most of the existing approaches use derivatives to find the optimal order quantity. The well-known formula for the optimal order quantity is the critical fractile, which is the ratio based on the unit underestimating cost of demand, the unit overestimating cost of demand, and the unit shortage cost of unsatisfied demand. However, in this paper we use riskless profit, which is the sum of the expected profit and expected cost, to analyze the newsvendor problems and reveal their structural properties and relationship between the expected profits. One of the popular trends in analyzing the inventory problem is to derive the optimal order quantity without using derivatives, such as in determining economic order quantity (EOQ) [5, 28]. Our approach is motivated by such a new trend for analysis of the inventory problem because the analytic framework for the derivative-free approach to the newsvendor problem has been little developed in the literature.

The rest of the paper is organized as follows. In section 2, we use the riskless profit to re-address the basic properties of expected profit and cost of underestimating and overestimating demand in the context of the standard newsvendor problems. In section 3, we use the riskless profit to analyze the extended newsvendor problem under a progressive multiple discounts. In section 4, we conduct extended experiments and suggest two conjectures regarding the relationship of the expected profits between the extended newsvendor problem under a multiple discounts schedule and the standard newsvendor problem under a no-discounts schedule. Finally, the paper concludes with a summary of the present research.

2. Newsvendor Problem under a No-Discounts Schedule

To compare the expected profits of the extended newsvendor problem under a multiple discounts schedule and the standard newsvendor problem under a no-discounts schedule, we use a closed-form formula of riskless profit to restate the standard newsvendor problem under a no-discount schedule. To formulate the standard newsvendor problem, we use the following notation :

a = regular selling price per unit
b = regular purchasing cost per unit
c = salvage value per unit for unsold products
s = shortage penalty cost per unit for unsatisfied demand
X = random variable denoting the demand, $0 \le X < \infty$
$f(X)$ = probability density function of X

$F(X)$ = cumulative distribution function of X
$E(X)$ = expected demand
Q = order quantity
Q^* = optimal order quantity
$ER^I(Q)$ = expected profit from order quantity Q
$EC^I(Q)$ = expected total cost of order quantity Q
$ES^I(Q)$ = sum of expected profit and cost of order quantity Q

From the definition of the nth partial moment with an upper limit Q of a random variable X given a probability density function $f(X)$, we can let $E_Q(X) = \int_0^Q Xf(X)dX$ for a specific order quantity Q [16]. Most researchers consider two cases of the newsvendor problem separately-the case of a shortage penalty cost s = 0 and the case of s > 0 [16, 18]. But we assume that s = 0 without loss of generality. The expected profit for a specific order quantity Q is then given by

$$ER^I(Q) = (a-c)E_Q(X) + (a-b)Q - (a-c)QF(Q) \quad (1)$$

and the expected total cost associated with underestimating and overestimating demand is given by

$$EC^I(Q) = (a-b)E(X). \quad (2)$$
$$-[(a-c)E_Q(X) + (a-b)Q - (a-c)QF(Q)]$$

Mathematical details for equations (1) and (2) can be found in Appendix. From equations (1) and (2), it can easily be noted that the sum of the expected profit and total cost of underestimating and overestimating demand is constant for every order quantity and is given as $(a-b)E(X)$. That is, for every order quantity Q we obtain

$$ES^I(Q) = ER^I(Q) + EC^I(Q) = (a-b)E(X). \quad (3)$$

The term $(a-b)E(X)$ in equation (3) is called as the riskless profit [22] or maximum profit [3] for a given unit price and cost. It can easily be shown that the aforementioned property regarding the sum of both expectations still holds true for the case s > 0. Therefore, we immediately obtain the well-known result that both the profit-based approach and the cost-based approach for the standard newsvendor problem yield the same optimal order quantity Q^* if an identical probability distribution of demand is assumed for both the expected profit maximization problem and the expected cost minimization problem [4, 27].

Equation (3) offers a quick way of calculating the expected cost for the standard newsvendor problem under a no-discount schedule if the expected profit for a specific order quantity is found, because the sum of expected profit and cost is simply given as (unit margin×expected demand) for all order quantities. Based on this property of the balance equation for expected profit and cost, the newsvendor can obtain useful insight using knowledge of the behavior of expected profit and cost for every order quantity, including the optimal order quantity. For example, if the optimal profit or cost varies steeply in the vicinity of the optimal order quantity, a closer inspection of the profit and cost for various order quantity values based on the balance equation of equation (3) can provide especially useful insight for a decision-maker who needs to reach the practically best profit (cost), which may not be optimal, under additional constraints such as budget limits.

3. Newsvendor Problem under a Multiple Discounts Schedule

As extensions to the standard newsvendor problem, certain discount schedules have been considered in the literature ([10], [18]). Of these, the two most common are the all-units quantity discounts and progressive multiple discounts schedules [15]. In this section, a progressive multiple discounts schedule will be considered because the expected profit (cost) under an all-unit quantity discounts schedule is only valid for certain specific ranges of order quantities [12]. For convenience of analysis and comparative purposes, salvage value and shortage penalty cost will not be considered without loss of generality.

To formulate the extended newsvendor problem under a progressive multiple discounts schedule, we use the following notation, some of which is adopted from Khouja [11] :

a_0 = regular selling price per unit, i.e., $a_0 = a$

b_0 = regular purchasing cost per unit, i.e., $b_0 = b$

n = the number of discounts offered

a_i = unit selling price during the ith discount period, $a_0 > a_1 > \cdots > a_i > \cdots > a_n$

b_i = unit purchasing cost during the ith discount period, $b_0 > b_1 > \cdots > b_i > \cdots > b_n$

t_i = fraction of realized demand at the regular price that can be additionally sold by discounting the product to price a_i.

$ER^{II}(Q)$ = expected profit for order quantity Q

$EC^{II}(Q)$ = expected total cost for order quantity Q

$ES^{II}(Q)$ = sum of expected cost and profit for order quantity Q

Following Khjouja [11], we assume that $t_0 = 1$ and $t_n = \infty$. Suppose the first $(r-1)$ discounts are profitable and the rest are not. Let

α_i = cost of underestimating demand, i.e., $\alpha_i = a_i - b_i$

β_i = cost of overestimating demand, i.e., $\beta_i = b_i - a_i$

$V_j = \sum_{i=0}^{j} t_i$

$h_j = Q/V_j$

$W_j = \sum_{i=0}^{j} \alpha_i t_i$

$S_j = \sum_{i=r}^{j} \beta_i t_i$

$F(h_j) = \int_0^{h_j} f(X)dX$

$E_{h_j}(X) = \int_0^{h_j} Xf(X)dX$

Assume that $\alpha_i \geq 0,\ \beta_i = 0$ for $i = 0, \cdots, r-1$ and $\alpha_i = 0,\ \beta_i \geq 0$ for $i = r, \cdots, n$. W_j is the profit for the total quantity demanded if the product is offered at all prices until the jth discount is reached. S_j is the profit for the total quantity demanded if the product is offered at all prices until the jth discount is reached after the $(r-1)$ discount period. Note that since $V_0 = 1$ and $V_n = \infty$, $h_0 = Q,\ h_n = 0,\ E_{h_0}(X) = E_Q(X)$, and $E_{h_n}(X) = 0$. Let $E_{h_{-1}}(X) = E(X)$ and $V_{-1} = W_{-1} = S_{-1} = 0$.

The expected profit over all discount periods for a specific order quantity Q in the extended newsvendor problem is given by

$$\begin{aligned} ER^{II}(Q) &= \sum_{j=0}^{r-1}(W_{j-1} - \alpha_j V_{j-1})\left[E_{h_{j-1}}(X) - E_{h_j}(X)\right] \\ &+ Q\left(\sum_{j=0}^{r-1}\alpha_j\left[F(h_{j-1}) - F(h_j)\right]\right) \\ &+ \sum_{j=r}^{n}(W_{r-1} - S_{j-1} + \beta_j V_{j-1})\left[E_{h_{j-1}}(X) - E_{h_j}(X)\right] \\ &- Q\left(\sum_{j=r}^{n}\beta_j\left[F(h_{j-1}) - F(h_j)\right]\right) \end{aligned} \quad (4)$$

and the expected total cost of underestimating demand and overestimating demand is given by

$$\begin{aligned} EC^{II}(Q) = &\sum_{j=0}^{r-1}\left[W_{r-1}+\sum_{i=0}^{j}(\alpha_j-\alpha_i)t_i\right]\left[E_{h_{j-1}}(X)-E_{h_j}(X)\right] \\ &-Q\left(\sum_{j=0}^{r-1}\alpha_j\left[F(h_{j-1})-F(h_j)\right]\right) \\ &+\sum_{j=r}^{n}(S_{j-1}-\beta_j V_{j-1})\left[E_{h_{j-1}}(X)-E_{h_j}(X)\right] \\ &+Q\left(\sum_{j=r}^{n}\beta_j\left[F(h_{j-1})-F(h_j)\right]\right). \end{aligned} \quad (5)$$

Mathematical details for equations (4) and (5) are provided in Appendix. From equations (4) and (5), we have the following balance equation for the extended newsvendor problem under a progressive multiple discounts schedule :

Observation 1 : The sum of the expected profit and cost of underestimating demand and overestimating demand for the extended newsvendor problem under a progressive multiple discount schedule is *constant* for every order quantity and is given by $W_{r-1}E(X)$. That is, for all Q's we have

$$ES^{II}(Q) = ER^{II}(Q) + EC^{II}(Q) = W_{r-1}E(X). \quad (6)$$

Proof : Combining equations (4) and (5) and simplifying gives

$$\begin{aligned} ES^{II}(Q) &= ER^{II}(Q) + EC^{II}(Q) \\ &= W_{r-1}\left(\sum_{j=0}^{n}\left[E_{h_{j-1}}(X)-E_{h_j}(X)\right]\right)+\sum_{j=0}^{r-1}\left[\sum_{i=0}^{j}(\alpha_j-\alpha_i)t_i \right. \\ &\quad \left. + W_{j-1}-\alpha_j V_{j-1}\right]\left[E_{h_{j-1}}(X)-E_{h_j}(X)\right] \\ &= W_{r-1}\left[E_{h_{-1}}(X)-E_{h_n}(X)\right]+\sum_{j=0}^{r-1}\left[\sum_{i=0}^{j}(\alpha_j-\alpha_i)t_i \right. \\ &\quad \left. + W_{j-1}-\alpha_j V_{j-1}\right]\left[E_{h_{j-1}}(X)-E_{h_j}(X)\right] \\ &= W_{r-1}E(X)+\sum_{j=0}^{r-1}\left[\sum_{i=0}^{j-1}(\alpha_j-\alpha_i)t_i + W_{j-1}-\alpha_j V_{j-1}\right] \\ &\quad \times\left[E_{h_{j-1}}(X)-E_{h_j}(X)\right] \\ &= W_{r-1}E(X)+\sum_{j=0}^{r-1}\left[\alpha_j V_{j-1}-W_{j-1}+W_{j-1}-\alpha_j V_{j-1}\right] \\ &\quad \times\left[E_{h_{j-1}}(X)-E_{h_j}(X)\right] \\ &= W_{r-1}E(X) \end{aligned}$$

since $E_{h_{-1}}(X)=E(X)$ and $E_{h_n}(X)=0$. Because W_{r-1} and $E(X)$ are independent of order quantity Q, the proof is complete. ■

Observation 1 immediately leads to the following result :

Observation 2 : Both the profit-based approach and the cost-based approach for the extended newsvendor problem under a progressive multiple discounts schedule yield the same optimal order quantity Q^* if an identical probability demand distribution is assumed for both the expected profit maximization problem and the expected cost minimization problem.

Observation 2 also offers a quick method of computing the expected cost (profit) if the expected profit (cost) for a specific order quantity can be found because the sum of the expected profit and cost for the extended newsvendor problem under progressive multiple discounts schedule is simply given as [the profit for the total quantity demanded if the product is offered at all prices until the (r-1)th discount is reached × expected demand]. Observation 2 states that the equivalence of the profit-based approach and the cost-based approach in terms of the optimal order quantity still holds even if a progressive multiple discounts schedule is introduced.

From the observation above, we obtain additional observation regarding the relationship between the riskless profits for the standard newsvendor problem under a no-discounts schedule and for the extended newsvendor problem under a progressive multiple discounts schedule. From observations 1 and 2, we immediately obtain the following observation :

Observation 3 : The riskless profit for the extended newsvendor problem under a multiple discounts schedule is no less than the one for the standard newsvendor problem under a no-discounts schedule for every order quantity if an identical probability distribution of demand and an identical pricing are assumed for both problems, i.e., $ES^{II}(Q) \geq ES^{I}(Q)$.

Proof : From the definition of W_{r-1}, the proof is trivial.

Observation 3 implies that if an identical probability distribution of demand and an identical pricing are assumed for both problems, the riskless profit for the extended newsvendor problem under a progressive multiple discounts schedule is no less than the equivalent metric for the standard newsvendor problem under a no-discounts schedule for any order quantity. However, we cannot yet say that $ER^{II}(Q) \geq ER^{I}(Q)$ for every order quantity Q.

We can establish the relationship between the discount schedules with different fractions of realized demand at the regular price that can be additionally sold by introducing a multiple discounts schedule. From observation 3 and the definition of W_{r-1}, we immediately obtain the following observation :

Observation 4 : Consider two discount schedules A and B under the condition that $t_i^A \geq t_i^B$ for all the discount periods i. An identical probability distribution for demand and an identical pricing are assumed for both discount schedules. Then, $ES_A^{II}(Q) \geq ES_B^{II}(Q)$.

Proof : From the definition of W_{r-1}, the proof is trivial.

Observation 4 can be applied to the situation where an identical discount schedule that is implemented under an identical probability distribution of demand and an identical pricing yields a different effect if the fraction of realized demand at the regular price that can be additionally sold by introducing the discount schedule varies under different market conditions in spite of an identical discount schedule. Observation 4 implies that as the effect in terms of the fraction of realized demand at the regular price that can be additionally sold by introducing a discount schedule increases, the riskless profit due to the combined effect of expected profit and cost also increases. However, we cannot say that the discounts schedule with a higher value of t_i for all discount periods yields a higher expected profit than the one for the discounts schedule with a lower value of t_i for all discount periods, i.e., $ER_A^{II}(Q) \geq ER_B^{II}(Q)$.

4. Insights and Conjectures from Computational Experiments

In the previous section, we have established a closed-form formula for the riskless profit of the newsvendor problem under a multiple discounts schedule and shown that both the expected profit maximization problem and the expected cost minimization problem still yield an identical optimal order quantity even if a progressive multiple discounts schedule is introduced because the riskless profit is constant for every order quantity. Furthermore, we have revealed the general relationship regarding the riskless profits between the standard newsvendor problem under no-discounts schedule and the extended newsvendor problem under a multiple discounts schedule.

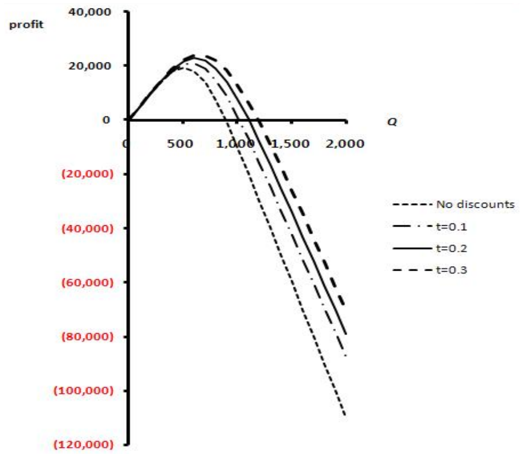
However, the expected profit eventually provides newsvendor with more meaningful information than the riskless profit. Khouja [11] argues that the optimal expected profit of the newsvendor problem under a multiple discounts schedule should be greater than the one of the newsvendor problem under no-discounts schedule but does not provide the rigorous proof. As can be seen in equation (4), it seems that it is very difficult to establish a general proof for Khouja's argument which holds even for every order quantity as well as the optimal order quantity. Therefore, in this section we conduct extended experiments for generalization of Khouja's argument and establish certain conjectures as a generalization over Khouja's argument based on the experimental results.

To establish insights and conjectures that may be valuable for the newsvendor faced with an identical discounting policy that features different parameters for the newsvendor inventory problem, we compare the expected profits of both types of newsvendor problems under various parameters and with a variety of probability demand distributions. Our goal was to establish strong arguments regarding expected profits under different values of t.

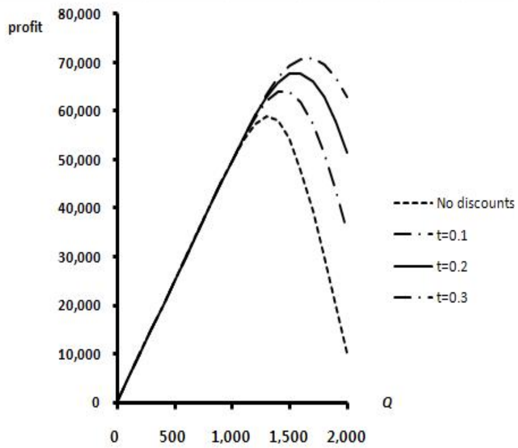
To implement the experiment, we selected three kinds of binomial distribution with parameters p = 0.3, p = 0.5, and p = 0.7, where p denotes the probability of success in a binomial experiment that featured 20 trials, and a uniform distribution. Continuous probability distribution of demand such as exponential or normal distribution can be considered for numerical experiment. In this study, the binomial distribution and uniform distribution are selected since they can easily be discretized. The use of the discrete demand distribution is advocated because it is easier to specify and it can be subjectively adjusted to account for the cases when the demand exceeded the order quantity [13]. Assume that the demand varies in increments of 100 from 0 to 2,000 for both distributions. The binomial distribution with p = 0.5 is close to the normal distribution that is frequently adopted by researchers to justify their solution approaches. The binomial distribution with p = 0.3 can be regarded as a situation where, on average, a small number of products is ordered frequently. The scenario with p = 0.7 features a significant quantity of products being ordered frequently. The uniform distribution represents even chance of order for every demand. Even if the experiments are conducted with a limited set of probability distribution of demand, such diverse patterns of demand due to the different shapes of probability distribution function of demand are very close to the representative types of demand that frequently occur in reality. For the extended experiments, with a regular unit selling price of $a_0 = 150$, three discounts of $a_1 = 120$, $a_2 = 40$, and $a_3 = 10$ off the regular unit selling price are progressively used to sell any excess inventory. For the sake of simplicity, assume a constant unit cost of b = 100. That is, only the first discount is profitable and the rest are not. We consider an identical discounting schedule with the same unit selling prices and costs as the standard newsvendor problem but with different values of $t_1 = t_2 = 0.1$, $t_1 = t_2 = 0.2$, and $t_1 = t_2 = 0.3$. Assume $t_3 = \infty$ in all problems.

[Figure 1]~[Figure 4] show plots of the expected profits in the case of a multiple discounts schedule with different values of t and in the case of a no-discounts schedule. However, from closer examination of figures some remarks follow which can give newsvendor valuable insights for implementation of multiple discounts :

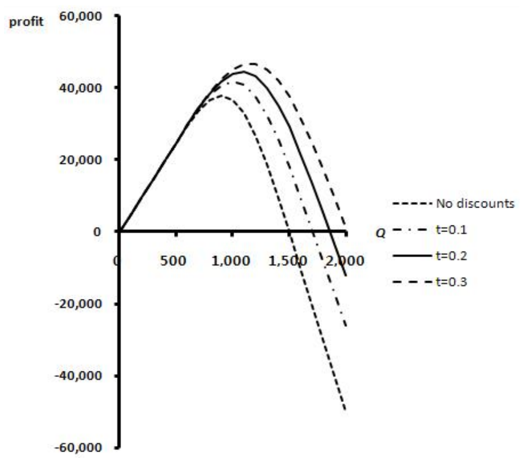
- As the value of t increases, the expected profit associated with a specific order quantity increases. In addition, we can strongly infer that



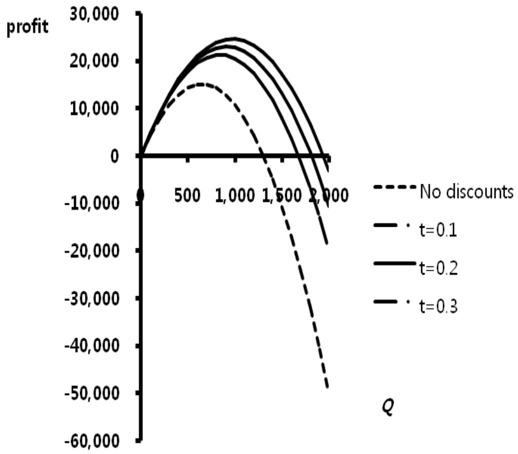
[Figure 1] Plots of the Expected Profits for Various Values of t Under the Binomial Distribution with p = 0.3



[Figure 3] Plots of the Expected Profits for Various Values of t Under the Binomial Distribution with p = 0.7



[Figure 2] Plots of the Expected Profits for Various Values of t Under the Binomial Distribution with p = 0.5



[Figure 4] Plots of the Expected Profits for Various Values of t Under the Uniform Distribution higher value of t will yield a higher expected profit for every order quantity.

- When the order quantity is small, introducing multiple discounts does not yield significant benefit compared with using no-discounts. For instance, in [Figure 1], multiple discounts schedule does not provide higher expected profits than using no-discounts for order quantity less than or equal to 300 units. As the value of p increases, more quantity needs to be ordered to justify offering the customer a multiple discounts schedule. Similar statement can be applied to the case of uniform distribution of demand. This phenomenon has not been addressed in Khouja [11].
- It is evident that the expected profits due to the multiple discounts schedule and no discounts schedule decrease when the order quantities exceed the optimal order quantities for the two schedules, respectively. But it is interesting to note that the more the newsvendor offering multiple discounts orders, the greater profit she can expect for the same order quantity as the newsvendor offering no discounts. When the order quantities under the two schedules increase simultaneously, the gap between the expected profits of the two schedules increases even if the order quantity is larger than the optimal order quantity.

In short, implementing multiple discounts is always advantageous to the retailer regardless of the order quantity than offering no discounts. These results can be established as the following conjectures :

Conjecture 1 : Given an identical probability distribution of demand, introducing a multiple discounts schedule yields higher expected profits than does a no-discounts schedule for any given order quantity. The more the newsvendor introducing a multiple discounts schedule orders, the larger the gap between the expected profits of both schedules when the order quantities under the two schedules increase at the same time.

Conjecture 2 : Given an identical multiple discounts schedule with the same unit selling prices and costs, a higher value of t yields a higher expected profit for every order quantity.

These conjectures encourage newsvendors to introduce a discounting schedule if the discounting policy does not decrease demand, at least. If theses conjectures are proved, the newsvendor considering implementation of a multiple discounts for customers can safely choose to implement her discounts schedule even if the optimal order quantity cannot be ordered due to additional restrictions such as budget or warehouse capacity constraint. But even if the above conjectures are true, this does not imply that given an identical multiple discounts schedule, but with the same unit selling prices and costs, a higher value of t will yield a lower expected cost for any given order quantity.

5. Conclusion

In this paper, we have investigated properties of the newsvendor problem under a multiple discounts schedule through the analysis of riskless profit and revealed the general relationship of the riskless profits between the standard newsvendor problem under no-discounts schedule and the newsvendor problem under a multiple discounts schedule. Based on extended numerical experiments, as a generalization over Khouja's argument [11] we propose valuable conjectures that encourage the newsvendor to introduce a multiple discounts schedule in place of a no-discount schedule regardless of the order quantity. Even if the numerical experiments are provided with a limited set of probability distribution of demand, the experimental results show consistent advantage of a multiple-discounts schedule compared with a no-discounts schedule in terms of the expected profit for ever order quantity. However, it has been found from the experiments that the advantage of a multiple discounts schedule over a no-discounts schedule is not significantly large for small order quantity and therefore, sufficiently large order quantity is needed to justify the implementation of a multiple discounts schedule. The complete proof of the conjectures raised in this paper is a challenging work for future research. The newsvendor model considered in this study adopts the original version in Khouja [11] which does not consider realistic factors such as price-dependent demand function, multi-periods, multiple products, and warehouse capacity limit. The complete proof of the conjectures raised in this study under the extension incorporating these factors into the newsvendor model with multiple discounts will improve applicability of the conjectures.

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〈Appendix〉

1. Derivation of equations (1) and (2)

Since in the literature the cases of s = 0 and s > 0 are usually discussed separately, our presentation also follows this convention.

(1) Case of s = 0

The profit for order quantity Q under demand X is

$$R^I(X,\ Q)=\begin{cases}(a-b)Q, & \text{if } X\geq Q,\\ (a-c)X-(b-c)Q, & \text{if } X<Q\end{cases}$$

The expected profit is then

$$\begin{aligned}ER^I(Q)&=(a-b)\int_Q^{\infty}Qf(X)dX+(a-c)\int_0^{\infty}Xf(X)dX-(b-c)\int_0^{\infty}Qf(X)dX\\ &=(a-c)E_Q(X)+(a-b)Q-(a-c)QF(Q).\end{aligned}$$

The total cost of underestimating and overestimating demand for order quantity Q under demand X is

$$C^I(X,\ Q)=(a-b)\times\text{Max}(X-Q,\ 0)+(b-c)\times\text{Max}(Q-X,\ 0).$$

The expected total cost is then

$$\begin{aligned}EC^I(Q)&=(a-b)\int_Q^{\infty}(X-Q)f(X)dX+(b-c)\int_0^{\infty}(Q-X)f(X)dX\\ &=(a-b)E(X)-\lfloor(a-c)E_Q(X)+(a-b)Q-(a-c)QF(Q)\rfloor.\end{aligned}$$

(2) Case of s > 0

For the model with s > 0, the profit for order quantity Q given demand X is

$$R^I(X,\ Q)=\begin{cases}(a-b+s)Q-sX, & \text{if } X\geq Q,\\ (a-c)X-(b-c)Q, & \text{if } X<Q\end{cases}$$

The expected profit is then

$$ER^I(Q)=(a-b+s)\int_Q^{\infty}Qf(X)dX-s\int_Q^{\infty}Xf(X)dX+(a-c)\int_0^{Q}Xf(X)dX-(b-c)\int_0 Qf(X)dX$$

$$= (a-b+s)Q[1-F(Q)] - s\lfloor E(X) - E_Q(X)\rfloor + (a-c)E_Q(X) - (b-c)QF(Q)$$
$$= (a-c)E_Q(X) - s\lfloor E(X) - E_Q(X)\rfloor + (a-b+s)Q - (a-c+s)QF(Q)$$

Since the unit underestimating cost of demand is $(a-b+s)$, the total cost of underestimating and overestimating demand for order quantity Q under demand X is given by

$$C^I(X, Q) = (a-b+s)\times \mathrm{Max}(X-Q, 0)\times \mathrm{Max}(Q-X, 0).$$

The expected total cost is then

$$ER^I(Q) = (a-b+s)\int_Q^\infty (X-Q)f(X)dX + (b-c)\int_0^Q (Q-X)f(X)dX$$
$$= (a-b+s)\lfloor E(X) - E_Q(X)\rfloor - (a-b+c)Q[1-F(Q)] + (b-c)QF(Q) - (b-c)E_Q(X)$$
$$= (a-b)E(X) - \lfloor (a-c)E_Q(X) - s\{E(X) - E_Q(X)\} + (a-b+s)Q - (a-c+s)QF(Q)\rfloor$$

Adding $ER^I(Q)$ and $EC^I(Q)$ leads to equation (3).

2. Derivation of equations (4) and (5)

To establish the balance equation for the newsvendor problem under a progressive multiple discounts schedule, we ignore the shortage penalty cost for convenience. To establish equations (4) and (5), the equations and notation in Khouja [11] are adopted with modifications. Let

$R_j^{II}(X, Q)$ = profit for order quantity Q under demand X in the jth discount period

$ER_j^{II}(Q)$ = expected profit for order quantity Q in the jth discount period

$ER^{II}(Q)$ = expected profit for order quantity Q over all discount periods

$CU_j^{II}(X, Q)$ = cost of underestimating demand for order quantity Q under demand X in the jth discount period

$ECU_j^{II}(Q)$ = expected cost of underestimating demand for order quantity Q in the jth discount period

$ECU^{II}(Q)$ = expected cost of underestimating demand for order quantity Q over all discount periods

$CO_j^{II}(X, Q)$ = cost of overestimating demand for order quantity Q under demand X in the jth discount period

$ECO_j^{II}(Q)$ = expected cost of overestimating demand for order quantity Q under demand X in the jth discount period

$ECO^{II}(Q)$ = expected cost of overestimating demand for order quantity Q over all discount periods.

From Khouja [11], the profit for order quantity Q under demand X in the jth discount period $(j = 0, 1, \cdots, r-1)$ is

$$R_j^{II}(X,\ Q)=\left(\sum_{i=0}^{j-1}\alpha_i t_i\right)X+\alpha_j(Q-V_{j-1}X)=(W_{j-1}-\alpha_j V_{j-1})X+\alpha_j Q.$$

The expected profit for order quantity Q in the jth discount period is

$$\begin{aligned}ER_j^{II}(Q)&=\int_{h_j}^{h_{j-1}}(W_{j-1}-\alpha_j V_{j-1})Xf(X)dX+\int_{h_j}^{h_{j-1}}\alpha_j Qf(X)dX\\&=(W_{j-1}-\alpha_j V_{j-1})\lfloor E_{h_{j-1}}(X)-E_{h_j}(X)\rfloor+\alpha_j Q\lfloor F(h_{j-1})-F(h_j)\rfloor\end{aligned}\tag{7}$$

The profit for order quantity Q under demand X in the jth discount period $(j=r,\,r+1,\,\cdots,\,n)$ is

$$R_j^{II}(X,\ Q)=\left(\sum_{i=0}^{r-1}\alpha_i t_i\right)X-\left(\sum_{i=r}^{j-1}\beta_i t_i\right)X-\beta_j(Q-V_{j-1}X)=(W_{r-1}-S_{j-1}+\beta_j V_{j-1})X-\beta_j Q.$$

The expected profit for order quantity Q under in the jth discount period is

$$\begin{aligned}ER_j^{II}(Q)&=\int_{h_j}^{h_{j-1}}(W_{r-1}-S_{j-1}+\beta_j V_{j-1})Xf(X)dX-\int_{h_j}^{h_{j-1}}\beta_j Qf(X)dX\\&=(W_{r-1}-S_{j-1}+\beta_j V_{j-1})\lfloor E_{h_{j-1}}(X)-E_{h_j}(X)\rfloor-\beta_j Q\lfloor F(h_{j-1})-F(h(h_j)\rfloor.\end{aligned}\tag{8}$$

Summing equations (7) and (8) over each discount period j and combining the results yields equation (4) for the expected profit.

The cost of underestimating demand for order quantity Q given a demand X in the jth discount period $(j=0,1,\cdots,\,r-1)$ is

$$\begin{aligned}CU_j^{II}(X,\ Q)&=\alpha_j(V_jX-Q)+\left(\sum_{i=j+1}^{r-1}\alpha_i t_i\right)X=\left(\alpha_j V_j+\sum_{i=j+1}^{r-1}\alpha_i t_i\right)X-\alpha_j Q\\&=\left[\sum_{i=0}^{r-1}\alpha_i t_i+\sum_{i=0}^{j}(\alpha_j-\alpha_i)t_i\right]X-\alpha_j Q=\left[W_{r-1}+\sum_{i=0}^{j}(\alpha_j-\alpha_i)t_i\right]X-\alpha_j Q.\end{aligned}$$

The expected cost of underestimating demand for order quantity Q in the jth discount period is

$$\begin{aligned}ECU_j^{II}(Q)&=\int_{h_j}^{h_{j-1}}\left[W_{r-1}+\sum_{i=0}^{j}(\alpha_j-\alpha_i)t_i\right]Xf(X)dX-\int_{h_j}^{h_{j-1}}\alpha_j Qf(X)dX\\&=\left[W_{r-1}+\sum_{i=0}^{j}(\alpha_j-\alpha_i)t_i\right]\left[E_{h_{j-1}}(X)-E_{h_j}(X)\right]-\alpha_j Q\left[F(h_{j-1})-F(h_j)\right].\end{aligned}$$

Finally, the expected cost of underestimating demand for order quantity Q over the discount periods $j=0,1,\cdots,\,r-1$ is

$$ECU^{II}(Q) = \sum_{j=0}^{r-1}\left[W_{r-1} + \sum_{i=0}^{j}(\alpha_j - \alpha_i)t_i\right]\left[E_{h_{j-1}}(X) - E_{h_j}(X)\right] - Q\left(\sum_{j=0}^{r-1}\alpha_j\left[F(h_{j-1}) - F(h_j)\right]\right). \tag{9}$$

The cost of overestimating demand for order quantity Q under demand X in the jth discount period $(j = r, r+1, \cdots, n)$ is

$$CO_j^{II}(X, Q) = \beta_j(Q - V_{j-1}X) + \left(\sum_{i=r}^{j-1}\beta_i t_i\right)X = (S_{j-1} - \beta_j V_{j-1})X + \beta_j Q.$$

The expected cost of overestimating demand for order quantity Q in the jth discount period is

$$ECO_j^{II}(Q) = \int_{h_j}^{h_{j-1}}(S_{j-1} - \beta_j V_{j-1})Xf(X)dX + \int_{h_j}^{h_{j-1}}\beta_j Qf(X)dX.$$

Then, the expected cost of overestimating demand for order quantity Q over the discount periods $j = r, r+1, \cdots, n$ is

$$ECO^{II}(Q) = \sum_{j=r}^{n}(S_{j-1} - \beta_j V_{j-1})\left[E_{h_{j-1}}(X) - E_{h_j}(X)\right] + Q\left(\sum_{j=r}^{n}\beta_j\left[F(h_{j-1}) - F(h_j)\right]\right). \tag{10}$$

Summing equations (9) and (10) over each discount period j and combining the results yields equation (5) for the expected total cost.