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天元術과 記數法

TianYuanShu and Numeral Systems in Eastern Asia

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중국의 명수법은 기록은 구어체를 사용하고, 계산은 산대를 사용하는 이중 구조를 가지고 있었다. 또 산서는 실생활의 문제만 다루는 과정에서 수학적 구조를 나타내는 방법을 택하여 계산 과정을 제외하면 이들에서 취급한 수는 모두 명수(名數)들이어서 순수한 수론의 발전을 이룰 수 없었다. 송대에 0의 도입과 함께, 천원술의 표현에서 나타나는 계수를 산대로 표시하는 방법을 통하여, 산대가 계산 도구와 함께 추상수의 기수법(記數法)이 되는 과정을 밝힌다. 수량의 단위를 사용한 소수의 표현도 이 과정에서 산대 표현으로 대치되었다. 그러나 명대에 산대 계산이 주산으로 대치되고 천원술이 잊히게 되어 추상수의 개념도 함께 잊히게 되었다. 청대의 산학자 심사계(沈士桂)가 저서 간첩이명산법(簡捷易明算法)에서 분수의 소수표시와 계산을 하는 과정에서 순환소수를 인지하고 이들의 계산법을 확립한 것도 보인다.

In Chinese mathematics, there have been two numeral systems, namely one in spoken language for recording and the other by counting rods for computations. They concerned with problems dealing with practical applications, numbers in them are concrete numbers except in the process of basic operations. Thus they could hardly develop a pure theory of numbers. In Song dynasty, 0 and TianYuanShu were introduced, where the coefficients were denoted by counting rods. We show that in this process, counting rods took over the role of the numeral system in spoken language and hence counting rod numeral system plays the role of that for abstract numbers together with the tool for calculations. Decimal fractions were also understood as denominate numbers but using the notions by counting rods, decimals were also admitted as abstract numbers. Noting that abacus replaced counting rods and TianYuanShu were lost in Ming dynasty, abstract numbers disappeared in Chinese mathematics. Investigating JianJie YiMing SuanFa(簡捷易明算法) written by Shen ShiGui(沈士桂) around 1704, we conclude that Shen noticed repeating decimals and their operations, and also used various rounding methods.

Keywords: abstract and concrete numbers, TianYuanShu, counting rod numeral system, decimal fractions, repeating decimals, Shen ShiGui(沈士桂), JanJie YiMing SuanFa(簡捷易明算法, ca. 1704)

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0 Introduction

The numeral systems in Chinese mathematics were divided into two systems, namely one in spoken language for recording and the other by counting rods(籌, 산대) for calculations. Furthermore, mathematical structures in Chinese mathematics were revealed through JinYou(今有) type problems, i.e., those dealing with practical affairs. Thus the numbers in the Chinese mathematical sources are concrete numbers which are mostly denominate numbers. Indeed, abstract numbers appear only in the theory of fractions in the first chapter of JiuZhang SuanShu(九章算術) and in quoting Chinese numeral systems notably in WuJing SuanShu(五經算術, 6th C.) written by Zhen Luan(甄鸞) and SuanXue QiMeng(算學啓蒙, 1299) by Zhu ShiJie(朱世傑). Although problems are given by concrete numbers, basic operations including extraction of roots in the process of solving them are practiced in the setting of abstract numbers which were done by counting rods. Because of these dual systems, development of pure number theory has been hindered. We note that the method of representing numbers by counting rods is precisely a positional numeral system but in the process of basic operations, they disregard the final zeros indicating number of digits. In Song dynasty(960 - 1279), TianYuanShu(天元術) to represent $\sum_{k=-m}^{n} a_k x^k$ $(m, n \geq 0)$ by the sequence of its coefficients was introduced. The terms of the sequence are denoted by counting rods. In this case, they use counting rods representation in dual roles, namely abstract numbers of coeffeicients and their operations. Further, the representation records the process of constructing and solving equations and hence it should be precise to indicate every place value including zeros. Investigating mathematics books written in Song and Yuan(1271 - 1368) dynasties, we show that the counting rods in the dynasties could combine the traditional numeral systems and thus they could develop one system like Hindu-Arabic one for abstract numbers including decimal fractions.

As is well known, counting rods were replaced by abacus and TianYuanShu has been almost forgotten for the next 400 years. We take JanJie YiMing SuanFa(簡捷易明算法, ca. 1704) written by Qing mathematician Shen ShiGui(沈士桂). Although he returned to the traditional system, he noticed repeating decimals and their basic operations and also developed various rounding methods.

For the Chinese sources, we refer to ZhongGuo KeXue JiShu DianJi TongHui

ShuXueJuan(中國科學技術典籍通彙 數學卷, [4]) and ZhongGuo LiDai SuanXue JiCheng (中國歷代算學集成, [5]). Those books appeared in them will not be numbered as an indivisual reference. For Chinese numeral systems, we refer to ([1], [7], [8]).

1 Numeral systems in Song - Yuan dynasties

Since notations of numbers had already appeared in oracle-bone scripts made in around 14 - 11th century B.C., the system has been retained as a numeral system for recording in China. But the system is somewhat inconvenient for basic operations for it is made of spoken language. Thus for calculations, they introduced a new system by counting rods which is a decimal positional numeral system and leads to the dual numeral systems as mentioned in the previous section. In the mathematics books before Song dynasty(960 - 1279), methods of basic operations were presumed and they first appeared in Yang Hui SuanFa(楊輝算法, 1274 - 1275). Zhu ShiJie systemized computaions in his SuanXue QiMeng. As in Yang Hui's book, Zhu disregards the zeros indicating number of digits when he discusses multiplications by 300, 700, 80, 90 in the first section dealing with those by single digits(因乘). This shows that counting rod numeral system for calculations is slightly different from that in spoken language because of obvious reasons. It is well known that the most important result in Song dynasty, probably in the whole history of Chinese mathematics, is the theory of equations. The one is the method of solving polynomial equations, called ZengCheng KaiFangFa(增乘開方法) and the other is TianYuanShu for the construction of equations. In the process, they use counting rods to represent coefficients of $\sum_{k=-m}^{n} a_k x^k$ $(m, n \geq 0)$ and unlike the case of calculations, precise numbers should be recorded for the process of obtaining equations. Hence they begin to use the numeral system by counting rods for recording and calculations. In this case, they need the notion of zero for counting rods for which they use 0. In the following, this system with zeros will be called the counting rod numeral system.

The counting rod numeral system appears first in ShuShu JiuZhang(數書九章, 1247) written by Qin JiuShao(秦九韶, 1202 - 1261) and then in CeYuan HaiJing(測圓海鏡, completed in 1248, 1282) and YiGu YanDuan(益古演段, 1259) by Li Ye(李冶, 1192 - 1279). Qin uses mostly the counting rod numeral system as an auxiliary presentation for the spoken language system.

But Li adopts zeros even for spoken language system as "股弦和 一千0八十" for the sum 1080 of the height and hypotenus in Book 1 of CeYuan HaiJing which means that he took 0 as a place value. He also uses the counting rod numeral system without any reference to the spoken language system in both books. Furthermore, Qin JiuShao represents negative numbers as counting rods representation added with Yi(益) or Fu(負) and positive numbers as that with Zheng(正). For the latter case, Zheng is mostly deleted by the obvious reason. Instead of this method, Li uses an easier method for negative number to add a slash on the last non-zero digit of the counting rod representation. In all, one may say that Li Ye is the first mathematician in China to establish counting rod representations for integers as a numeral system. Zhu ShiJie follows Li's system in his SuanXue QiMeng.

We now look upon the counting rod numeral system for rational numbers in China. As we have mentioned earlier, numbers in Chinese mathematics books are denominate numbers and hence decimal fractions are also indicated by units in weights and measures. For abstract decimals, Zhu quoted spoken language units for decimals(小數) up to 10^{-128} as those for natural numbers(大數) in SuanXue QiMeng but they have never been used in problems in the book. We quote the following statement in Book 2 of SuanFa TongZong(算法統宗, 1592) written by Cheng DaWei(程大位, 1533 - 1606).

> 假如四分兩之一者則二錢五分也 此所謂有盡者也
> 若數如三分兩之一者三錢三分三厘以至于三三之無窮
> 此所謂數之不盡者也 必須以分通之乃可算也
> 不然則畸零之不盡終無可置位矣

It says that $\frac{1}{4} = 0.25$ is a finite decimal but $\frac{1}{3} = 0.333\ldots$ is an infinite decimal which can not be represented as a decimal. Cheng includes this statement twice for the reason to introduce fractions although it is not relevant to the sections dealing with the reduction of fractions(約分) and the process of mixed numbers into fractions(通分). The other mathematicians should also have known the infinite decimals for rational numbers are understood by fractions from JiuZhang SuanShu. For TianYuanShu presentations, decimals are more convenient than fractions. Thus they must have the decimal point separating the integral part from the fractional part of a number. Since they deal with denominate numbers, they use the corresponding units like cun(寸), chi(尺) or bu(步) under or above the first digit in the counting rod numeral system. In case of the integral part being zero, they just put 0 without the unit. Using these, they can extend the counting rod numeral system to numbers with finite fractional parts. This method appears in ShuShuJiuZhang, CeYuan HaiJing and YiGu YanDuan. For repeating decimals, we will discuss them in the next section.

We note that Qin JiuShao also uses the counting rods representations for degrees in Book 3 of ShuShu JiuZhang. His system for degrees is not sexagesimal but centesimal as 1 degree(度) = 100Fen(分) and then subdivisions go down in the order of Miao(秒), XiaoFen(小分), XiaoMiao(小秒), WeiFen(微分), WeiMiao(微秒). Thus a digit of fractional part for this system is a two digit number and Qin uses 00 for the usual zero in the counting rod numeral system and indicates unit under the last digit of each place value. Since he uses this representation for an auxiliary one, there are many cases where he omits units. Although the numeral system with the base 100 in the counting rods representations is almost same with the decimal system, we may say that Qin did know the structure of numeral systems with various bases.

In XiangMing SuanFa(詳明算法, 1373), An ZhiZhai(安止齋) also includes that the digits of basic numers are determined by the number of zeros as follows:

凡大小數相減若空者皆爲大數
十一 百二 千三 萬四 十萬五 百萬六 千萬七 萬萬八
分一 釐二 毫三 絲四 忽五 微六 纖七 沙八 塵九 埃十
升一 合二 勺三 抄四 撮五 圭六 粟七 粒八

The first two lines - natural numbers from 10 to 100,000,000 and decimals from 0.1 to 10^{-10} - indicate the number of zeros of spoken language abstract numbers and the third one is those of the traditional decimals given by the units of volumes in the counting rods representations. This shows that the counting rod numeral system includes the traditional spoken language system and is exactly same with Hindu-Arabic system without the decimal point.

2 Shen ShiGui and his theory of decimal fractions

In the previous section, we show that mathematicians in Song - Yuan era have completed the counting rod numeral system for recording and calculations which is also a positional system in the process of developing TianYuanShu.

In Ming dynasty(1368 - 1643), abacus replaced counting rods and TianYuanShu was almost forgotten so that the traditional numeral system in spoken language was restored in place of the counting rod numeral system.

In this section, we deal with decimal fractions in JanJie YiMing SuanFa(簡捷易明算法, ca 1704) written by a Qing mathematician Shen ShiGui(沈士桂) in the setting of the traditional numeral system. We don't have any information about his life. But by the preface of JanJie YiMing SuanFa, he served as an accountant probably in ZheJiangXing(浙江省) and wrote a book SuanFa DaQuan TongJian(算法大全統鑑, 1681). He revised and renamed it JanJie YiMing SuanFa, published in around 1704(甲申之後). It consists of 4 books and mostly follows SuanFa TongZong. Although western mathematics had a great influence to Chinese mathematics in the 17th century, Shen belongs to mathematicians who were not influenced by western mathematics([3]). He deals with most of the traditional subjects in the book which does not contain any new results except his theory of decimal fractions. Indeed, the book begins with basic rules of abacus, called DuiDuoFa(堆垛法) and hence one can gather that he prefers decimal fractions to fractions.

We note that our source for JanJie YiMing SuanFa is a copy transcribed in the 19th century Chosun which contains many errors in transcription([2]).

Shen ShiGui quotes names of the spoken language numeral system for natural numbers in SuanXue QiMeng. But for decimals, he adds Miao(渺) and Mo(漠) to the system in XiangMing SuanFa quoted in the previous section and his system is decimal. We note that the system in SuanXue QiMeng is same with those up to Sha(沙) in Shen's system but from Chen(塵), they are not decimal. Although Shen includes the system for decimals, he does not use it in his problems but use the monetary units in ZheXing(浙省錢糧尾數) as follows:

兩, 錢, 分, 釐, 毫, 絲, 忽, 微. 塵, 渺, 漠, 纖, 埃, 沙

Thus his decimal system goes down to 10^{-13}. Furthermore, in the first example of Section JieMing GuiChu FaYi(解明歸除法義) of Book 1, $10,000 \div 1248 = 8.0\dot{1}2820\dot{5}$, which is denoted as

八兩一分二釐八毫二絲五微.一塵二渺八漠二纖五沙一二八二0五.

We note that the last digit is 5×10^{-19} and after the last spoken language notion 沙, he uses the present day decimal notion including zeros.

Incidentally, Shen states that the monetary units in JiangNanXing(江南省) are exactly same with the ordinary spoken language system for decimals. This indicates strongly that Shen served as an official in ZheJiangXing.

As in the above example, Shen has results with decimal fractions with many number of digits including repeating decimals, he has to round them. Mostly he just drops digits after some suitable digits as above but he also mentions the process of rounding off in Section ZuanQuanShuFa(纂全書法) of Book 4 as follows:

至沙後 如有一二三四之零 便可去去
若是五六七八之零 可收成一沙之數合 總目自然無錯

Although he uses the process of rounding off after 沙($= 10^{-13}$), it is precisely the modern method.

In an example of the same section, Shen has to calculate ax, bx for various x, $a = 3,691,122.226,168,978,818,7$ and $b = 0.0\dot{2}357\dot{7}$. He does not denote the latter as the present form with dots indicating repeating decimals but repeats four times the repeating cycle and put only digits downward. The notion is precisely the positional numeral system without the decimal point. Incidentally, he adds five more notions Fu(浮), Jin(盡), Zong(宗), Zhi(止), Dao(到) after 沙 for b which we can not find so far except in this book. For this example, he first calculates na, nb for $n = 1, 2, \ldots, 8, 9$ and put the results with the change of digits and then using this table, he obtains ax, bx.

As we mentioned above, Shen converts fractions to decimal fractions for solving problems. Sometimes he goes too far as the following examples show. They appear in Section SiLu ChaFen(四六差分) and SanQi ChaFen(三七差分) of Book 3. We recall the problem p, q ChaFen: find terms of a sequence $a_k (k = 1, 2, \ldots, n)$ under the conditions that $\sum a_k$ is given and $a_k : a_{k+1} = p : q$ for any k. The problem is easily solved by ChaFenFa when one has the ratio $a_1 : a_2 : \cdots : a_n$.

Using $p^m q^{n-1} : p^{m-1} q^n = p : q$, one has $a_1 : a_2 : \dots a_n = p^{n-1} : p^{n-2}q : \dots : q^{n-1}$ where each term is a natural number. Instead of this method, Shen observes that $p : q = 1 : \dfrac{q}{p}$ and then he converts $\dfrac{q}{p}$ into a decimal α. Thus he has the ratio $a_1 : a_2 : \dots : a_n = p : p\alpha : p\alpha^2 : \dots p\alpha^{n-1}$. For SiLu ChaFen, $\dfrac{6}{4} = 1.5$, his method may contain easier calculation to get the ratio. For SanQi ChaFen, he takes $\dfrac{7}{3} = 0.23333$. Clearly his method involves a lot more calculations than the usual method.

As shown in examples mentioned above, one can easily conclude that Shen's conversion from fractions to decimals gives rise to repeating decimals. He has not the notion to indicate repeating cycles but he must notice the structure of repeating decimals. Indeed, whenever he has the result involving repeating decimals, he presents four times the cycle. Furthermore, in the example $10,000 \div 1248$ in Book 1 which we discussed earlier, he mentioned the following:

沙後餘數收之則多去之則少必要留六檔方合總數無盈縮矣

We note that the period of the repeating decimal is 6. Shen checks the result $a \div b = q$ by $b \times q = a$ from Book 1. In this example, he also checks the division. The quotation says that in order to check the division, he adds one more cycle although the long division never ends and the error is small. Incidentally, Shen uses the term YingSuoFa(盈縮法) for YingBuZuShu(盈不足術).

3 Conclusion

In Chinese mathematics books, there are dual numeral systems, one in spoken language for recording and the other by counting rods for calculations. The latter is a positional system but in the earlier books, calculations are presumed so that the notions by counting rods are omitted. When TianYuanShu is introduced in Song dynasty, they have the notion of the zero, use the counting rod numeral system for coefficients in TianYuanShu and have to record the process to get equations. In the process, they also have the way to represent finite decimal fractions. We note that Abu'l-Hasan al-Uqlidisi invents the decimal point in his book The Book of Chapters on Hindu Arithmetic(952 - 953) in the form of the short vertical mark pointing out the unit's place([6]) but in Chinese sources, there is no such a simple decimal mark but the unit's places are indicated by the corresponding unit in the system of weights and measures. This process leads to that the counting rod numeral system can replace the numeral system in spoken language and plays the role of both recording and computations. Thus they may succeed in having the numeral system for abstract numbers.

Chosun mathematician Hong JungHa(洪正夏, 1684 - ?) studied thoroughly TianYuanShu in SuanXue QiMeng, he used the counting rod numeral system in his book GuIlJib(九一集, 1724) and then Lee SangHyuk(李尙爀, 1810 - ?) also adoped it([9]).

During the Ming dynasty, TianYuanShu and counting rods were almost forgotten, the counting rod numeral system also disappeared. For decimal fractions, discarding inconvenient system of spoken language, they changed them into a decimal system following the monetary units and hence they could easily deal with really small numbers.

Once they have suitable notions for decimals with large digits, then they can easily reveal structures of decimals for they have already enough informations on algebraic structures of rational numbers. Furthermore, the basic computations of fractions by abacus are very awkward so that conversion of fractions into decimals may be widely used in Ming and Qing dynasties. In this vein, Shen ShiGui obtains basic structures of decimals including repeating decimals in his JanJie YiMing SuanFa. Repeating deccimals in Chosun were first introduced in a text book GeunI SanSulSeo(近易算術書, 1895)([10]).

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