ON OPIAL INEQUALITIES INVOLVING HIGHER ORDER DERIVATIVES

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Abstract. In the present paper we establish some new Opial-type inequalities involving higher order partial derivatives. The results in special cases yield some of the recent results on Opial's inequality and provide new estimates on inequalities of this type.

1. Introduction

In the year 1960, Opial [21] established the following inequality:

Theorem A. *Suppose $f \in C^1[0, a]$ satisfies $f(0) = f(a) = 0$ and $f(x) > 0$ for all $x \in (0, a)$. Then the inequality holds*

$$(1.1) \qquad \int_0^a |f(x)f'(x)| \, dx \le \frac{a}{4} \int_0^a (f'(x))^2 dx,$$

where this constant $a/4$ is best possible.

Opial's inequality and its generalizations, extensions and discretizations play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations [2, 3, 7, 18, 20]. The inequality (1.1) has received considerable attention and a large number of papers dealing with new proofs, extensions, generalizations, variants and discrete analogues of Opial's inequality have appeared in the literature [9–13, 15, 19, 22–29, 31].

The first natural extension of Opial's inequality (1.1) involving $x^{(n)}(t), n \ge 1$ instead of $x'(t)$ is embodied in the following:

Theorem B ([30]). *Let $x(t) \in C^n[0, a]$ be such that $x^{(i)}(0) = 0, 0 \le i \le n-1$. Then the inequality holds*

$$(1.2) \qquad \int_0^a |x(t)x^{(n)}(t)| dt \le \frac{1}{2} a^n \int_0^a |x^{(n)}(t)|^2 dt.$$

A sharper version of (1.2) is the following:

Theorem C ([14]). *Let $x(t) \in C^{n-1}[0,a]$ be such that $x^{(i)}(0) = 0, 0 \leq i \leq n-1$. Further, let $x^{(n-1)}(t)$ be absolutely continuous, and $\int_0^n |x^{(n)}(t)|^2 dt < \infty$. Then the inequality holds*

$$(1.3) \qquad \int_0^a |x(t)x^{(n)}(t)| dt \leq c_n a^n \int_0^a |x^{(n)}(t)|^2 dt,$$

with equality if and only if $n = 1$ and $x^{(n)}(t) = c$, where $c_n = \frac{1}{2n!} \left(\frac{n}{2n-1} \right)^{\frac{1}{2}}$.

A result involving two functions and their higher order derivatives is embodied in the following:

Theorem D ([23]). *For $j = 1, 2$, let $x_j(t) \in C^{n-1}[0,a]$ be such that $x_j^{(i)}(0) = 0, 0 \leq i \leq n-1$. Further, let $x_j^{(n-1)}(t)$ be absolutely continuous, and*

$$\int_0^n |x_j^{(n)}(t)|^2 dt < \infty.$$

Then the inequality holds

$$(1.4)$$
$$\int_0^a \left(\left| x_1(t) \cdot x_2^{(n)}(t) \right| + \left| x_2(t) \cdot x_1^{(n)}(t) \right| \right) dt \leq c_n a^n \int_0^a \left(\left| x_1^{(n)}(t) \right|^2 + \left| x_2^{(n)}(t) \right|^2 \right) dt,$$

with equality if and only if $n = 1, x_j^{(n)}(t) = c$, where the constant c_n is defined in Theorem C.

For Opial type integral inequalities involving high-order partial derivatives see [1,4,6,17,33]. For an extensive survey on these inequalities, see [3]. The first aim of this paper is to establish the following Opial-type inequality involving two functions and their higher order partial derivatives based on applications of improvements of Das [14] and Pachpatte's ideas [23].

Theorem 1.1. *Let a, b, s, t, σ, τ be real numbers, $0 \leq \sigma \leq s \leq a$ and $0 \leq \tau \leq t \leq b$. For $j = 1, 2, n \geq 1, k \geq 1$, let $x_j(s,t) \in C^{(n-1)}[0,a] \times C^{(k-1)}[0,b]$ be such that $\frac{\partial^i}{\partial \sigma^i} x_j(0,\tau) = 0, \frac{\partial^{i'}}{\partial \tau^{i'}} x_j(\sigma,0) = 0, \sigma \in [0,s], \tau \in [0,t], 0 \leq i \leq n-1, 0 \leq i' \leq k-1$, and $x_j^{(n,k)}(s,\tau) = 0, x_j^{(n,k)}(\sigma,t) = 0$. Further, let $x_j^{(n-1,k-1)}(s,t)$ be absolutely continuous, and $\int_0^a \int_0^b \left| x_j^{(n,k)}(s,t) \right|^2 ds dt < \infty$. Then*

$$\int_0^a \int_0^b \left(\left| x_1(s,t) \cdot x_2^{(n,k)}(s,t) \right| + \left| x_2(s,t) \cdot x_1^{(n,k)}(s,t) \right| \right) ds dt$$

$$(1.5) \qquad \leq c_{n,k} a^n b^k \int_0^a \int_0^b \left(\left| x_1^{(n,k)}(s,t) \right|^2 + \left| x_2^{(n,k)}(s,t) \right|^2 \right) ds dt,$$

with equality if and only if $n = k = 1, x_j^{(n,k)}(s,t) = c$, *where*

$$x_j^{(n,k)}(s,t) = \frac{\partial^n}{\partial s^n}\left(\frac{\partial^k}{\partial t^k}x_j(s,t)\right),$$

and

$$c_{n,k} = \frac{1}{4n!k!}\left(\frac{2nk}{(2n-1)(2k-1)}\right)^{\frac{1}{2}}.$$

We also prove an interesting Opial-type inequality involving two functions and their higher order partial derivatives as follows:

Theorem 1.2. *Let* a,b,s,t,σ,τ *be real numbers,* $0 \le \sigma \le s \le a$ *and* $0 \le \tau \le t \le b$. *Let* $l > 0, 0 < m < 1$ *and satisfying* $l + m > 1$. *For* $j = 1, 2, n \ge 1, k \ge 1$, *let* $x_j(s,t) \in C^{(n-1)}[0,a] \times C^{(k-1)}[0,b]$ *be such that* $\frac{\partial^i}{\partial\sigma^i}x_j(0,\tau) = 0, \frac{\partial^{i'}}{\partial\tau^{i'}}x_j(\sigma,0) = 0$, $\sigma \in [0,s], \tau \in [0,t]$, $0 \le i \le n-1, 0 \le i' \le k-1$. *Further, let* $x_j^{(n-1,k-1)}(s,t)$ *be absolutely continuous, and* $\int_0^a \int_0^b \left|x_j^{(n,k)}(s,t)\right|^{l+m} dsdt < \infty$. *Then*

(1.6)

$$\int_0^a \int_0^b \left[|x_1(s,t)|^l|x_2^{(n,k)}(s,t)|^m + |x_2(s,t)|^l|x_1^{(n,k)}(s,t)|^m\right] dsdt$$

$$\le c_{n,k}^* a^{nl}b^{kl}\left[\left(\int_0^a \int_0^b \left|x_1^{(n,k)}(s,t)\right|^{l+m} dsdt\right)^{\frac{l}{m}} \cdot \int_0^a \int_0^b \left|x_2^{(n,k)}(s,t)\right|^{m+l} dsdt\right.$$

$$\left. + \left(\int_0^a \int_0^b \left|x_2^{(n,k)}(s,t)\right|^{l+m} dsdt\right)^{\frac{l}{m}} \cdot \int_0^a \int_0^b \left|x_1^{(n,k)}(s,t)\right|^{m+l} dsdt\right]^{\frac{m}{l+m}},$$

where

$$c_{n,k}^* = \left(\frac{nk(1-\xi)^2}{(n-\xi)(k-\xi)}\right)^{l(1-\xi)} \frac{2^{\xi l}\xi^{2\xi l}}{(n!k!)^l}, \ \xi = \frac{1}{l+m}.$$

2. Main results and their proofs

Our main results are given in the following theorems.

Theorem 2.1. *Let* a,b,s,t,σ,τ *be real numbers,* $0 \le \sigma \le s \le a$ *and* $0 \le \tau \le t \le b$. *For* $j = 1, 2, n \ge 1, k \ge 1$, *let* $x_j(s,t) \in C^{(n-1)}[0,a] \times C^{(k-1)}[0,b]$ *be such that* $\frac{\partial^i}{\partial\sigma^i}x_j(0,\tau) = 0, \frac{\partial^{i'}}{\partial\tau^{i'}}x_j(\sigma,0) = 0$, $\sigma \in [0,s], \tau \in [0,t]$, $0 \le i \le n-1, 0 \le i' \le k-1$, *and* $x_j^{(n,k)}(s,\tau) = 0$, $x_j^{(n,k)}(\sigma,t) = 0$. *Further, let* $x_j^{(n-1,k-1)}(s,t)$ *be*

absolutely continuous, and $\int_0^a \int_0^b \left| x_j^{(n,k)}(s,t) \right|^2 dsdt < \infty$. *Then*

(2.1)
$$\int_0^a \int_0^b \left(\left| x_1(s,t) \cdot x_2^{(n,k)}(s,t) \right| + \left| x_2(s,t) \cdot x_1^{(n,k)}(s,t) \right| \right) dsdt$$
$$\leq c_{n,k} a^n b^k \int_0^a \int_0^b \left(\left| x_1^{(n,k)}(s,t) \right|^2 + \left| x_2^{(n,k)}(s,t) \right|^2 \right) dsdt,$$

with equality if and only if $n = k = 1, x_j^{(n,k)}(s,t) = c$, *where* $x_j^{(n,k)}(s,t)$ *and* $c_{n,k}$ *are defined in Theorem 1.1, respectively.*

Proof. For σ, integration by parts $(n-1)$-times in right side of the following first formula and in view of $\frac{\partial^i}{\partial \sigma^i} x(0,\tau) = 0$, $\frac{\partial^{i'}}{\partial \tau^{i'}} x(\sigma,0) = 0$, $0 \leq i \leq n-1, 0 \leq i' \leq k-1$, we have

(2.2) $x_1(s,t)$

$$= \frac{(-1)^n}{(n-1)!} \int_s^0 (\sigma-s)^{n-1} \frac{\partial^n}{\partial \sigma^n} x_1(\sigma,t) d\sigma$$

$$= \frac{(-1)^{2n-1}}{(n-1)!} \int_s^0 (s-\sigma)^{n-1} \frac{\partial^n}{\partial \sigma^n} x_1(\sigma,t) d\sigma$$

$$= \frac{1}{(n-1)!} \int_0^s (s-\sigma)^{n-1} \frac{\partial^n}{\partial \sigma^n} x_1(\sigma,t) d\sigma$$

$$= \frac{1}{(n-1)!(k-1)!} \int_0^s (s-\sigma)^{n-1} \frac{\partial^n}{\partial \sigma^n} \left(\int_0^t (t-\tau)^{k-1} \frac{\partial^k}{\partial \tau^k} x_1(\sigma,\tau) d\tau \right) d\sigma$$

$$= \frac{1}{(n-1)!(k-1)!} \int_0^s \int_0^t (s-\sigma)^{n-1}(t-\tau)^{k-1} \cdot x_1^{(n,k)}(\sigma,\tau) d\sigma d\tau.$$

Multiplying both sides of (2.2) by $x_2^{(n,k)}(s,t)$ and using the Cauchy-Schwarz inequality, we have

$$\left| x_1(s,t) \cdot x_2^{(n,k)}(s,t) \right|$$

$$\leq \frac{\left| x_2^{(n,k)}(s,t) \right|}{(n-1)!(k-1)!} \left(\int_0^s \int_0^t (s-\sigma)^{2n-2}(t-\tau)^{2k-2} d\sigma d\tau \right)^{\frac{1}{2}}$$

(2.3) $$\times \left(\int_0^s \int_0^t \left| x_1^{(n,k)}(\sigma,\tau) \right|^2 d\sigma d\tau \right)^{\frac{1}{2}}$$

$$= \frac{1}{(n-1)!(k-1)!\sqrt{(2n-1)(2k-1)}} \cdot s^{n-\frac{1}{2}} t^{k-\frac{1}{2}} \left| x_2^{(n,k)}(s,t) \right|$$

$$\times \left(\int_0^s \int_0^t \left| x_1^{(n,k)}(\sigma,\tau) \right|^2 d\sigma d\tau \right)^{\frac{1}{2}}.$$

Thus, integrating both sides of (2.3) over t from 0 to b first and then integrating the resulting inequality over s from 0 to a and applying the Cauchy-Schwarz

inequality again, we obtain

$$\int_0^a \int_0^b \left| x_1(s,t) \cdot x_2^{(n,k)}(s,t) \right| ds dt$$

$$\leq \frac{1}{(n-1)!(k-1)!\sqrt{(2n-1)(2k-1)}} \left(\int_0^a \int_0^b s^{2n-1} t^{2k-1} ds dt \right)^{\frac{1}{2}}$$

$$\times \left(\int_0^a \int_0^b \left| x_2^{(n,k)}(s,t) \right|^2 \left(\int_0^s \int_0^t \left| x_1^{(n,k)}(\sigma,\tau) \right|^2 d\sigma d\tau \right) ds dt \right)^{\frac{1}{2}}$$

$$= \frac{1}{2n!} \left(\frac{2n}{2n-1} \right)^{\frac{1}{2}} \frac{1}{2k!} \left(\frac{2k}{2k-1} \right)^{\frac{1}{2}} a^n b^k$$

$$(2.4) \qquad \times \left(\int_0^a \int_0^b \left| x_2^{(n,k)}(s,t) \right|^2 \left(\int_0^s \int_0^t \left| x_1^{(n,k)}(\sigma,\tau) \right|^2 d\sigma d\tau \right) ds dt \right)^{\frac{1}{2}}.$$

Similarly, we obtain

$$\int_0^a \int_0^b \left| x_2(s,t) \cdot x_1^{(n,k)}(s,t) \right| ds dt$$

$$\leq \frac{1}{2n!} \left(\frac{2n}{2n-1} \right)^{\frac{1}{2}} \frac{1}{2k!} \left(\frac{2k}{2k-1} \right)^{\frac{1}{2}} a^n b^k$$

$$(2.5) \qquad \times \left(\int_0^a \int_0^b \left| x_1^{(n,k)}(s,t) \right|^2 \left(\int_0^s \int_0^t \left| x_2^{(n,k)}(\sigma,\tau) \right|^2 d\sigma d\tau \right) ds dt \right)^{\frac{1}{2}}.$$

Thus, from (2.4), (2.5) and in view of the elementary inequality $a^{\frac{1}{2}} + b^{\frac{1}{2}} \leq [2(a+b)]^{\frac{1}{2}}$, $a,b \geq 0$, we obtain

$$\int_0^a \int_0^b \left(\left| x_1(s,t) \cdot x_2^{(n,k)}(s,t) \right| + \left| x_2(s,t) \cdot x_1^{(n,k)}(s,t) \right| \right) ds dt$$

$$\leq 2c_{n,k} a^n b^k \left[\int_0^a \int_0^b \left\{ \left| x_2^{(n,k)}(s,t) \right|^2 \left(\int_0^s \int_0^t \left| x_1^{(n,k)}(\sigma,\tau) \right|^2 d\sigma d\tau \right) \right. \right.$$

$$(2.6) \qquad \left. \left. + \left| x_1^{(n,k)}(s,t) \right|^2 \left(\int_0^s \int_0^t \left| x_2^{(n,k)}(\sigma,\tau) \right|^2 d\sigma d\tau \right) \right\} ds dt \right]^{\frac{1}{2}}.$$

On the other hand

$$\frac{\partial^2}{\partial s \partial t} \left[\left(\int_0^s \int_0^t \left| x_1^{(n,k)}(\sigma,\tau) \right|^2 d\sigma d\tau \right) \left(\int_0^s \int_0^t \left| x_2^{(n,k)}(\sigma,\tau) \right|^2 d\sigma d\tau \right) \right]$$

$$= \frac{\partial}{\partial s} \left[\int_0^s |x_1(\sigma,t)|^2 d\sigma \cdot \int_0^s \int_0^t \left| x_2^{(n,k)}(\sigma,\tau) \right|^2 d\sigma d\tau \right.$$

$$\left. + \int_0^s \int_0^t \left| x_1^{(n,k)}(\sigma,\tau) \right|^2 d\sigma d\tau \cdot \int_0^s |x_2(\sigma,t)|^2 d\sigma \right]$$

$$= \left|x_1^{(n,k)}(s,t)\right|^2 \cdot \int_0^s \int_0^t \left|x_2^{(n,k)}(\sigma,\tau)\right|^2 d\sigma d\tau$$

$$+ \int_0^s |x_1(\sigma,t)|^2 d\sigma \cdot \int_0^t |x_2(s,\tau)|^2 d\tau$$

$$+ \left|x_2^{(n,k)}(s,t)\right|^2 \cdot \int_0^s \int_0^t \left|x_1^{(n,k)}(\sigma,\tau)\right|^2 d\sigma d\tau$$

$$+ \int_0^s |x_2(\sigma,t)|^2 d\sigma \cdot \int_0^t |x_1(s,\tau)|^2 d\tau$$

$$= \left|x_1^{(n,k)}(s,t)\right|^2 \cdot \int_0^s \int_0^t \left|x_2^{(n,k)}(\sigma,\tau)\right|^2 d\sigma d\tau$$

$$(2.7) \qquad + \left|x_2^{(n,k)}(s,t)\right|^2 \cdot \int_0^s \int_0^t \left|x_1^{(n,k)}(\sigma,\tau)\right|^2 d\sigma d\tau.$$

From (2.6), (2.7) and in view of the elementary inequality $2(ab)^{\frac{1}{2}} \le a+b$, $a,b \ge 0$, we obtain,

$$\int_0^a \int_0^b \left(\left|x_1(s,t) \cdot x_2^{(n,k)}(s,t)\right| + \left|x_2(s,t) \cdot x_1^{(n,k)}(s,t)\right| \right) dsdt$$

$$\le 2c_{n,k}a^n b^k \left[\int_0^a \int_0^b \frac{\partial^2}{\partial s \partial t} \left\{ \left(\int_0^s \int_0^t \left|x_1^{(n,k)}(\sigma,\tau)\right|^2 d\sigma d\tau \right) \right. \right.$$

$$\left. \left. \times \left(\int_0^s \int_0^t \left|x_2^{(n,k)}(\sigma,\tau)\right|^2 d\sigma d\tau \right) \right\} dsdt \right]^{\frac{1}{2}}$$

$$= 2c_{n,k}a^n b^k \left[\int_0^a \int_0^b \left|x_1^{(n,k)}(s,t)\right|^2 dsdt \cdot \int_0^a \int_0^b \left|x_2^{(n,k)}(s,t)\right|^2 dsdt \right]^{\frac{1}{2}}$$

$$\le c_{n,k}a^n b^k \int_0^a \int_0^b \left(\left|x_1^{(n,k)}(s,t)\right|^2 + \left|x_2^{(n,k)}(s,t)\right|^2 \right) dsdt. \qquad \square$$

Remark 2.2. Let $x(s,t)$ reduce to $x(t)$ by letting $s = s(t)$ and with suitable modifications in all intermediate steps of proof of Theorem 2.1, then Theorem 2.1 becomes Theorem D stated in the introduction which was established by Pachpatte [23].

Remark 2.3. Taking for $x_1(s,t) = x_2(s,t) = x(s,t)$ in (2.1), we have

$$(2.8) \quad \int_0^a \int_0^b \left|x(s,t) \cdot x^{(n,k)}(s,t)\right| dsdt \le c_{n,k}a^n b^k \int_0^a \int_0^b \left|x^{(n,k)}(s,t)\right|^2 dsdt,$$

with equality if and only if $n = 1, x^{(n)}(s,t) = c$.

Let $x(s,t)$ reduce to $x(t)$ by letting $s = s(t)$ and with suitable modifications, then (2.8) becomes the following inequality:

$$\int_0^a \left| x(t) \cdot x^{(n)}(t) \right| dt \le c_n a^n \int_0^a \left| x^{(n)}(t) \right|^2 dt,$$

with equality if and only if $n = 1, x^{(n)}(t) = c$, where the constant c_n is defined in (1.3).

This is just an inequality in Theorem C stated in the introduction established by Das [14].

Remark 2.4. Let $0 \le \alpha, \beta < n$, for $j = 1,2$ let $g_j(s,t) \in C^{(n-\alpha-1)}[0,a] \times C^{(k-\beta-1)}[0,b]$ be such that $\frac{\partial^i}{\partial s^i} g_j(0,t) = \frac{\partial^{i'}}{\partial t^{i'}} g_j(s,0) = 0, 0 \le i \le n-\alpha-1, 0 \le i' \le k-\beta-1$ and suppose that $g^{(n-\alpha-1,k-\beta-1)}(s,t)$ are absolutely continuous, and $\int_0^a \int_0^b \left| g_j^{(n-\alpha,k-\beta)}(s,t) \right|^2 dsdt < \infty$, then from (2.1) it follows that

$$\int_0^a \int_0^b \left[\left| g_1(s,t) \cdot g_2^{(n-\alpha,k-\beta)}(s,t) \right| + \left| g_1^{(n-\alpha,k-\beta)}(s,t) \cdot g_2(s,t) \right| \right] dsdt$$

$$\le c_{n-\alpha,k-\beta} a^{n-\alpha} b^{k-\beta} \int_0^a \int_0^b \left[\left| g_1^{(n-\alpha,k-\beta)}(s,t) \right|^2 + \left| g_2^{(n-\alpha,k-\beta)}(s,t) \right|^2 \right] dsdt.$$

Thus, for $g_j(s,t) = x_j^{(\alpha,\beta)}(s,t)$, where $x_j(s,t) \in C^{(n-1)}[0,a] \times C^{(k-1)}[0,b]$, $\frac{\partial^i}{\partial s^i} x(0,t) = 0, \frac{\partial^{i'}}{\partial t^{i'}} x(s,0) = 0, \alpha \le i \le n-1, \beta \le i' \le k-1$, and $x_j^{(n-1,k-1)}(s,t)$ are absolutely continuous, and $\int_0^a \int_0^b \left| x_j^{(n,k)}(s,t) \right|^2 dsdt < \infty$, then

$$\int_0^a \int_0^b \left[\left| x_1^{(\alpha,\beta)}(s,t) \cdot x_2^{(n,k)}(s,t) \right| + \left| x_1^{(n,k)}(s,t) \cdot x_2^{(\alpha,\beta)}(s,t) \right| \right] dsdt$$

$$(2.9) \quad \le c_{n-\alpha,k-\beta} a^{n-\alpha} b^{k-\beta} \int_0^a \int_0^b \left[\left| x_1^{(n,k)}(s,t) \right|^2 + \left| x_2^{(n,k)}(s,t) \right|^2 \right] dsdt.$$

Obviously, a special case of (2.9) is following inequality:

$$\int_0^a \int_0^b \left[\left| x_1^{(k,k)}(s,t) \cdot x_2^{(n,n)}(s,t) \right| + \left| x_1^{(n,n)}(s,t) \cdot x_2^{(k,k)}(s,t) \right| \right] dsdt$$

$$(2.10) \quad \le c_{n-k,n-k}(ab)^{n-k} \int_0^a \int_0^b \left[\left| x_1^{(n,n)}(s,t) \right|^2 + \left| x_2^{(n,n)}(s,t) \right|^2 \right] dsdt.$$

Let $x_j(s,t)$ reduce to $s_j(t)$ and with suitable modifications, then (2.10) becomes the following inequality:

$$\int_0^a \left[|x_1^{(k)}(t) x_2^{(n)}(t)| + |x_1^{(n)}(t) x_2^{(k)}(t)| \right] dt$$

$$\le c_{n-k} a^{n-k} \int_0^a \left[|x_1^{(n)}(t)|^2 + |x_1^{(n)}(t)|^2 \right] dt,$$

where the constant c_{n-k} is defined in (1.3). This is just an new inequality established by Pachpatte [24].

Theorem 2.5. *Let a, b, s, t, σ, τ be real numbers, $0 \le \sigma \le s \le a$ and $0 \le \tau \le t \le b$. Let $l > 0, 0 < m < 1$ and satisfying $l + m > 1$. For $j = 1, 2, n \ge 1, k \ge 1$, let $x_j(s, t) \in C^{(n-1)}[0, a] \times C^{(k-1)}[0, b]$ be such that $\frac{\partial^i}{\partial \sigma^i} x_j(0, \tau) = 0, \frac{\partial^{i'}}{\partial \tau^{i'}} x_j(\sigma, 0) = 0, \sigma \in [0, s], \tau \in [0, t], 0 \le i \le n-1, 0 \le i' \le k-1$. Further, let $x_j^{(n-1,k-1)}(s, t)$ be absolutely continuous, and $\int_0^a \int_0^b \left| x_j^{(n,k)}(s, t) \right|^{l+m} dsdt < \infty$. Then*

(2.11)

$$\int_0^a \int_0^b \left[|x_1(s,t)|^l |x_2^{(n,k)}(s,t)|^m + |x_2(s,t)|^l |x_1^{(n,k)}(s,t)|^m \right] dsdt$$

$$\le c_{n,k}^* a^{nl} b^{kl} \left[\left(\int_0^a \int_0^b \left| x_1^{(n,k)}(s,t) \right|^{l+m} dsdt \right)^{\frac{l}{m}} \cdot \int_0^a \int_0^b \left| x_2^{(n,k)}(s,t) \right|^{m+l} dsdt \right.$$

$$\left. + \left(\int_0^a \int_0^b \left| x_2^{(n,k)}(s,t) \right|^{l+m} dsdt \right)^{\frac{l}{m}} \cdot \int_0^a \int_0^b \left| x_1^{(n,k)}(s,t) \right|^{m+l} dsdt \right]^{\frac{m}{l+m}},$$

where $x_j^{(n,k)}(s, t)$ is defined in Theorem 1.1, and $c_{n,k}^*$ is defined in Theorem 1.2.

Proof. From (2.2), we have

$$|x_1(s,t)| \le \frac{1}{(n-1)!(k-1)!} \int_0^s \int_0^t (s-\sigma)^{n-1} (t-\tau)^{k-1} \left| x_1^{(n,k)}(\sigma, \tau) \right| d\sigma d\tau,$$

by Hölder's inequality with indices $l + m$ and $\frac{l+m}{l+m-1}$, it follows that

$$|x_1(s,t)| \le \frac{1}{(n-1)!(k-1)!} \left(\int_0^s \int_0^t [(s-\sigma)^{n-1}(t-\tau)^{k-1}]^{\frac{l+m}{l+m-1}} d\sigma d\tau \right)^{\frac{l+m-1}{l+m}}$$

$$\times \left(\int_0^s \int_0^t \left| x_1^{(n,k)}(\sigma, \tau) \right|^{l+m} d\sigma d\tau \right)^{\frac{1}{l+m}}$$

$$= A s^{n-\xi} t^{k-\xi} \left(\int_0^s \int_0^t \left| x_1^{(n,k)}(\sigma, \tau) \right|^{l+m} d\sigma d\tau \right)^{\xi},$$

where

$$A = \left(\frac{(1-\xi)^2}{(n-\xi)(k-\xi)} \right)^{1-\xi} \frac{1}{(n-1)!(k-1)!}.$$

Multiplying the both sides of above inequality by $|x_2^{(n,k)}(s,t)|^m$ and integrating both sides over t from 0 to b first and then integrating the resulting inequality over s from 0 to a, we obtain

$$\int_0^a \int_0^b |x_1(s,t)|^l \left| x_2^{(n,k)}(s,t) \right|^m dsdt$$

$$\leq A^l \int_0^a \int_0^b s^{l(n-\xi)} t^{l(k-\xi)} \left| x_2^{(n,k)}(s,t) \right|^m \left(\int_0^s \int_0^t \left| x_1^{(n,k)}(\sigma,\tau) \right|^{l+m} d\sigma d\tau \right)^{l\xi} dsdt.$$

Now, applying Hölder's inequality with indices $\frac{l+m}{l}$ and $\frac{l+m}{m}$ to the integral on the right side, we obtain

(2.12)

$$\int_0^a \int_0^b |x_1(s,t)|^l \left| x_2^{(n,k)}(s,t) \right|^m dsdt$$

$$\leq A^l \left(\int_0^a \int_0^b s^{(n-\xi)(l+m)} t^{(k-\xi)(l+m)} dsdt \right)^{\frac{l}{l+m}}$$

$$\times \left(\int_0^a \int_0^b \left| x_2^{(n,k)}(s,t) \right|^{m+l} \left(\int_0^s \int_0^t \left| x_1^{(n,k)}(\sigma,\tau) \right|^{l+m} d\sigma d\tau \right)^{\frac{l}{m}} dsdt \right)^{\frac{m}{l+m}}$$

$$= A^l \left(\frac{\xi^2}{nk} \right)^{\xi l} a^{nl} b^{kl}$$

$$\times \left(\int_0^a \int_0^b \left| x_2^{(n,k)}(s,t) \right|^{m+l} \left(\int_0^s \int_0^t \left| x_1^{(n,k)}(\sigma,\tau) \right|^{l+m} d\sigma d\tau \right)^{\frac{l}{m}} dsdt \right)^{\frac{m}{l+m}}.$$

Similarly, we obtain

(2.13)

$$\int_0^a \int_0^b |x_2(s,t)|^l \left| x_1^{(n,k)}(s,t) \right|^m dsdt$$

$$\leq A^l \left(\frac{\xi^2}{nk} \right)^{\xi l} a^{nl} b^{kl}$$

$$\times \left(\int_0^a \int_0^b \left| x_1^{(n,k)}(s,t) \right|^{m+l} \left(\int_0^s \int_0^t \left| x_2^{(n,k)}(\sigma,\tau) \right|^{l+m} d\sigma d\tau \right)^{\frac{l}{m}} dsdt \right)^{\frac{m}{l+m}}.$$

Thus, from (2.12), (2.13) and in view of the elementary inequality $a^\lambda + b^\lambda \leq 2^{1-\lambda}(a+b)^\lambda, 0 \leq \lambda \leq 1, a, b \geq 0$, we obtain

$$\int_0^a \int_0^b \left[|x_1(s,t)|^l |x_2^{(n,k)}(s,t)|^m + |x_2(s,t)|^l |x_1^{(n,k)}(s,t)|^m \right] dsdt$$

$$\leq c_{n,k}^* a^{nl} b^{kl} \left[\int_0^a \int_0^b \left| x_2^{(n,k)}(s,t) \right|^{m+l} \left(\int_0^s \int_0^t \left| x_1^{(n,k)}(\sigma,\tau) d\sigma d\tau \right| \right)^{\frac{l}{m}} \right.$$

$$\left. + \int_0^a \int_0^b \left| x_1^{(n,k)}(s,t) \right|^{m+l} \left(\int_0^s \int_0^t \left| x_2^{(n,k)}(\sigma,\tau) \right|^{l+m} d\sigma d\tau \right)^{\frac{l}{m}} dsdt \right]^{\frac{m}{l+m}}$$

$$\leq c_{n,k}^* a^{nl} b^{kl} \left[\left(\int_0^a \int_0^b \left| x_1^{(n,k)}(s,t) \right|^{l+m} dsdt \right)^{\frac{l}{m}} \cdot \int_0^a \int_0^b \left| x_2^{(n,k)}(s,t) \right|^{m+l} dsdt \right.$$

$$\left. + \left(\int_0^a \int_0^b \left| x_2^{(n,k)}(s,t) \right|^{l+m} dsdt \right)^{\frac{l}{m}} \cdot \int_0^a \int_0^b \left| x_1^{(n,k)}(s,t) \right|^{m+l} dsdt \right]^{\frac{m}{l+m}}.$$

This completes the proof. □

Remark 2.6. Taking for $x_1(s,t) = x_2(s,t) = x(s,t)$ in (2.11), we have

$$\int_0^a \int_0^b |x(s,t)|^l \cdot |x^{(n,k)}(s,t)|^m dsdt \leq c_{n,k}^* a^{nl} b^{kl} \int_0^a \int_0^b \left| x^{(n,k)}(s,t) \right|^{l+m} dsdt.$$

Remark 2.7. Let $x(s,t)$ reduce to $x(t)$ by letting $s = s(t)$ and with suitable modifications in all intermediate steps of proof of Theorem 2.5, then (2.11) becomes the following result.

$$\int_0^a \left[|x_1(t)|^l |x_2^{(n)}(t)|^m + |x_2(t)|^l |x_1^{(n)}(t)|^m \right] dt$$

$$\leq c_n^* a^{nl} \left[\left(\int_0^a \left| x_1^{(n)}(t) \right|^{l+m} dt \right)^{\frac{l}{m}} \cdot \int_0^a \left| x_2^{(n)}(t) \right|^{m+l} dt \right.$$

$$(2.14) \qquad \left. + \left(\int_0^a \left| x_2^{(n)}(t) \right|^{l+m} dt \right)^{\frac{l}{m}} \cdot \int_0^a \left| x_1^{(n,k)}(t) \right|^{m+l} dt \right]^{\frac{m}{l+m}},$$

where

$$(2.15) \qquad c_n^* = \left(\frac{n(1-\xi)}{n-\xi} \right)^{l(1-\xi)} \frac{2^{\xi l} \xi^{\xi l}}{(n!)^l}, \ \xi = \frac{1}{l+m}.$$

Taking for $x_1(t) = x_2(t) = x(t)$ in (2.14), we have the following interesting inequality.

$$\int_0^a |x(t)|^l |x^{(n)}(t)|^m dt \leq c_n^* a^{nl} \int_0^a \left| x^{(n)}(t) \right|^{l+m} dt.$$

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