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확률적 수요를 갖는 단일구매자와 단일공급자 시스템의 다품목 통합발주문제

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Joint Replenishment Problem for Single Buyer and Single Supplier System Having the Stochastic Demands

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■ Abstract ■

In this paper, we analyze a logistic system involving a supplier who produces and delivers multiple types of items and a buyer who receives and sells the products to end customers. The buyer controls the inventory level by replenishing each product item up to a given order-up-to-level to cope with stochastic demand of end customers. In response to the buyer's order, the supplier produces or outsources the ordered item and delivers them at the start of each period.

For the system described above, a mathematical model for a single type of item was developed from the buyer's perspective. Based on the model, an efficient method to find the cycle length and safety factor which correspond to a local minimum solution is proposed. This single product model was extended to cover a multiple item situation. From the model, algorithms to decide the base cycle length and order interval of each item were proposed. The results of the computational experiment show that the algorithms were able to determine the global optimum solution for all tested cases within a reasonable amount of time.

Keyword : Joint Replenishment Problem, Stochastic Demand, Optimization

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1. Introduction

The joint replenishment problem (JRP) concerns the inventory control of multiple item types with the objective of minimizing the sum of relevant costs. This paper analyzes the JRP in a logistic system involving a supplier who produces and delivers multiple items and a buyer who receives and sells these items to end customers. In this system, the buyer controls the inventory level by replenishing each item type up to a given order-up-to level in order to meet the stochastic demands of the end customers. In response to the buyer's order, the supplier produces or outsources the ordered items and delivers them at the beginning of each period.

Previous research related to the JRP can be categorized into two groups. The first group of research concerns the JRP under a deterministic demand. Goyal [7] developed an enumeration algorithm to find the global optimum for a cyclic policy. Silver [21] developed an efficient heuristic algorithm that was later improved by Goyal and Belton [8]. Kaspi and Rosenblatt [12] studied a JRP with similar settings to introduce an algorithm with superior performance. Jackson et al. [9] introduced a power-of-two policy and demonstrated that the policy's error was within six percent of the optimum. Rosenblatt and Lee [20] discussed a JRP in a system with imperfect production processes. Fogarty and Barringer [5] analyzed joint order release decisions under dependent demand. Viswanathan [25] proposed a modification of the previous algorithm suggested by Fung and Ma [6] and ensured that the modified algorithm obtained the optimal strict cycle policy. Lee and Yao [16] derived a global optimum search algorithm under the power-of-two policy. Khouja et al. [14] proposed an algorithm for solving the JRP for products that may experience unit cost change. For a general survey of the JRP including the deterministic demand cases, refer to Khouja and Goyal [13].

The second group of research pertains to stochastic but stationary demands. The policies in this group can be divided into two sub-groups : continuous review policy and periodic review policy. Balintfy [2] initiated research on the continuous policy by proposing a can-order policy. Under this policy, an item must be ordered when its inventory position reaches a must-order level. At the same time, all other items whose inventory positions are at or below a can-order level are also ordered. This policy was further analyzed by Silver [22] and Federgruen et al. [4]. Ohno and Ishigaki [19], Melchiors [17], and Nielsen and Larsen [18] analyzed a JRP with a Poisson or compound Poisson process. Later, Larsen [15] suggested a $Q(s, S)$ policy for a system with a compound correlated Poisson process.

The periodic review policies review inventory status at every deterministic interval or at a stochastic interval determined by cumulative demand. Atkins and Iyogun [1] proposed two policies. The first policy orders each item type to be replenished up to an individual base stock level. In the second policy, each item is replenished at an interval that is an integer multiple of a base period. Visnawathan [24] developed a policy called the $Q(s, S)$ policy, in which each item is reviewed at a fixed interval, and an independent order-up-to-level policy is used to control each item type. Johansen and Melchiors [11] suggested a near-optimal can-order policy applicable to a periodic review environment and showed that the policy performed well under conditions of irregular demand.

Eynan and Kropp [3] examined a periodic review system with variable out of stock costs. They proposed a simple heuristic for determining both the base cycle and order interval of each product type, as well as a service level that minimizes the total cost. The heuristic they proposed was shown to find a solution close to the global optimum. Jeong and Kim [10] presented a model and solution methodology for the same problem.

The present paper extends the paper by Jeong and Kim [10] to solve a problem similar to the one analyzed by Eynan and Kropp [3] but with the added complexity that a target customer service constraint must first be satisfied. This modification reflects a real practice in which management considers the customer service rate as a primary management target, followed by the second objective of profit maximization. Our paper introduces three kinds of algorithms for calculating the base cycle, cycle multiplier, and safety factor of each item type that satisfies given service targets at the minimum cost. The results of the computational experiment show that the algorithms are able to identify the global optimum solution for all tested cases within a reasonable amount of time.

Section 2 defines our problem and introduces a mathematical model and an algorithm for a single product problem. Section 3 extends the result of the single item problem to a multiple item problem, for which a mathematical model and algorithms are introduced. Section 4 presents the results of the computational experiments. Finally, Section 5 concludes this paper with a summary and comments on the direction of future research.

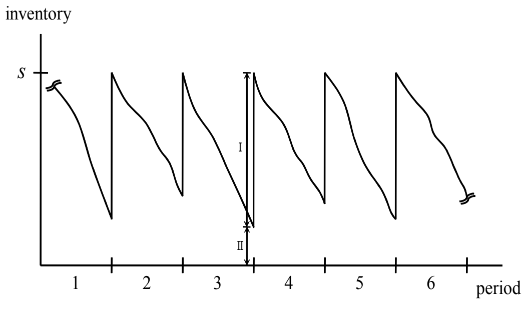
2. Single Item Problem

This section introduces a mathematical model for a buyer who orders a single type of item according to a periodic order-up-to-level policy called $(R,\ S)$ policy.

The following is a list of necessary notations.

R : review interval, in year,
L : lead time needed for a replenishment order to arrive at the buyer's location, in year,
k : safety factor used for setting the safety stock level of the buyer,
D : annual demand rate of the end customers, in units per year, in units per year,
σ : standard deviation of D, in units per year,
A : buyer's ordering cost per order, in dollars,
h : buyer's holding cost per unit per year, in dollars,
b : buyer's shortage cost per unit, in dollars,
s : buyer's order-up-to level, in units,
TC : total cost per year incurred by the buyer, in dollars.

The buyer's total cost consists of ordering, holding, and shortage costs. The total ordering cost per year is the ordering cost per period divided by R, i.e.,



[Figure 1] Inventory Change of the Buyer

$$\frac{A}{R}. \tag{1}$$

[Figure 1] shows a plot of the buyer's inventory change; the average length of I is the average demand per cycle and may be expressed as DR. The height denoted by II is the average inventory remaining before a replenishment delivery arrives. By definition, II represents the safety stock level of the buyer. When k is used to denote the safety factor of the buyer, the safety stock level is expressed as

$$k\sigma\sqrt{R+L} \tag{2}$$

Thus, the average height of the inventory maintained per year is

$$\mathrm{E}\left(\frac{I}{2}+II\right)=\frac{DR}{2}+k\sigma\sqrt{R+L}. \tag{3}$$

The holding cost per year is the value determined by multiplying the result of (3) by the per unit holding cost; this value can be expressed as

$$\left(\frac{DR}{2}+k\sigma\sqrt{R+L}\right)h. \tag{4}$$

A shortage occurs when the end customer's demand exceeds the order-up-to level of the buyer. Thus, the expected number of shortages per cycle is

$$\int_s^\infty (y-s)f(y)dy, \tag{5}$$

where $f(y)$ is the probability density function (pdf) of y, which denotes the customer demand during $R+L$. The expected shortage cost per year is

$$\frac{b}{R}\int_s^\infty (y-s)f(y)dx. \tag{6}$$

In (6),

$$s=D(R+L)+k\sigma\sqrt{R+L}. \tag{7}$$

When we assume, as in Eynan and Kropp [3], that the customer demand during an year is distributed normally with mean D and standard deviation σ, demand during $R+L$ time grid is normally distributed with a mean $D(R+L)$ and standard deviation $\sigma\sqrt{R+L}$. When y is used to denote the demand during $R+L$ time grid, its pdf is expressed as

$$f(y)=\frac{1}{\sqrt{2\pi}\,\sigma\sqrt{R+L}}e^{-\frac{(y-D(R+L))^2}{2(\sigma\sqrt{R+L})^2}}. \tag{8}$$

By setting

$$z=\frac{y-D(R+L)}{\sigma\sqrt{R+L}}, \tag{9}$$

we obtain

$$dy=\sigma\sqrt{R+L}\,dz. \tag{10}$$

Using (8) through (10), (6) can be rewritten as

$$\frac{b}{R}\int_s^\infty (y-s)f(y)dy \tag{11}$$

$$=\frac{b}{R}\int_{D(R+L)+k\sigma\sqrt{R+L}}^\infty \left(y-D(R+L)-k\sigma\sqrt{R+L}\right)f(y)dy$$

$$=\frac{b}{R}\int_k^\infty \left(z\sigma\sqrt{R+L}-k\sigma\sqrt{R+L}\right)$$

$$\times\frac{1}{\sigma\sqrt{2\pi}\sqrt{R+L}}e^{-\frac{z^2}{2}}\sigma\sqrt{R+L}dz$$

$$=\frac{b}{R}\sigma\sqrt{R+L}\int_k^\infty (z-k)g(z)dz$$

$$=\frac{b}{R}\sigma\sqrt{R+L}\left(\int_k^\infty zg(z)dz-k\int_k^\infty g(z)dz\right)$$

$$=\frac{b}{R}\sigma\sqrt{R+L}\left(g(k)-k(1-F(k))\right)$$

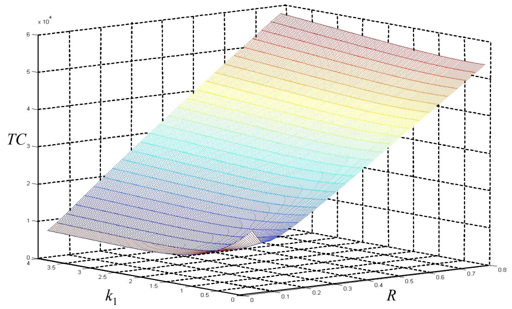
$$=\frac{b}{R}\sigma\sqrt{R+L}\,G_u(k),$$

where $g(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$, that is the pdf of the standard normal distribution, $F(k)=\int_{-\infty}^{k} g(z)dz$, that is the cumulative distribution function of the standard normal distribution, and $G_u(k)=g(k)-k(1-F(k))$, that is the unit normal loss function (See for reference page 721 of Silver et al. [23]).

Thus the total expected cost to the buyer is formulated as the function in equation (12).

$$TC(R,k)=\frac{A}{R}+h\left(\frac{DR}{2}+k\sigma\sqrt{R+L}\right)+\frac{b}{R}\sigma\sqrt{R+L}\,G_u(k). \quad (12)$$

As shown in <Appendix A>, the cost function in (12) is not guaranteed to be a convex function. However, it is expected to be a convex function for parameter values in reasonable ranges or at least behave like a convex function. [Figure 2] showing a typical shape of the cost function when the convexity condition is not met support our expectation. It was thus hoped that R, k values that minimize the cost function in (12) can be found by taking advantage of the first necessary condition of stationary point. In this regard, a partial differentiation with respect to k is performed. This will yield



[Figure 2] Contour Graph of the Cost Function in k_1, R axis

$$\frac{\partial TC}{\partial k}(R,k)=h\sigma\sqrt{R+L}-\frac{b}{R}\sigma\sqrt{R+L}(1-F(k)). \quad (13)$$

By setting

$$\frac{\partial TC}{\partial k}(R,k)=0, \quad (14)$$

we obtain the following relationship that must be satisfied by the optimal R, k values :

$$1-F(k)=\frac{h}{b}R, \quad (15)$$

or

$$R=\frac{b}{h}(1-F(k)). \quad (16)$$

Partial differentiation of (12) with respect to R yields

$$\frac{\partial TC}{\partial R}(R,k)=-\frac{A}{R^2}+\frac{Dh}{2}+\frac{k\sigma h}{2\sqrt{R+L}}-\frac{b\sigma}{2}(f(k)-k(1-F(k)))\left(\frac{R+2L}{R^2\sqrt{R+L}}\right). \quad (17)$$

By setting

$$\frac{\partial TC}{\partial R}(R,k)=0, \quad (18)$$

it is obtained that

$$\frac{k\sigma h}{2\sqrt{R+L}}-\frac{b\sigma}{2}(f(k)-k(1-F(k)))\left(\frac{R+2L}{R^2\sqrt{R+L}}\right)=\frac{A}{R^2}-\frac{Dh}{2}. \quad (19)$$

Substituting (16) into (19) yields

$$\frac{k\sigma h}{2\sqrt{(b/h)(1-F(k))+L}}-\frac{b\sigma}{2}(f(k)-k(1-F(k)))\times\left(\frac{(b/h)(1-F(k))+2L}{((b/h)(1-F(k)))^2\sqrt{(b/h)(1-F(k))+L}}\right)-\frac{A}{((b/h)(1-F(k)))^2}+\frac{Dh}{2}=0. \quad (20)$$

The value of k that satisfies (20) can be obtained using either a general search method or an appropriate software package. During our computational experiment, Microsoft Excel was used to solve (20). The search area for k was confined to practically feasible values of k from 0 to 4.0. A safety factor of 4.0 corresponds to a 100% customer service rate. Inserting this k value into (16) yields the corresponding R value. This R, k set is the optimal cycle length and safety factor for the buyer considering only cost minimization.

However, the buyer's use of this R, k set (which seeks only cost minimization) may lead to an unacceptably low customer service level, e.g., a fill rate of 60%, a situation in which 40% of the incoming customers are unable to purchase the product due to inventory shortage. As such, it is common practice for retailers to pursue a cost minimizing objective as a second priority, attempting to first achieve a target service level [23]. Considering this practice, the final solution for the buyer is the R, k set that satisfies the target customer service level at the minimum cost.

An algorithm to determine the final solution that satisfies a preset target service level at minimum cost is introduced below. The minimum cost k value is denoted as $k^{\min}$, and the target customer service level is represented by k^{target}.

RK_Single Algorithm

Step 1 : Determine the minimum cost k value according to (20) and save the value as $k^{\min}$.

Step 2 : If $k^{\min} \geq k^{\text{target}}$, go to Step 3. Otherwise, go to Step 4.

Step 3 : Set $k \leftarrow k^{\min}$ and use (16) to calculate the corresponding R value. Generate an R, k set as the solution to the problem. Stop.

Step 4 : Set $k \leftarrow k^{\text{target}}$ and use (16) to calculate the corresponding R value. Generate an R, k set as the solution to the problem. Stop.

A check is performed in Step 2 to determine if the minimum cost k value is greater than the target customer service level. If so, then the R, k set that yields the minimum cost also satisfies the target service level. Since the cost function is convex with respect to k for a given value of R, this set can be accepted as the final solution. Step 4 represents a case in which the minimum cost k does not satisfy the target service level. In such a case, the given target customer service level is the level that satisfies the target at minimum cost. As shown in <Appendix A>, the cost function is convex for wide range of input parameters. Thus the solution generated by the RK_Single algorithm is expected to locate close to the global optimum solution for most cases.

3. Multiple Item Problem

This section introduces a method for periodic coordinated replenishment of two or more types of items. Additional notations are as follows.

i : item type, $i = 1, 2, \cdots, I$,

A : major ordering cost for the family of items, in dollars,

a_i : minor ordering cost for item i, in dollars,

k_i : safety factor used for setting the safety stock level of item i,

D_i : demand rate of item i, in units per year,

σ_i : standard deviation of D_i, in units per year,
R : length of base cycle, in year,
L_i : lead time of item i, in year,
m_i : the integer multiple of R intervals in which item i is ordered.

The general assumptions of the JRP also apply to our problem. Thus, it is assumed that a major ordering cost of A is incurred when one or more items is ordered. Also, there is a minor ordering cost of σ_i when item type i is ordered. There is an order of at least one item type for every R interval, and item type i is included in an order for every $m_i R$ time interval. The relevant costs of the inventory control include ordering, holding, and shortage costs.

Traditional JRP has been studied extensively with assumption that item demand is independent with each other, which is also accepted for our multiple item problem.[1] More specifically, it is assumed that demand of item i is normally distributed with mean D_i and standard deviation σ_i and is independent of demands of other items. With this assumption, the total cost in Equation (21) can be derived using similar reasoning as in the single item case.

$$TC(R, \tilde{k}, \tilde{m}) = \frac{A}{R} + \frac{1}{R}\sum_{i=1}^{I}(a_i/m_i) + \sum_{i=1}^{I}[(D_i m_i R/2 + k_i \sigma_i \sqrt{m_i R + L_i})h_i + \frac{b_i}{m_i R}\sigma_i \sqrt{m_i R + L_i}\, G_u(k_i)]. \tag{21}$$

In (21), $\tilde{k} = (k_1, k_2, \cdots, k_I) \in R^n$, $\tilde{m} = (m_1, m_2, \cdots, m_I)$, and $G_u(k_i)$ is the unit normal loss function.

1) A few exceptional papers analyzed the JRP with correlated demands between items. For more details, please refer to Larsen [15].

Our problem is to find the R, $\tilde{k}$, $\tilde{m}$ set that minimizes the cost function under the constraint that the given target service rate of each item should be satisfied. Partial differentiation of the total cost function with respect to k_i yields the following relationship.

$$\frac{\partial TC}{\partial k_i}(R, \tilde{k}, \tilde{m}) = \sigma_i h_i \sqrt{m_i R + L_i} - \left(\frac{b_i}{m_i R}\sigma \sqrt{m_i R + L_i}\,(1 - F(k_i))\right), \tag{22}$$

where $F(k_i)$ is the cumulative distribution function of the standard normal distribution. Thus, $\frac{\partial TC}{\partial k_i}(R, \tilde{k}, \tilde{m}) = 0$ yields the following relationship between R, k_i, m_i values on a local optimum point.

$$1 - F(k_i) = (h_i/b_i)m_i R,\ \ i = 1, 2, \cdots, I. \tag{23}$$

Equation (23) can be rearranged to

$$R = (b_i/m_i h_i)(1 - F(k_i)),\ \ i = 1, 2, \cdots, I. \tag{24}$$

From (24), it is observed that the following relationship holds between m_i and k_i.

$$(b_1/m_1 h_1)(1 - F(k_1)) = (b_2/m_2 h_2)(1 - F(k_2)) = \cdots = (b_I/m_I h_I)(1 - F(k_I)). \tag{25}$$

Rearranging (25) yields

$$k_i = F^{-1}(1 - m_i h_i b_1 (1 - F(k_1)/m_1 h_1 b_i),\ \ i = 1, 2, \cdots, I. \tag{26}$$

Partial differentiation of the total cost function with respect to R yields the following relationship.

$$\frac{\partial TC}{\partial R}(R, \tilde{k}, \tilde{m}) = -\frac{A}{R^2} - \sum_{i=1}^{I}(a_i/m_i)/R^2 \tag{27}$$

$$+\sum_{i=1}^{I}\left[\frac{D_i m_i h_i}{2}+\frac{k_i\sigma_i h_i m_i}{2\sqrt{m_iR+L_i}} -b_i\sigma_i G_u(k_i)\left(\frac{m_iR+2L_i}{2m_iR^2\sqrt{m_iR+L_i}}\right)\right].$$

Thus, $\frac{\partial TC}{\partial R}(R,\tilde{k},\tilde{m})=0$ yields the following relationship between R, $\tilde{k}$, $\tilde{m}$ values satisfying the first order necessary condition of a local minimum.

$$\sum_{i=1}^{I}\left[\frac{D_i m_i h_i}{2}+\frac{k_i\sigma_i h_i m_i}{2\sqrt{m_iR+L_i}} -b_i\sigma_i G_u(k_i)\left(\frac{m_iR+2L_i}{2m_iR^2\sqrt{m_iR+L_i}}\right)\right] =\frac{A}{R^2}+\sum_{i=1}^{I}(a_i/m_i)/R^2. \tag{28}$$

Plugging (24) into (28) yields the equation

$$\sum_{i=1}^{I}\left[\frac{D_i m_i h_i}{2}+\frac{F^{-1}(1-m_ih_ib_1(1-F(k_1)/m_1h_1b_i)\sigma_ih_im_i}{2\sqrt{m_i(b_1/h_1)(1-F(k_1))+L_i}} -b_i\sigma_iG_u(F^{-1}(1-m_ih_ib_1(1-F(k_1))/m_1b_1b_i)) \left(\frac{m_i(b_1/h_1)(1-F(k_1))+2L_i}{2m_i((b_1/h_1)(1-F(k_1)))^2\sqrt{m_iR+L_i}}\right)\right] =\frac{A}{((b_1/h_1)(1-F(k_1)))^2}+\frac{\sum_{i=1}^{I}(a_i/m_i)}{((b_1/h_1)(1-F(k_1)))^2}. \tag{29}$$

Equation (29) can be numerically solved for k_1 if the $\tilde{m}$ is fixed at given values. It took less than one second for a program coded into Microsoft Excel to obtain a k_1 value that satisfied (29) during our experiment. Plugging the obtained k_1 value into (24) for $i=1$ yields the R value. The values for $k_2, k_3, \cdots, k_I$ can be found by plugging the k_1 value into (26). This procedure can generate R and $\tilde{k}$ values that minimize the total cost function when $\tilde{m}$ values are given.

Since it is difficult to find a method that can simultaneously determine the R, $\tilde{k}$, $\tilde{m}$ values, a sequential approach was chosen for our method. The $\tilde{m}$ values minimizing the cost function is first determined. Based on these $\tilde{m}$ values, R and $\tilde{k}$ values, which minimize the cost function, can be determined. Finally, an iterative search is conducted to improve the solution found by this sequential approach.

The method used to find the $\tilde{m}$ values minimizing the cost is based on the following ideas :

(1) Find a group of items that should be ordered at every cycle.
(2) Estimate a tentative cycle length.
(3) Based on the tentative cycle length, evaluate each item that is not in group (1) to find an integer multiple of the cycle length that is suitable to the item.

A method implementing these ideas is introduced below.

Algorithm for finding $\tilde{m}$ values (MI_Algorithm)

Step 1 : Using the RK_Single algorithm introduced in Section 2, find an optimal cycle length for each item at a given service target and save it in R_i.

Step 2 : Sort the R_i values in ascending order.

Step 3 : Let $\hat{j}$ be the smallest j that satisfies $R_j' < R_j$, where R_j' is the optimal cycle length when items 1 to j are replenished at the same time.

Step 4 : The items i such that $i \le \hat{j}$ have an m_i of 1. For item types $i=\hat{j}+1, \hat{j}+2, \cdots, I$ will have $m_i=q$, where q satisfies

$$\sqrt{(q-1)q} \le \frac{R_i}{R_{\hat{j}}'} \le \sqrt{(q+1)q}. \tag{30}$$

In Step 3, the optimal cycle length when the item types 1 to j are replenished at the same time (R_j') is determined by solving (29) for k_1, after substituting 1 for m_1 to m_j. This k_1 value is then plugged into (24) to yield a cost minimizing cycle length for the joint replenishment of items 1 to j, i.e., R_j'. Equation (30) is based on the logic used by Eynan and Kropp [3].

After calculating the $\tilde{m}$ values using the MI_Algorithm, it is possible to determine the cost minimizing values of R and $\tilde{k}$. However, it is still possible that this R, $\tilde{k}$, $\tilde{m}$ set will not satisfy the given target service rates. Using the following algorithm, however, we can find, for preset $\tilde{m}$, the R, $\tilde{k}$ set that satisfies the given target service rate of each item type at minimum total cost.

Multiple Item (Product) Algorithm (MP_Algorithm)

Step 1 : Run MI_Algorithm to obtain $\tilde{m}$ values.

Step 2 : Solve (29) for k_1 using the $\tilde{m}$ values obtained in Step 1.

Step 3 : Plug k_1 from Step 2 into (26) to get $k_2, k_3, \cdots, k_I$.

Step 4 : If $k_i \geq k_i^{\text{target}}$ for all i, go to Step 5; otherwise, go to Step 6.

Step 5 : Plug k_1 into (24) to obtain R. Report R, k_i, m_i as the solution. Stop.

Step 6 : Let $i^* = \arg\max_i \left(k_i^{\text{target}} - k_i \mid k_i^{\text{target}} - k_i > 0\right)$. Reset $k_{i^*} \leftarrow k_{i^*}^{\text{target}}$.

Step 7 : Plug k_{i^*} from Step 6 into (24) to obtain a new cycle length R.

Step 8 : Plug R from Step 7 into (23) to obtain new k_i values for all $i (i \neq i^*)$.

Step 9 : If $k_i \geq k_i^{\text{target}}$ for all i, report R, k_i, m_i as the solution and stop; otherwise, go to Step 6.

In Step 4, if $k_i \geq k_i^{\text{target}}$ for all i, then the current k_1 values are the safety factors that satisfy the given target safety rates at the minimum cost. Otherwise, the item with the largest deviation from the given service rate is chosen in Step 6, and its k_1 value is replaced by its target value. This k_1 value change requires the current cycle length to be adjusted to make the cycle feasible. A new feasible cycle length is calculated in Step 7, and, based on this new cycle, the safety factors of all other items are updated in Step 8. During the iterative search of the MP_Algorithm, $\tilde{m}$ values are fixed to the one obtained by the MI_Algorithm.

Since the cost function in (21) is not a convex function, solution generated by the MP_Algorithm may not be a local minimum. In this regard, a computational experiment to estimate how close the solution by the MP_Algorithm locates to the global minimum solution will be performed in the next section.

4. Computational Experiments

In this section, the accuracy of our method is tested in a variety of settings. The experiments were conducted on a computer with a Microsoft XP operating system, a 2.0 MHz CPU, and 2 GB of RAM. The algorithms were programmed into Microsoft Excel. A complete enumeration was used to obtain the global optimal solution. A percent deviation of the total cost was used as a measure of performance; it is defined as :

$$\text{Percent deviation} = 100(TC - TC^*)/TC^*, \quad (31)$$

where TC denotes the total cost of the MP_Algorithm and TC^* is the one found by the complete enumeration.

The data in <Table 1>, which was used in Eynan and Kropp [3], was used for our initial test, which was conducted with the minimum target values set to zero. This setting was necessary to produce the testing conditions used in Eynan and Kropp [3], which solved a JRP without the minimum service target constraint.

The output generated from the MP_Algorithm is summarized in <Table 2>, showing that the total cost of the MP_Algorithm's solution is equal to the cost of the global optimum solution. Thus, we conclude that our algorithm found the global optimum solution for the first problem. The computational time required by the MP_Algorithm was 0.41 CPU second and the complete enumeration took 3044.47 CPU seconds.

The second test was conducted for a problem set that was prepared using the input parameters randomly generated from probability distributions. The probability distributions for generating input parameters and other relevant data are summarized in <Table 3>. The problems with three different problem sizes of 4, 6, and 8 items were prepared for the test. For each problem size, ten different problems were solved using the MP_Algorithm. The complete enumeration was also performed to determine the global optimum.

The output summarized in <Table 4> shows that the MP_Algorithm successfully found the global optimum solutions in all 30 problems. Each number in <Table 4> was rounded in fourth digit. The reason why our algorithm never failed to find the global optimal was analyzed by examining contour graphs of the total cost function.

As shown in [Figure 1], the cost function looks like a convex function of two variables select for each axis. Thus, a probable cause is that there

<Table 1> Input Data for the First Test

item number	minor ordering cost	holding cost	demand per year	standard deviation of per year demand	lead time	per unit shortage cost
1	1.8	0.4	2,900	500	0.05	0.8
2	2.0	1.0	1,850	500	0.05	2.0
3	1.2	0.8	2,750	500	0.05	1.6
4	3.2	0.2	1,600	500	0.05	0.4
5	3.1	0.8	3,200	500	0.05	1.6
6	2.7	0.2	1,400	500	0.05	0.4

<Table 2> Result of the First Test

item number	MP_Algorithm				global optimum			
	m_i	k_i	cycle length	total cost	m_i	k_i	cycle length	total cost
1	1	1.915	0.0550	1,909.86	1	1.915	0.0550	1,909.86
2	1	1.915			1	1.915		
3	1	1.915			1	1.915		
4	2	1.594			2	1.594		
5	1	1.915			1	1.915		
6	2	1.594			2	1.594		

〈Table 3〉 Input Data for the Second Test

major ordering cost	minor ordering cost	holding cost	demand rate per year	standard deviation of per year demand	lead time	per unit shortage cost
U^*[10, 30]	U[1, 5]	U[5, 10]	U[1000, 5000]	U[50, 250]	U[0.01, 0.1]	U[10, 50]

Note) * Uniform distribution

〈Table 4〉 Output of the Second Test

problem size	percent deviation				number of global optimumfound / number of trial
	average	maximum	minimum	standard deviation	
4	0.000	0.000	0.000	0.000	10/10
6	0.000	0.000	0.000	0.000	10/10
8	0.000	0.000	0.000	0.000	10/10

〈Table 5〉 Result for the Bigger Sized Problems

problem size	percent deviation				average CPU second
	average	maximum	minimum	standard deviation	
10	-2.869	-4.091	-0.998	0.826	0.68
20	-5.274	-6.882	-3.165	1.017	1.29
30	-7.715	-9.352	-5.263	1.332	2.04

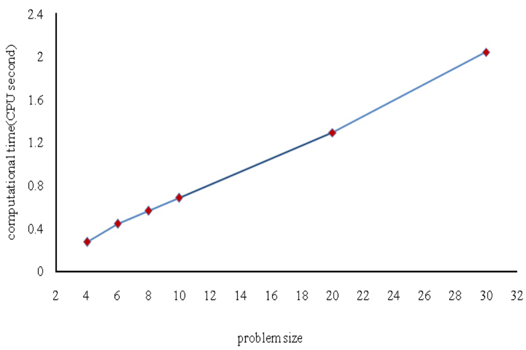
exists only one k_1 value satisfying the necessary condition of a local optimum in (29) in the region where k_1 value is less than 4.0. As a result, the single k_1 value found by (29) leads to the best solution in that region, which is the global optimum solution for our problem.

The average computational time required by the MP_Algorithm was 0.27, 0.44 and 0.59 CPU second for 4, 6 and 8 sized problems. By the complete enumeration, it took 3386.72, 4811.54 and 6842.29 CPU seconds to find the global optimum.

A test for realistic bigger sized problems was also carried out. For the test, 10, 20 and 30 item problem sets were generated from the probability distributions explained in <Table 3>. Since it was not possible to get the global optimum solution by the enumeration due to excessive computational time, the best solution found by the enumeration in two hours limit was used for a comparison purpose. Performance measure was the percent deviation in (31) used previously. Ten problems were solved for each problem size to get an averaged performance measure.

The result shown in <Table 5> reveals that the MP_Algorithm's solution is 2.869, 5.274, 7.715 percent better than the one found by the enumeration in two hour limit. It took on average 0.68, 1.29, 2.04 CPU seconds to get the solution for each problem size by our algorithm. As shown in [Figure 3], the trend of computational time for growing problem size looks to be linear. This is a desired property for real application, where a larger size problem should be routinely solved.

From the results found during the experiments, it was concluded that our algorithms proved their accuracy by finding the global optimum for all tested problems. They also showed their efficiency by finding a good solution in less than 3.0 CPU seconds for more realistic larger sized problems.



[Figure 3] Trend of Computational Time

5. Conclusion

This paper analyzes a stochastic joint replenishment problem with multiple type of items. A mathematical model was developed from a buyer's perspective. Using a condition of a local optimum solution, an algorithm to determine the cycle length and safety factor that satisfy a given target service rate at the minimum cost was proposed.

A multiple item model was constructed by extending the single item model and conditions for a local optimum was analyzed. Based on the findings, three kinds of algorithms were developed to determine the base cycle length and the multiplier and safety factor of each item that satisfy each item's target service rate at the minimum total expected cost.

The results of the computational experiment showed that the algorithms were able to find the global optimum solution for all tested cases within a reasonable amount of time. By exploiting the results and insights obtained from this research, buyers faced with a stochastic JRP will be able to more effectively handle the problem and will be able to identify the most profitable means of joint replenishment.

Further research may be necessary to incorporate multiple suppliers into the current problem, so that it becomes a multi-item, multi-supplier joint replenishment problem. Also, analysis of a problem with correlated demands between items seems to be necessary.

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〈Appendix A〉 Convexity of the Cost Function in (12)

Hessian matrix of the buyer's expected cost function in (12) is

$$\begin{bmatrix} \dfrac{8A(R+L)\sqrt{R+L}-\sigma khR^3+\sigma bG_u(k)(3R^2+12RL+8L^2)}{4R^3(R+L)\sqrt{R+L}} & \dfrac{h\sigma R^2+b\sigma(1-F(k))(R+2L)}{2R^2\sqrt{R+L}} \\ \dfrac{h\sigma R^2+b\sigma(1-F(k))(R+2L)}{2R^2\sqrt{R+L}} & \dfrac{b}{R}\sigma\sqrt{R+L}\,f(k) \end{bmatrix}.$$

All the elements except (1, 1)th element of the Hessian matrix are nonnegative. Thus, if the (1, 1)th element is nonnegative, which holds for many cases with parameter values in reasonable ranges, the buyer's cost function is convex.

〈Appendix B〉 Convexity of the Cost Function in (21)

Taking the second-order partial derivative of (21) with respect to k_i, we obtain

$$\frac{\partial TC}{\partial k_i}(R,\tilde{k},\tilde{m}) = \sigma_i\sqrt{m_iR+L_i}\,h_i - \frac{b_i}{m_iR}\sigma_i\sqrt{m_iR+L_i}\,(1-F(k_i)),$$

$$\frac{\partial^2 TC}{\partial k_i^2}(R,\tilde{k},\tilde{m}) = \frac{b_i}{m_iR}\sigma_i\sqrt{m_iR+L_i}\,f(k_i) \ge 0.$$

Thus, the total cost function, $TC(R,\tilde{k},\tilde{m})$, is a convex function with respect to k_i for given values of other parameters.