SEMIPRIME SUBMODULES OF GRADED MULTIPLICATION MODULES

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Abstract. Let $G$ be a group. Let $R$ be a $G$-graded commutative ring with identity and $M$ be a $G$-graded multiplication module over $R$. A proper graded submodule $Q$ of $M$ is semiprime if whenever $I^nK \subseteq Q$, where $I \subseteq h(R)$, $n$ is a positive integer, and $K \subseteq h(M)$, then $IK \subseteq Q$. We characterize semiprime submodules of $M$. For example, we show that a proper graded submodule $Q$ of $M$ is semiprime if and only if $\text{grad}(Q) \cap h(M) = Q \cap h(M)$. Furthermore if $M$ is finitely generated, then we prove that every proper graded submodule of $M$ is contained in a graded semiprime submodule of $M$. A proper graded submodule $Q$ of $M$ is said to be almost semiprime if

\[
(\text{grad}(Q) \cap h(M)) \setminus (\text{grad}(0_M) \cap h(M)) = (Q \cap h(M)) \setminus (\text{grad}(0_M) \cap Q \cap h(M)).
\]

Let $K, Q$ be graded submodules of $M$. If $K$ and $Q$ are almost semiprime in $M$ such that $Q + K \neq M$ and $Q \cap K \subseteq M_g$ for all $g \in G$, then we prove that $Q + K$ is almost semiprime in $M$.

1. Introduction

Let $G$ be a group. Then we define a $G$-graded ring $R$ and a $G$-graded module over $R$ in the same way as in [2], [3], and [5]. The notations which the authors use are slightly different but basically the same.

Throughout this paper $G$ is a group, $R$ is a $G$-graded commutative ring with identity and $M$ is a $G$-graded module over $R$. From now on, by graded we mean $G$-graded, unless otherwise indicated.

Lemma 1.1. Let $R$ be a graded ring.

(i) If $a$ and $b$ are graded ideals of $R$, then $a + b$, $a \cap b$, and $ab$ are graded ideals of $R$.
(ii) If $a$ is an element of $h(R)$, then the cyclic ideal $aR$ of $R$ is graded.

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Let $M = \oplus_{g \in G} M_g$ be a graded $R$-module. Let $N$ be a submodule of $M$. The factor $R$-module $M/N$ becomes a $G$-graded module over $R$ with $g$-component $(M/N)_{g} = (M_{g} + N)/N$ for $g \in G$. A submodule $N$ of $M$ is called to be graded if $N = \oplus_{g \in G} N_{g}$ where $N_{g} = N \cap M_{g}$ for $g \in G$. Clearly, $0$ is a graded submodule of $M$.

If $N$ and $K$ are submodules of an $R$-module $M$, the set of all elements $r \in R$ satisfying $rK \subseteq N$ becomes an ideal of $R$ and is denoted by $(N :_{R} K)$ as usual.

**Lemma 1.2.** Let $R$ be a graded ring and $M$ be a graded $R$-module.

(i) If $N$ and $K$ are graded submodules of $M$, then $N + K$ and $N \cap K$ are graded submodules of $M$.

(ii) If $a$ is an element of $h(R)$ and $x$ is an element of $h(M)$, then $aM$ and $Rx$ are graded submodules of $M$.

(iii) If $N$ is a graded submodule of $M$ and $K$ is a graded submodule of $M$, then $(N :_{R} K)$ is a graded ideal of $R$.

**Proof.** Clearly, (i) holds. See [3, Lemma 2.2] for (ii). For the proof of (iii), see [2, Lemma 2.1] and [5, Lemma 1(ii)]. We give a proof of (iii) for our record.

To show that $(N :_{R} K)$ is a graded ideal of $R$, let $I = (N :_{R} K)$. We show $I = \oplus_{g \in G} I_{g}$. For all $g \in G$, $I_{g} = I \cap R_{g} \subseteq I$. Hence $\oplus_{g \in G} I_{g} \subseteq I$. Conversely, let $x$ be any element of $I$. Since $R$ is graded, there exist $g_{1}, g_{2}, \ldots, g_{n} \in G$ such that $x = \sum_{j=1}^{n} x_{g_{j}}$. To show that $I \subseteq \oplus_{g \in G} I_{g}$, it suffices to show that $x_{g_{j}} \in I$ since then $x_{g_{j}} \in R_{g_{j}} \cap I \subseteq I_{g_{j}}$. In turn, it suffices to show that $x_{g_{j}} K \subseteq N$.

Since $K$ is graded, $xK \subseteq N$, and $N$ is graded, we have
\[
x_{g_{j}} K = x_{g_{j}} (\oplus_{h \in G} K_{h}) = \oplus_{h \in G} x_{g_{j}} K_{h} \subseteq \oplus_{h \in G} (xK)_{g_{j} h} \subseteq \oplus_{h \in G} N_{g_{j} h} \subseteq N,
\]
as required. $\square$

**Corollary 1.3.** Let $R$ be a graded ring. If $a$ and $b$ are graded ideals of $R$, then $(a :_{R} b)$ is a graded ideal of $R$.

Let $R$ be a graded ring and $M$ be a graded $R$-module. We recall that a proper graded submodule $P$ of $M$ is prime if whenever $rm \in P$, where $r \in h(R)$ and $m \in h(M)$, then either $r \in (P :_{R} M)$ or $m \in P$.

**Definition 1.4.** Let $R$ be a graded ring and $M$ be a graded $R$-module. A proper graded submodule $Q$ of $M$ is semiprime if whenever $I^{n} K \subseteq Q$, where $I \subseteq h(R)$, $n$ is a positive integer, and $K \subseteq h(M)$, then $IK \subseteq Q$.

**Remark 1.5.** It is easy to check that a proper graded ideal $I$ of a graded ring $R$ is semiprime if and only if whenever $x^{t} y \in I$, where $x, y \in h(R)$ and $t$ is a positive integer, then $xy \in I$.

**Proposition 1.6.** Let $R$ be a graded ring and $M$ be a graded $R$-module. Then every graded prime submodule of $M$ is semiprime. Moreover, every graded prime ideal of $R$ is semiprime.
Proof. Assume that $I^n K \subseteq N$, where $n$ is a positive integer, $I \subseteq h(R)$ and $K \subseteq h(M)$. Now, since $N$ is a graded prime, we have either $I \subseteq (N : M) \subseteq (N : K)$ or $I^{n-1} K \subseteq N$. In the first case $IK \subseteq N$ and we are done. If $I^{n-1} K \subseteq N$, then $I \subseteq (N : M)$ or $I^{n-2} K \subseteq N$. In this way we have $IK \subseteq N$. Hence $N$ is a graded semiprime submodule of $M$. \qed

For basic properties of a multiplication module one may refer to [1], [4] and [6].

A graded $R$-module $M$ is said to be a graded multiplication module if for every graded submodule $N$ of $M$, there exists a graded ideal $a$ of $R$ such that $N = aM$. Let $M$ be a graded $R$-module. Assume that $M$ is a graded multiplication module. If $N$ and $K$ are graded submodules of $M$, then there exist graded ideals $a$ and $b$ of $R$ such that $N = aM$ and $K = bM$. Then the product of $N$ and $K$ is defined to be $(ab)M$ and is denoted by $N \cdot K$. It is well-known in [1, Theorem 3.4] and [5, Theorem 4] that the product is well-defined. In fact, $ab$ is a graded ideal of $R$ by Lemma 1.1 and $N \cdot K$ is independent of the choices of $a$ and $b$. Also, for every positive integer $k$, $N^k$ is defined to be

$$k \text{ times } \overbrace{N \cdot N \cdots N}.$$

Let $R$ be a graded ring and $M$ be a graded multiplication module over $R$. The graded radical of a graded submodule $N$ of $M$ is the set of all elements $m$ of $M$ such that $(Rm)^k \subseteq N$ for some positive integer $k$ and is denoted by grad$(N)$.

Remark 1.7. There were several authors who would like to define the product $x \cdot y$ of two elements $x$ and $y$ of $M$ to be $Rx \cdot Ry$ and then they used the notation “$x^n \subseteq N$ for some positive integer $n$” in their papers, such as in [1, Theorem 3.13] and in [5, Corollary 4 to Theorem 12]. If $n = 1$, then $x \subseteq N$. This does not make sense, because $x \in M$. Hence it is natural not to define the product of two elements of $M$. However, we define the product of two submodules of $M$ as in the second paragraph just posterior to the proof of Proposition 1.6.

Let $R$ be a graded ring and $M$ be a graded multiplication module over $R$. A graded submodule $N$ of $M$ is called nilpotent if $N^t = 0$ for some positive integer $t$. If a graded submodule $N$ of $M$ is nilpotent, then grad$(0) = \text{grad}(N)$.

A nonempty subset $S$ of $M$ is said to be multiplicatively closed if $(Rx)^n \cap S \neq \emptyset$ for each positive integer $n$ and each $x \in S$.

The present paper will proceed as follows. Let $R$ be a graded ring and $M$ be a graded multiplication module over $R$.

In Section 2, we characterize graded semiprime submodules of $M$ as follows.

(1) (Theorem 2.1 and its corollary) The following ten statements are equivalent for a proper graded submodule $P$ of $M$.

(i) $P$ is semiprime.
(ii) If $(Rx)^n \subseteq P$, where $x \in h(M)$ and $n$ is a positive integer, then $x \in P$. 
(iii) If $K^n \subseteq P$, where $K$ is a graded submodule of $M$ and $n$ is a positive integer, then $K \subseteq P$.
(iv) If $L$ is a graded submodule of $M$ such that $P \subseteq L \subseteq M$, then $(P :_R L)$ is a graded semiprime ideal of $R$.
(v) $(P :_R M)$ is a graded semiprime ideal of $R$.
(vi) $\text{grad}(P) = P$.
(vii) If $Rx \cdot Ry \subseteq P$, where $x, y \in h(M)$, then $Rx \cap Ry \subseteq P$.
(viii) The factor $R$-module $M/P$ has no nonzero nilpotent submodule.
(ix) There exits a graded semiprime ideal $p$ of $R$ with $(0 :_R M) \subseteq p$ such that $P = pM$.
(x) $M \setminus P$ is multiplicatively closed.

Moreover, if $M$ is regular, then we show that every proper graded submodule of $M$ is semiprime.

We give an example showing that the condition “$M$ being a multiplication module” cannot be omitted.

Using the result above, we show that the three statements are true.

(2) (Theorem 2.6) If $K$ is a graded submodule of $M$ and $S$ is a multiplicatively closed subset of $M$ such that $K \cap S = \emptyset$, then there is a graded semiprime submodule $P$ of $M$ which is maximal with respect to the properties that $K \subseteq P$ and $P \cap S = \emptyset$.

(3) (Proposition 2.8) If $N$ is a graded semiprime submodule of $M$, then it contains a minimal graded semiprime submodule.

(4) (Theorem 2.9) If $N$ is a proper graded submodule of $M$ and $M$ is finitely generated, then there exists a graded semiprime submodule of $M$ that contains $N$.

In Section 3, we define an almost semiprime submodule of $M$.

(5) (Theorem 3.5) Let $Q, K$ be graded submodules of $M$. If $Q$ and $K$ are almost semiprime in $M$ such that $Q + K \neq M$ and $Q \cap K \subseteq M_g$ for all $g \in G$, then we prove that $Q + K$ is almost semiprime in $M$.

2. Semiprime submodules

In this section, we deal with graded multiplication modules over graded rings. We define a semiprime submodule of a graded multiplication module over a graded ring to characterize it. And then we discuss several properties of semiprime submodules.

Let $M$ be a multiplication module over a ring $R$. Let $K$ be a submodule of $M$. Then there exists an ideal $I$ of $R$ such that $K = IM$. Consider the following descending chain of ideals of $R$:

$$I \supseteq I^2 \supseteq \cdots.$$ 

Then we can get a descending chain of submodules of $M$

$$K \supseteq K^2 \supseteq \cdots.$$
From this, we can see the following: if $K \subseteq N$, where $N$ is a submodule of $M$, then $K^n \subseteq N$ for every positive integer $n$. In view of this it is natural to ask a question: when $K^n \subseteq N$, where $n$ is a positive integer, under what conditions can we get $K \subseteq N$? The following result deals with this question.

**Theorem 2.1.** Let $M$ be a graded multiplication module over $R$ and $P$ be a proper graded $R$-submodule of $M$. Then the following statements are equivalent.

(i) $P$ is semiprime.

(ii) If $(Rx)^n \subseteq P$, where $x \in h(M)$ and $n$ is a positive integer, then $x \in P$.

(iii) If $K^n \subseteq P$, where $K$ is a graded submodule of $M$ and $n$ is a positive integer, then $K \subseteq P$.

(iv) If $L$ is a graded submodule of $M$ such that $P \subseteq L \subseteq M$, then $(P :_R L)$ is a graded semiprime ideal of $R$.

(v) $(P :_R M)$ is a graded semiprime ideal of $R$.

(vi) $\text{grad}(P) = P$.

(vii) If $Rx \cdot Ry \subseteq P$, where $x, y \in h(M)$, then $Rx \cap Ry \subseteq P$.

(viii) The factor $R$-module $M/P$ has no nonzero nilpotent submodule.

(ix) There exists a graded semiprime ideal $p$ of $R$ with $(0 :_R M) \subseteq p$ such that $P = pM$.

**Proof.** (i) $\Rightarrow$ (ii) Let $P$ be a graded semiprime submodule of $M$. Assume that $(Rx)^n \subseteq P$, where $x \in h(M)$ and $n$ is a positive integer. Since $M$ is a multiplication module, there exists a graded ideal $a$ of $R$ such that $Rx = aM$. Then

$$a^nM = (aM)^n = (Rx)^n \subseteq P.$$ 

Since $P$ is a graded semiprime submodule of $M$, we have $Rx = aM \subseteq P$. Therefore $x \in P$.

(ii) $\Rightarrow$ (iii) Assume that $K^n \subseteq P$, where $K$ is a graded submodule of $M$ and $n$ is a positive integer. To show that $K \subseteq P$, it suffices to show that every element $x$ of $h(K)$ belongs to $P$. Let $x$ be an arbitrary element of $h(K)$. Then $x \in h(M)$ and $(Rx)^n \subseteq K^n \subseteq P$. By (ii), $x \in P$.

(iii) $\Rightarrow$ (iv) Assume that (iii) is true. Assume that $L$ is a graded submodule of $M$ such that $P \subseteq L \subseteq M$. Then $(P :_R L)$ is proper. By Lemma 1.2, $(P :_R L)$ is graded.

Also, assume that $a^n b \subseteq (P :_R L)$, where $n$ is a positive integer and $a$ and $b$ are graded ideals of $R$. Then

$$(ab)L^n = (ab)^nL = b^{n-1}((a^n b)L) \subseteq b^{n-1}P \subseteq P.$$ 

Notice that $(ab)L$ is a graded submodule of $M$. Then by (iii) we have $(ab)L \subseteq P$. This shows that $ab \subseteq (P :_R L)$. Hence $(P :_R L)$ is a semiprime ideal.

(iv) $\Rightarrow$ (v) Assume that (iv) is true. Taking $L$ by $M$, we can see that $(P :_R M)$ is a graded semiprime ideal of $R$.

(v) $\Rightarrow$ (vi) Assume that (v) is true. Clearly, $P \subseteq \text{grad}(P)$. Conversely, assume that $(Rx)^n \subseteq P$ for some positive integer $n$. Then we need to show
that \( x \in P \). If \( n = 1 \), then \( x \in P \); we are done. Assume that \( n > 1 \). Since \( M \) is a graded multiplication module, there is a graded ideal \( \alpha \) of \( R \) such that \( Rx = \alpha M \). Then
\[
\alpha^n M = (Rx)^n \subseteq P.
\]
So, \( \alpha^{n-1} \alpha = \alpha^n \subseteq (P :_R M) \). Since \( (P :_R M) \) is graded semiprime, we get \( \alpha \subseteq (P :_R M) \). Hence
\[
x \in Rx = \alpha M \subseteq (P :_R M)M = P,
\]
as required.

(vi) \( \Rightarrow \) (vii) Assume that (vi) is true. Assume that \( Rx \cdot Ry \subseteq P \), where \( x, y \in h(M) \). Let \( m \) be an arbitrary element of \( Rx \cap Ry \). Then \( Rm \subseteq Rx \) and \( Rm \subseteq Ry \). Hence
\[
(Rm)^2 \subseteq (Rx) \cdot (Ry) \subseteq P.
\]
By (vi), \( Rm \subseteq P \). Hence \( m \in P \). This shows that \( Rx \cap Ry \subseteq P \).

(vii) \( \Rightarrow \) (viii) Assume that (vii) is true. Let \( x + P \) be an arbitrary nilpotent element of \( M/P \). Then there exists a positive integer \( n \) such that \( ((Rx + P)^{n}/P) = 0 \) in \( M/P \). There exists a graded ideal \( \alpha \) of \( R \) such that \( Rx = \alpha M \). So,
\[
((Rx)^n + P)/P = (\alpha^n M + P)/P = \alpha^n(M/P) = ((Rx + P)^{n}/P) = 0.
\]
This implies that \( (Rx)^n \subseteq P \). By (vii),
\[
x \in Rx = n \text{ times}
\]
Hence \( x + P = 0 + P \).

(viii) \( \Rightarrow \) (ix) Assume that (viii) is true. Since \( M \) is a graded multiplication module, there exists a graded ideal \( p \) of \( R \) such that \( P = pM \). To show that \( p \) is semiprime, assume that \( \alpha^n b \subseteq p \), where \( \alpha \) and \( b \) are graded ideals of \( R \). Then \( (\alpha b)^n \subseteq p \). So,
\[
((\alpha b)M)^n = (\alpha b)^n M \subseteq pM = P.
\]
This means that
\[
((\alpha b)M + P)/P)^n = ((\alpha b)M)^n + P)/P = \{0 + P\}.
\]
By (viii), \( ((\alpha b)M + P)/P = \{0 + P\} \). This implies that
\[
(\alpha b)M \subseteq ((\alpha b)M + P = P = pM.
\]
Since \( M \) is multiplication, it follows that \( \alpha b \subseteq p \). Therefore \( p \) is semiprime.

Also, let \( \alpha \) be an arbitrary element of \( 0 :_R M \). Then \( \alpha M = 0 \subseteq pM \). Since \( M \) is multiplication, it follows that \( \alpha \in p \). Hence \((0 :_R M) \subseteq p \).

(ix) \( \Rightarrow \) (i) Assume that (ix) is true. To show that \( P \) is semiprime, assume that \( \alpha^n K \subseteq P \), where \( \alpha \) is a graded ideal of \( R \) and \( K \) is a graded submodule of \( M \), and \( n \) is a positive integer. Since \( M \) is a graded multiplication module, there exists a graded ideal \( b \) of \( R \) such that \( K = bM \). Then
\[
(\alpha^n b)M = \alpha^n K \subseteq P = pM.
\]
Since \( p + (0 :_R M) = p \), it follows from [6, Theorem 9, p. 231] that either \( a^n b \subseteq p \) or \( M = (p :_R a^n b)M \). If \( a^n b \subseteq p \), then we have \( ab \subseteq p \) since \( p \) is semiprime. Hence \( aK = a(bM) = (ab)M \subseteq pM = P \); we are done. Or, assume that \( M = (p :_R a^n b)M \). Notice that
\[
(a^n(p :_R a^n b)b) = (p :_R a^n b)a^n b \subseteq p.
\]
Since \( p \) is semiprime, we have \( (p :_R a^n b)ab \subseteq p \). Hence
\[
aK = a(bM) = (ab)M = ((p :_R a^n b)ab)M \subseteq pM = P.
\]
Hence \( P \) is semiprime. \( \square \)

**Corollary 2.2.** Let \( R \) be a graded ring and \( M \) be a graded multiplication module over \( R \). Then a proper graded submodule \( P \) of \( M \) is semiprime if and only if \( M \setminus P \) is multiplicatively closed.

**Proof.** Let \( P \) be a graded semiprime submodule of \( M \) and let \( x \in M \setminus P \). Since \( P \) is graded semiprime, it follows from Theorem 2.1 that \( (Rx)^n \subseteq P \) for every positive integer \( n \). Hence \( (Rx)^n \cap (M \setminus P) \neq \emptyset \). This shows that \( M \setminus P \) is multiplicatively closed.

Conversely, assume that \( M \setminus P \) is multiplicatively closed. To show that \( P \) is semiprime, assume that \( (Rx)^n \subseteq P \), where \( n \) is a positive integer and \( x \in h(M) \). We need to show that \( x \notin P \). Suppose on the contrary that \( x \notin P \). Then \( x \in M \setminus P \). By our assumption, \( (Rx)^n \cap (M \setminus P) \neq \emptyset \). Take \( y \in (Rx)^n \cap (M \setminus P) \). Then \( y \in (Rx)^n \subseteq P \). This contradiction shows that \( x \in P \); as needed. \( \square \)

Let \( M \) be a graded multiplication module over a graded ring \( R \). Then \( N \cdot K \subseteq N \cap K \) for each pair of graded submodules \( N \) and \( K \) of \( M \). \( M \) is said to be **regular** if for each pair of graded submodules \( N \) and \( K \) of \( M \), \( N \cdot K = N \cap K \).

**Corollary 2.3.** Let \( R \) be a graded ring and \( M \) be a regular graded multiplication module over \( R \). Then every proper graded submodule of \( M \) is semiprime.

The condition “\( M \) being multiplication” in Theorem 2.1 cannot be omitted. The example of this is given below.

**Example 2.4.** First, consider the set \( \mathbb{Z} \) of all integers. Then \( (\mathbb{Z}, +) \) is a group with additive identity \( 0 \) and \( (\mathbb{Z}, +, \cdot) \) is a commutative ring with identity \( 1 \). Take \( G = (\mathbb{Z}, +) \) and \( R = (\mathbb{Z}, +, \cdot) \). Define
\[
R_g = \begin{cases} 
\mathbb{Z} & \text{if } g = 0 \\
0 & \text{otherwise.}
\end{cases}
\]
Then each \( R_g \) is an additive subgroup of \( R \) and \( R \) is their internal direct sum. In fact, \( 1 \in R_0 \) and \( R_g R_h \subseteq R_{g+h} \). That is, \( R = \bigoplus_{g \in G} R_g \). Hence \( R \) is a \( G \)-graded ring. In other words, the ring \( (\mathbb{Z}, +, \cdot) \) of integers is a \( (\mathbb{Z}, +) \)-graded ring.
Next, let $M$ be the set $\mathbb{Z} \times \mathbb{Z}$. Then $M$ can be given a $\mathbb{Z}$-module structure. Define
\[
M_g = \begin{cases} 
\mathbb{Z} \times 0 & \text{if } g = 0 \\
0 \times \mathbb{Z} & \text{if } g = 1 \\
0 \times 0 & \text{otherwise.}
\end{cases}
\]
Then $M = \bigoplus_{g \in G} M_g$. Hence $M$ is a $G$-graded $R$-module. In other words, the $\mathbb{Z}$-module $(\mathbb{Z} \times \mathbb{Z}, +, \cdot)$ is a $\mathbb{Z}$-graded $\mathbb{Z}$-module.

Now, consider a submodule $N = 9\mathbb{Z} \times 0$ of $M$. Then it is a graded submodule. $(N :_R M) = 0$ and so it is a graded semiprime ideal of $R$. But the graded submodule $N$ is not graded semiprime in $M$, since $3^2(2, 0) \in N$ but $3(2, 0) \notin N$.

By Theorem 2.1, we can see that the $\mathbb{Z}$-module $(\mathbb{Z} \times \mathbb{Z}, +, \cdot)$ is not a multiplication module.

**Lemma 2.5.** Let $R$ be a graded ring and $M$ be a graded $R$-module. If $P$ is a graded submodule of $M$ and $x \in h(M)$, then both $Rx$ and $P + Rx$ are graded submodules of $M$.

**Proof.** This follows from Lemma 1.2. □

**Theorem 2.6.** Let $R$ be a graded ring and $M$ be a graded multiplication module over $R$. Let $K$ be a graded submodule of $M$ and $S$ be a multiplicatively closed subset of $M$ such that $K \cap S = \emptyset$. Then there is a graded semiprime submodule $P$ of $M$ which is maximal with respect to the properties that $K \subseteq P$ and $P \cap S = \emptyset$.

**Proof.** Let $\Omega$ be the set of all graded submodules $L$ of $M$ such that $K \subseteq L$ and $L \cap S = \emptyset$. $K \in \Omega$, so in particular $\Omega \neq \emptyset$. By the Zorn lemma $\Omega$ has a maximal element, say $P$. It is enough to show that $P$ is semiprime. To show that $P$ is semiprime, assume that $(Rx)^n \subseteq P$, where $n$ is a positive integer and $x \in h(M)$. Then we need to show that $x \in P$. Suppose on the contrary that $x \notin P$. Then $P \subset P + Rx$. By Lemma 2.5, $P + Rx$ is graded. By the maximality of $P$, $P + Rx \notin \Omega$. Hence $(P + Rx) \cap S = \emptyset$. Take $y \in (P + Rx) \cap S$. Then $y \in P + Rx$ and $y \in S$. Since $M$ is a multiplication module and $(Rx)^n \subseteq P$, we can show that
\[
(P + Rx)^n \subseteq P + (Rx)^n = P.
\]
Also, since $S$ is multiplicatively closed and $y \in S$, we have $(Ry)^n \cap S \neq \emptyset$. Hence
\[
\emptyset \neq (Ry)^n \cap S \subseteq (P + Rx)^n \cap S \subseteq P \cap S,
\]
contradicting the disjointness of $P$ and $S$. This shows that $x \in P$. Therefore $P$ is a graded semiprime submodule. □

**Lemma 2.7.** Let $R$ be a graded ring and $M$ be a graded multiplication module over $R$. Let $\Omega$ be a nonempty family of graded submodules of $M$.

(i) If each member of $\Omega$ is semiprime in $M$, then so is $\bigcap_{Q \in \Omega} Q$.

(ii) If each member of $\Omega$ is semiprime in $M$, $\Omega$ is totally ordered by inclusion, and $\bigcup_{Q \in \Omega} Q \neq M$, then $\bigcup_{Q \in \Omega} Q$ is a proper graded semiprime submodule of $M$.\]
Proof. (i) Assume that each member of $\Omega$ is semiprime in $M$. Then by Theorem 2.1,

$$\text{grad}(\bigcap_{Q \in \Omega} Q) \cap h(M) \subseteq (\bigcap_{Q \in \Omega} \text{grad}(Q)) \cap h(M)$$

$$= \bigcap_{Q \in \Omega} (\text{grad}(Q) \cap h(M))$$

$$= \bigcap_{Q \in \Omega} (Q \cap h(M))$$

$$= (\bigcap_{Q \in \Omega} Q) \cap h(M).$$

It is clear that the converse inclusion holds. Hence by Theorem 2.1 again, $\bigcap_{Q \in \Omega} Q$ is semiprime.

(ii) Assume that $\Omega$ is totally ordered by inclusion and $\cup_{Q \in \Omega} Q \neq M$. Then it is clear that $\cup_{Q \in \Omega} Q$ is a proper graded submodule of $M$. Now assume that each member of $\Omega$ is semiprime in $M$. Then by Theorem 2.1,

$$\text{grad}(\cup_{Q \in \Omega} Q) \cap h(M) \subseteq (\cup_{Q \in \Omega} \text{grad}(Q)) \cap h(M)$$

$$= \cup_{Q \in \Omega} (\text{grad}(Q) \cap h(M))$$

$$= \cup_{Q \in \Omega} (Q \cap h(M))$$

$$= (\cup_{Q \in \Omega} Q) \cap h(M).$$

It is clear that the converse inclusion holds. Hence by Theorem 2.1 again, $\cup_{Q \in \Omega} Q$ is semiprime. □

A graded semiprime submodule $P$ of a graded $R$-module $M$ is said to be minimal if whenever $N \subseteq P$ and $N$ is graded semiprime, then $N = P$.

**Proposition 2.8.** Let $R$ be a graded ring and $M$ be a graded multiplication module over $R$. If $N$ is a graded semiprime submodule of $M$, then it contains a minimal graded semiprime submodule.

**Proof.** Consider the set $\Sigma$ of all graded semiprime submodules $P$ of $M$ such that $N \supseteq P$. Since $N \in \Sigma$ we see that $\Sigma$ is not empty. Also $\Sigma$ is a partial order on $\Sigma$. Let $\Omega$ be a non-empty subset of $\Sigma$ which is totally ordered by $\supseteq$. Therefore by Lemma 2.7(i), $\bigcap_{P \in \Omega} P$ is a graded semiprime submodule of $M$. Now the result holds by applying the Zorn lemma. □

**Theorem 2.9.** Let $R$ be a graded ring and $M$ be a graded multiplication module over $R$. If $N$ is a proper graded submodule of $M$ and if $M$ is finitely generated, then there exists a graded semiprime submodule of $M$ that contains $N$.

**Proof.** Assume that $N$ is a proper graded submodule of $M$ and $M$ is finitely generated. Let $\Sigma$ be the collection of all proper graded submodules of $M$ that contains $N$. Then $N \in \Sigma$. In particular, $\Sigma \neq \emptyset$. Order $\Sigma$ by inclusion. Then $\Sigma$ is partially ordered. Let $\Omega$ be any chain of $\Sigma$. Take $Q^* = \cup_{Q \in \Omega} Q$. Then by Lemma 2.7(ii), $Q^* \in \Sigma$. $\Omega$ has an upper bound in $\Sigma$. By the Zorn lemma, $\Sigma$ has a maximal member, say $P$. It remains to prove that $P$ is semiprime.

Suppose that $\text{grad}(P) \cap h(M) \neq P \cap h(M)$. Then we can take an element $x \in (\text{grad}(P) \cap h(M)) \setminus (P \cap h(M))$. Then $x \notin P$, so $P \subset P + Rx$. By
Lemma 2.7(ii) and by the maximality of $P$, we must have $P + Rx = M$. Since $x \in \text{grad}(P)$, there exists a positive integer $n$ such that $x^n \in P$. Hence
\[ M = M^n = (P + Rx)^n \subseteq P + (Rx)^n \subseteq P, \]
so $M = P$. This contradiction shows that $\text{grad}(P) \cap h(M) = P \cap h(M)$. Therefore it follows from Theorem 2.1 that $P$ is semiprime. \qed

3. Almost semiprime submodules

In this section we define an almost semiprime submodule of a graded multiplication module over a graded ring and discuss the sum of two almost semiprime submodules.

Let $R$ be a graded ring and $M$ be a graded multiplication module over $R$. Let $Q$ be a proper graded submodule of $M$. Then $Q \cap h(M) \subseteq \text{grad}(Q) \cap h(M)$. The following two statements are true:
\[
\text{grad}(0_M) \cap h(M) \subseteq \text{grad}(Q) \cap h(M),
\]
\[
\text{grad}(0_M) \cap Q \cap h(M) \subseteq Q \cap h(M).
\]

More precisely, we can draw their lattice diagram as follows:

\[ \begin{align*}
\text{grad}(Q) \cap h(M) & \\ & \text{grad}(0_M) \cap h(M) \\
\text{grad}(0_M) \cap Q \cap h(M) & \\ & Q \cap h(M)
\end{align*} \]

Then it is easy to see that
\[
(Q \cap h(M)) \setminus (\text{grad}(0_M) \cap Q \cap h(M)) \\
\subseteq (\text{grad}(Q) \cap h(M)) \setminus (\text{grad}(0_M) \cap h(M)).
\]

**Remark 3.1.** This statement is the same as the following one but the following one is much easier for us to make sure if it is true.
\[
(Q \setminus \text{grad}(0_M)) \cap h(M) \subseteq (\text{grad}(Q) \setminus \text{grad}(0_M)) \cap h(M).
\]

**Definition 3.2.** Let $R$ be a graded ring and $M$ be a graded multiplication module over $R$. A proper graded submodule $Q$ of $M$ is said to be **almost semiprime** if
\[
(\text{grad}(Q) \cap h(M)) \setminus (\text{grad}(0_M) \cap h(M)) \\
= (Q \cap h(M)) \setminus (\text{grad}(0_M) \cap Q \cap h(M)).
\]

Let $g \in G$. Likewise, a proper graded submodule $Q_g$ of the $R_e$-module $M_g$ is said to be **almost $g$-semiprime** if
\[
(\text{grad}(Q_g) \cap M_g) \setminus (\text{grad}(0_{M_g}) \cap M_g) = Q_g \setminus (\text{grad}(0_{M_g}) \cap Q_g).
\]
It is immediate that the zero submodule of a graded multiplication module is graded and almost semiprime.

Let $R$ be a graded ring and $M$ be a graded multiplication module over $R$. Let $Q$ be a proper graded submodule of $M$. Assume that $Q$ is semiprime. Then it follows from Theorem 2.1 that $\text{grad}(Q) \cap h(M) = Q \cap h(M)$, so that $\text{grad}(0_M) \cap h(M) = \text{grad}(0_M) \cap Q \cap h(M)$. Hence $Q$ is almost semiprime. This shows that every semiprime submodule of $M$ is almost semiprime. Conversely, if $Q$ is almost semiprime and $\text{grad}(0_M) \cap h(M) = \text{grad}(0_M) \cap Q \cap h(M)$, then $Q$ is semiprime.

Proposition 3.3. Let $R$ be a graded ring, $M$ be a graded multiplication module over $R$ and $Q$ be a proper graded submodule of $M$. If $Q$ is almost semiprime, then for every $g \in G$, $Q_g$ is almost $g$-semiprime in $M_g$.

Proof. Assume that $Q$ is almost semiprime. Then the equality (3.1) holds. Let $g \in G$. Note that $Q = \oplus_{g \in G} Q_g$. Then taking the intersection of the equation (3.1) with $M_g$, we can get (3.2). Hence $Q_g$ is almost semiprime. □

Lemma 3.4. Let $R$ be a graded ring, $M$ a graded multiplication module over $R$ and $K, Q$ graded submodules of $M$ such that $K \subseteq Q$. Then the following statements are true.

(i) If $Q$ is almost semiprime such that $K \subseteq M_g$ for all $g \in G$, then $Q/K$ is almost semiprime in $M/K$.

(ii) If $K$ and $Q/K$ are almost semiprime in $M$ and $M/K$, respectively, then $Q$ is almost semiprime in $M$.

Proof. If $K \subseteq Q$, then we have already known that $M/K$ and $Q/K$ are $G$-graded.

(i) Assume that $Q$ is almost semiprime such that $K \subseteq M_g$ for all $g \in G$. Then $K \subseteq \bigcup_{g \in G} M_g = h(M)$ and

$$h(M/K) = \bigcup_{g \in G} ((M_g + K)/K) = \bigcup_{g \in G} (M_g/K) = h(M)/K.$$ 

Now since the equality (3.1) holds, direct computation gives

$$\text{grad}(Q/K) \cap h(M/K) \backslash \text{grad}(0_M/K) \cap h(M/K) = (Q/K \cap h(M/K)) \backslash (\text{grad}(0_M/K) \cap Q/K \cap h(M/K)).$$

Hence $Q/K$ is almost semiprime.

(ii) In order to show that $Q$ is almost semiprime, we show that (3.1) holds. Let $x$ belong up in the equality (3.1). Then $(Rx)^s \subseteq Q$ for some positive integer $s$. This implies that $Q(x + K)^s = ((Rx)^s + K)/K$ is in $Q/K$. Hence $x + K \in \text{grad}(Q/K)$. Now, there are two cases to consider.

Case 1. Assume that $x + K$ is in $\text{grad}(0_M/K)$. Then there exists a positive integer $t$ such that $(Rx + K)^t = 0$ in $M/K$. So, $(Rx)^t \subseteq K$. This implies that $x \in \text{grad}(K)$. Since $K$ is almost semiprime, we have

$$x \in (\text{grad}(K) \cap h(M)) \backslash (\text{grad}(0_M) \cap h(M)) = (K \cap h(M)) \backslash (\text{grad}(0_M) \cap K \cap h(M)).$$
Hence since $K \subseteq Q$, $x$ belongs down in the equality (3.1).

Case 2. Assume that $x + K$ is not in $\text{grad}(0_{M/K})$. Then $x + K$ belongs up in the equality (3.3). Since $Q/K$ is almost semiprime, the equality (3.3) holds. Hence $x + K$ belongs down in the equality (3.3). This implies that $x + K \in Q/K$. Then there exists an element $y \in Q$ such that $x + K = y + K$. This implies that $x = (x-y) + y \in K + Q = Q$ since $K \subseteq Q$. Hence $x$ belongs down in the equality (3.1). This shows that the equality (3.1) holds. Therefore $Q$ is almost semiprime.

Theorem 3.5. Let $R$ be a graded ring, $M$ be a graded multiplication module over $R$ and $K, Q$ be graded submodules of $M$. If $K$ and $Q$ are almost semiprime in $M$ such that $Q + K \neq M$ and $Q \cap K \subseteq M_g$ for all $g \in G$, then $Q + K$ is almost semiprime in $M$.

Proof. Assume that $Q$ and $K$ are almost semiprime in $M$ such that $Q + K \neq M$ and $Q \cap K \subseteq M_g$ for all $g \in G$. Then Lemma 3.4(i), $Q/(Q \cap K)$ is also almost semiprime in $M/(Q \cap K)$. Notice that $Q/(Q \cap K) \cong (Q + K)/K$ by the second isomorphism theorem for modules. Then $(Q + K)/K$ is almost semiprime in $M/K$. Hence by Lemma 3.4(ii), $Q + K$ is almost semiprime.

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