On Two Dimensional $q$-Hölder’s Inequality

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Abstract. In this article, the reverse $q$-Hölder type inequality and two dimensional $q$-Hölder’s inequality are established. We also obtain some $q$-integral inequalities by using $q$-Hölder’s inequality which give $q$-Hardy’s inequalities as spacial cases.

1. Introduction

Throughout this paper, we will fit $q \in (0,1)$. We denote by $I$ one of the following sets: (1) $\mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}$; (2) $[0, b]_q = \{bq^n : n \in \mathbb{Z}\}$, $b > 0$; (3) $[a, b]_q = \{bq^k : 0 \leq k \leq n\}$, $b > 0$, $a = bq^n$, $n \in \mathbb{Z}$. Due to restrictions on the number of pages, the basic definitions and theorems of $q$-integral were omitted, and the reader was referred to [3, 2, 4]. And we note $\int_I f(x)\,dq_x$ the $q$-integral of $f$ on the correspondent $I$.

Let $p$ and $p'$ be two positive reals satisfying $p > 1$ and $1/p + 1/p' = 1$, and $f$ and $g$ be two functions defined on $I$. Then

\begin{equation}
\left|\int_I f(x)g(x)\,dq_x\right| \leq \left(\int_I |f(x)|^p\,dq_x\right)^{\frac{1}{p}} \left(\int_I |g(x)|^{p'}\,dq_x\right)^{\frac{1}{p'}}.
\end{equation}

The above inequality was given by Fitouhi and Brahim [3], but the condition $p > 0$ was not added. According to the definition of $q$-integral, we have $\int_I f(x)g(x)\,dq_x \leq \int_I |f(x)g(x)|\,dq_x$. So, the above conditions hold, (1.1) is restated as follows:

\begin{equation}
\int_I |f(x)g(x)|\,dq_x \leq \left(\int_I |f(x)|^p\,dq_x\right)^{\frac{1}{p}} \left(\int_I |g(x)|^{p'}\,dq_x\right)^{\frac{1}{p'}}.
\end{equation}

Tuna and Kutukcu [5] and Ammi and Torres [1] gave two dimensional $\Delta$-Hölder’s inequalities and two dimensional $\diamondsuit$-Hölder’s inequalities, respectively. Motivated by [5] and [1], we will study the reverse $q$-Hölder type inequality and two dimensional $q$-Hölder’s inequality. We also obtain some $q$-integral inequalities by using $q$-Hölder’s inequality which give $q$-Hardy’s inequalities as spacial cases.

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2. Main results

**Theorem 2.1.** For two positive functions \( f \) and \( g \) satisfying \( 0 < m \leq f^p/g^p' \leq M < \infty \) on \( I \). If \( 1/p + 1/p' = 1 \) with \( p > 1 \), we have

\[
\left( \int_I f^p(x) dq_x \right)^{\frac{1}{p}} \left( \int_I g^{p'}(x) dq_x \right)^{\frac{1}{p'}} \leq \left( \frac{M}{m} \right)^{\frac{1}{pp'}} \int_I f(x) g(x) dq_x.
\]

**Proof.** Since \( f^p/g^p' \leq M \), then \( f^p = M g^{p'} \). Multiplying by \( f > 0 \), it follows that

\[
f^p = f^{1 + \frac{1}{p'}} \leq M \frac{p'}{p} f g.
\]

and so,

\[
\left( \int_I f^p(x) dq_x \right)^{\frac{1}{p}} \leq M \frac{p}{p'} \left( \int_I f(x) g(x) dq_x \right)^{\frac{1}{p'}}.
\]

On the other hand, since \( m \leq f^p/g^p' \), then \( f^p \geq m^{1/p} g^{p'/p} \), hence

\[
\int_I f(x) g(x) dq_x \geq m^{\frac{1}{pp'}} \int_I g^{p'}(x) dq_x = m^{\frac{1}{p}} \int_I g^{p'}(x) dq_x.
\]

We obtain that

\[
\left( \int_I f(x) g(x) dq_x \right)^{\frac{1}{p}} \geq m^{\frac{1}{pp'}} \left( \int_I g^{p'}(x) dq_x \right)^{\frac{1}{p'}}.
\]

Combining (2.2) and (2.3), we have the desired inequality (2.1). The proof is completed. \( \square \)

**Theorem 2.2.** Let \( f(x, y) \), \( g(x, y) \), and \( h(x, y) \) be three functions defined on \( I^2 \). If \( 1/p + 1/p' = 1 \) with \( p > 1 \), we have

\[
\int_I \int_I |h(x, y) f(x, y) g(x, y)| dq_x dq_y
\]

\[
\leq \left( \int_I \int_I |h(x, y)||f(x, y)|^{p} dq_x dq_y \right)^{\frac{1}{p}} \times \left( \int_I \int_I |h(x, y)||g(x, y)|^{p'} dq_x dq_y \right)^{\frac{1}{p'}}.
\]

**Proof.** Inequality (2.4) is trivially true in the case when \( f \) or \( g \) or \( h \) is identically zero. Suppose that

\[
\left( \int_I \int_I |h(x, y)||f(x, y)|^{p} dq_x dq_y \right) \left( \int_I \int_I |h(x, y)||g(x, y)|^{p'} dq_x dq_y \right) \neq 0.
\]

Apply the following Young’s inequality

\[
x^{\frac{1}{p'}} y^{\frac{1}{p}} \leq \frac{1}{p} x + \frac{1}{p'} y, \quad x, y \geq 0 \quad \text{and} \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad p > 1,
\]
Corollary 2.1. Let \( f(x, y) \) \( g(x, y) \) and \( h(x, y) \) be three functions defined on \( I^2 \). Then

\[
\left( \int_I \int_I |h(x, y)| f(x, y) g(x, y) d_q x d_q y \right)^2 \leq \left( \int_I \int_I |h(x, y)| |f(x, y)|^2 d_q x d_q y \right) \times \left( \int_I \int_I |h(x, y)| |g(x, y)|^2 d_q x d_q y \right).
\]

(2.5)

Proof. The Cauchy-Schwartz inequality (2.5) is the particular case \( p = p' = 2 \) of (2.4). \qed
Theorem 2.3. Let $K(x, y) f(x)$ and $g(y)$, $\varphi(x)$ and $\psi(y)$ be nonnegative functions. Let

$$F(x) = \int_I K(x, y) \psi^{-p}(y) dq_y$$

and

$$G(y) = \int_I K(x, y) \varphi^{-p'}(x) dx,$$

where $1/p + 1/p' = 1$ with $p > 1$. Then the two inequalities

\begin{equation}
\int_I \int_I K(x, y) f(x) g(y) dq_y dx_y 
\leq \left( \int_I \varphi^p(x) F(x) f^p(x) dx_x \right)^{\frac{1}{p}} \times \left( \int_I \psi^{p'}(y) G(y) g^{p'}(y) dq_y \right)^{\frac{1}{p'}}.
\end{equation}

and

\begin{equation}
\left( \int_I G^{1-p}(y) \psi^{-p}(y) \left( \int_I K(x, y) f(x) dq_x x_y \right)^{p-1} \right)^{\frac{1}{p}} \leq \left( \int_I \varphi^p(x) F(x) f^p(x) dx_x \right)^{\frac{1}{p}} \times \left( \int_I \psi^{p'}(y) G(y) g^{p'}(y) dq_y \right)^{\frac{1}{p'}}
\end{equation}

hold and are equivalent. Equation (2.7) is the $q$-Hardy’s inequality.

Proof. First, we prove that (2.6) hold. Write

$$\int_I \int_I K(x, y) f(x) g(y) dq_y dx_y = \int_I \int_I K(x, y) f(x) \frac{\varphi(x)}{\psi(y)} g(y) \frac{\psi(y)}{\varphi(x)} dq_y dx_y.$$ 

Applying $q$-Hölder’s inequality (1.2), we have

$$\int_I \int_I K(x, y) f(x) g(y) dq_y dx_y 
\leq \left( \int_I \varphi^p(x) F(x) f^p(x) dx_x \right)^{\frac{1}{p}} \times \left( \int_I \psi^{p'}(y) G(y) g^{p'}(y) dq_y \right)^{\frac{1}{p'}}.$$ 

Now we show that (2.6) is equivalent to (2.7). Suppose that inequality (2.6) is verified. Set

$$g(y) = G^{1-p}(y) \psi^{-p}(y) \left( \int_I K(x, y) f(x) dq_x x_y \right)^{p-1}.$$ 

Using (2.6) and taking into account that $1/p + 1/p' = 1$ with $p > 1$, we obtain

$$\int_I G^{1-p}(y) \psi^{-p}(y) \left( \int_I K(x, y) f(x) dq_x x_y \right)^{p} dq_y = \int_I \int_I K(x, y) f(x) g(y) dq_y dx_y 
\leq \left( \int_I \varphi^p(x) F(x) f^p(x) dx_x \right)^{\frac{1}{p}} \times \left( \int_I \psi^{p'}(y) G(y) g^{p'}(y) dq_y \right)^{\frac{1}{p'}}
= \left( \int_I \varphi^p(x) F(x) f^p(x) dx_x \right)^{\frac{1}{p}} \times \left( \int_I G^{1-p}(y) \psi^{-p}(y) \left( \int_I K(x, y) f(x) dq_x x_y \right)^{p} dq_y \right)^{\frac{1}{p'}}.$$
Inequality (2.7) is obtained by dividing both sides of the previous inequality by
\[
\left( \int_I G^{1-p}(y) \psi^{-p}(y) \left( \int_I K(x,y)f(x)d_qx \right)^p d_qy \right)^\frac{1}{p}.
\]
Reciprocally, suppose that inequality (2.7) is valid. From \(q\)-Hardy’s inequality we can write that
\[
\int_I \int_I K(x,y)f(x)g(y)d_qxd_qy \leq \left( \int_I \phi^p(x)F(x)f^p(x)d_qx \right)^\frac{1}{p} \times \left( \int_I \psi^p(y)G(y)g^p(y)d_qy \right)^\frac{1}{p},
\]
which completes the proof. \(\Box\)

As corollaries of Theorem 2.3 we have the following results. Without loss of generality, only take \(I = [a,b]_q\) for example.

**Corollary 2.2.** Let \(h(y) f(x)\) and \(g(y), \varphi(x)\) and \(\psi(y)\) be nonnegative functions, and \(1/p + 1/p' = 1\) with \(p > 1\). Setting \(H(y) = h(y)\psi^{-p}(y)\), then the two inequalities
\[
\int_a^b \int_a^y h(y) f(x)g(y)d_qxd_qy \leq \left( \int_a^b \varphi^p(x)F(x)f^p(x)d_qx \right)^\frac{1}{p} \times \left( \int_a^b \psi^p(y)G(y)g^p(y)d_qy \right)^\frac{1}{p}
\]
and
\[
\int_a^b H(y) \left( \int_a^y \varphi^{-p'}(y)d_qx \right)^{1-p} \left( \int_a^y f(x)d_qx \right)^p d_qy \leq \left( \int_a^b \varphi^p(x)F(x)f^p(x)d_qx \right)^\frac{1}{p}
\]
hold and are equivalent.
Proof. Use Theorem 2.3 with $K(x, y) = \begin{cases} h(y), & \text{if } x \leq y, \\ 0, & \text{if } x > y. \end{cases}$ \hfill $\square$

**Corollary 2.3.** Let $h(y) f(x)$ and $g(y)$, $\varphi(x)$ and $\psi(y)$ be nonnegative functions, and $1/p + 1/p' = 1$ with $p > 1$. Setting $H(y) = h(y)\psi^{-p}(y)$, then the two inequalities

$$
\int_a^b \int_y^b h(y) f(x) g(y) d_q x d_q y \leq \left( \int_a^b \varphi^p(x) f^p(x) \left( \int_a^x H(y) d_q y \right) d_q x \right)^{1/p} \left( \int_a^b \psi^p(y) g^p(y) h(y) \left( \int_y^b \varphi^{-p}(y) d_q y \right) d_q y \right)^{1/p'}
$$

and

$$
\int_a^b H(y) \left( \int_y^b \varphi^{-p}(y) d_q y \right)^{1-p} \left( \int_y^b f(x) d_q x \right)^p d_q y 
\leq \left( \int_a^b \varphi^p(x) f^p(x) \left( \int_a^x H(y) d_q y \right) d_q x \right)^{1/p} \left( \int_a^b \psi^p(y) g^p(y) h(y) \left( \int_y^b \varphi^{-p}(y) d_q y \right) d_q y \right)^{1/p'}
$$

hold and are equivalent.

Proof. Use Theorem 2.3 with $K(x, y) = \begin{cases} 0, & \text{if } x \leq y, \\ h(y), & \text{if } x > y. \end{cases}$ \hfill $\square$

It is interesting to consider the case when functions $F(x)$ and $G(y)$ of Theorem 2.3 are bounded. We then obtain the following:

**Theorem 2.4.** Let $K(x, y)$ $f(x)$ and $g(y)$, $\varphi(x)$ and $\psi(y)$ be nonnegative functions. Let

$$
F(x) = \int_I K(x, y)\psi^{-p}(y) d_q y \leq F_1(x)
$$

and

$$
G(y) = \int_I K(x, y)\varphi^{-p'}(y) d_q x \leq G_1(y),
$$

where $1/p + 1/p' = 1$ with $p > 1$. Then the two inequalities

$$
\int_I \int_I K(x, y) f(x) g(y) d_q x d_q y \leq \left( \int_I \varphi^p(x) F_1(x) f^p(x) d_q x \right)^{1/p} \left( \int_I \psi^p(y) G_1(y) g^p(y) d_q y \right)^{1/p'}
$$

and

$$
\int_I G_1^{-p}(y)\psi^{-p}(y) \left( \int_I K(x, y) f(x) d_q x \right)^p d_q y \leq \int_I \varphi^p(x) F_1(x) f^p(x) d_q x
$$
hold and are equivalent.

**Theorem 2.5.** Let \( F, G, L(f, g), M(f) \) and \( N(g) \) be positive functions, and \( 1/p + 1/p' = 1 \) with \( p > 1 \) such that

\[
0 < \int_I M^p(f(x))F^p(x)dx < \infty, \quad 0 < \int_I N^{p'}(g(x))G^{p'}(x)dx < \infty.
\]

then the two inequalities

\[
\int_I \int_I \frac{F(x)G(y)}{L(f(x), g(y))} dx dy 
\leq C \left( \int_I M^p(f(x))F^p(x)dx \right)^\frac{1}{p} \times \left( \int_I N^{p'}(g(x))G^{p'}(y)dy \right)^\frac{1}{p'}
\]

and

\[
\int_I N^{-p}(g(y)) \left( \int_I \frac{F(x)}{L(f(x), g(y))} dx \right)^p dy \leq C^p \int_I M^p(f(x))F^p(x)dx,
\]

where \( C \) is a constant, are equivalent.

**Proof.** Suppose that the inequality (2.9) is valid. Then we have

\[
\int_I \int_I \frac{F(x)G(y)}{L(f(x), g(y))} dx dy 
= \int_I N(g(y))G(y) \left( \int_I \frac{F(x)}{L(f(x), g(y))} dx \right) dy 
\leq \left( \int_I N^{p'}(g(y))G^{p'}(y)dy \right)^\frac{1}{p'} \left( \int_I N^{-p}(g(y)) \left( \int_I \frac{F(x)}{L(f(x), g(y))} dx \right)^p dy \right)^\frac{1}{p} 
\leq C \left( \int_I M^p(f(x))F^p(x)dx \right)^\frac{1}{p} \left( \int_I N^{p'}(g(y))G^{p'}(y)dy \right)^\frac{1}{p'},
\]

which implies inequality 2.8. Let us now suppose that the inequality (2.8) is valid. By setting \( G(y) = N^{-p}(g(y)) \left( \int_I \frac{F(x)}{L(f(x), g(y))} dx \right)^\frac{1}{p} \) and applying 2.8, then we ob-
which implies that
\[
\int_{I} N^{-p}(g(y)) \left( \int_{I} \frac{F(x)}{L(f(x), g(y))} d_q x \right)^p d_q y \\
\leq C \left( \int_{I} M^p(f(x)) F^p(x) d_q x \right)^{\frac{1}{p}} \\
\times \left( \int_{I} N^{p'}(g(y)) N^{-pp'}(g(y)) \left( \int_{I} \frac{F(x)}{L(f(x), g(y))} d_q x \right)^p d_q y \right)^{\frac{1}{p'}} \\
= C \left( \int_{I} M^p(f(x)) F^p(x) d_q x \right)^{\frac{1}{p}} \left( \int_{I} N^{-p}(g(y)) \left( \int_{I} \frac{F(x)}{L(f(x), g(y))} d_q x \right)^p d_q y \right)^{\frac{1}{p}} ,
\]

which implies that
\[
\int_{I} N^{-p}(g(y)) \left( \int_{I} \frac{F(x)}{L(f(x), g(y))} d_q x \right)^p d_q y \leq C^p \int_{I} M^p(f(x)) F^p(x) d_q x .
\]

The proof is complete. \qed

References


