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Approximation of $M/G/c$ Retrial Queue with $M/\mathrm{PH}/c$ Retrial Queue

Yang Woo Shin[1,a], Dug Hee Moon[b]

[a]Department of Statistics, Changwon National University
[b]Department of Industrial and Systems Engineering, Changwon National University

Abstract

The sensitivity of the performance measures such as the mean and the standard deviation of the queue length and the blocking probability with respect to the moments of the service time are numerically investigated. The service time distribution is fitted with phase type(PH) distribution by matching the first three moments of service time and the $M/G/c$ retrial queue is approximated by the $M/\mathrm{PH}/c$ retrial queue. Approximations are compared with the simulation results.

Keywords: $M/G/c$ retrial queue, phase type distribution, sensitivity, approximation, simulation.

1. Introduction

We consider an $M/G/c$ retrial queue in which customers arrive from outside according to a Poisson process with rate λ and there are c identical servers and no waiting positions in the service facility. When an arriving customer finds all the servers busy, the customer joins orbit and repeats its request after an exponential amount of time with rate γ until the customer gets into the service facility. Let S be the service time of a customer whose distribution function is $G(x)$ and assume that $G(0) = 0$, $m_k = E(S^k) < \infty$, $k = 1, 2, 3$. We also assume $\rho = \lambda m_1/c < 1$ for the stability of the system.

Retrial queues have been widely used to model the many practical situations in telephone systems and telecommunication networks. Even for the Markovian retrial queues with multiple servers, the exact results have not been obtained except for some special cases. Instead, attempts to develop algorithmic or approximation methods have been extensively made. However, there are few results for approximations of the $M/G/c$ retrial queues in general. For the literature of algorithms or approximation methods for retrial queues, we refer the monographs Artalejo and Gómes-Correl (2008), Falin and Templeton (1997) and references.

A distribution function $F(x)$ on $(0, \infty)$ is said to be of phase type with representation $(\boldsymbol{\alpha}, T)$ and denote it by $\mathrm{PH}(\boldsymbol{\alpha}, T)$ if $F(x) = 1 - \boldsymbol{\alpha} \exp(Tx)\boldsymbol{e}$, where $\boldsymbol{e}$ is the column m-vector whose components are all 1, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is a probability distribution and $T = (t_{ij})$ is the $m\times m$ matrix with $t_{ii} < 0$, $1 \le i \le m$ and $t_{ij} \ge 0$, $i \ne j$, and $T\boldsymbol{e} \le \mathbf{0}\,(\ne \mathbf{0})$. For more details about phase type(PH) distribution, see Chapter 2 of Neuts (1981). It is well known that the set of PH-distributions is dense (in the sense of weak convergence) in the set of all probability distributions on $(0, \infty)$ (*e.g.* see Asmussen, 2003, p.84).

There are some algorithmic methods for the retrial queue with PH distribution of service time *e.g.* Breuer *et al.* (2002), Diamond and Alfa (1999), Artalejo and Gómes-Correl (2008, Chapter 8).

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[1] Corresponding author: Professor, Department of Statistics, Changwon National University, Changwon, Gyeongnam 641-773, Korea. E-mail: ywshin@changwon.ac.kr

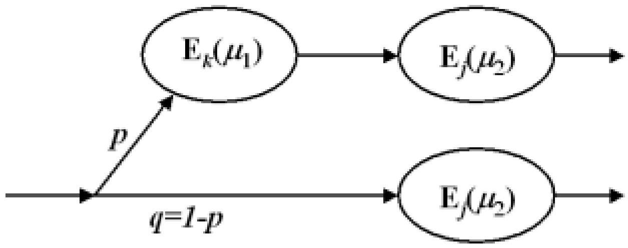

Figure 1: *Diagram for* $\mathrm{CE}_{k,j}(p;\mu_1,\mu_2)$ *distribution*

However, as far as the authors know, there is no literature on the method of how to choose the PH distribution for approximation of $M/G/c$ retrial queue. In this paper, the effects of the moments of the service time to the performance measures in $M/G/c$ retrial queues are investigated numerically. Based on the numerical experiments, the service time distribution is fitted with phase type distribution by matching the first three moments of service time and the $M/G/c$ retrial queue is approximated with the $M/\mathrm{PH}/c$ retrial queue.

In Section 2, we numerically investigate the sensitivity of some performance measures with respect to the moments of service time. An approximation method of service time distribution in $M/G/c$ retrial queue with PH distribution is proposed in Section 3. Some numerical results and concluding remarks are presented in Section 4 and Section 5, respectively.

2. Sensitivity of $M/G/c$ Retrial Queue

Let X_0 and X_1 be the number of busy servers and customers in orbit in steady state, respectively and set $L_0 = \mathbb{E}(X_0)$, $L_1 = \mathbb{E}(X_1)$. By V_0 and V_1, denote the variances of X_0 and X_1, respectively and set $\sigma_i = \sqrt{V_i}$, $i = 0, 1$. It follows from Little's formula that $L_0 = \lambda m_1$ does not depend on the second or the higher moments of the service time. For the single server case, it can be seen that that L_1 depend only on the arrival rate λ and the first two moments m_1 and m_2 of the service time and σ_1 is determined by λ and m_k, $k = 1, 2, 3$ (Falin and Templeton, 1997).

In this section, we investigate numerically how the performance measures are affected by the moments of the service time in multi-server case. For numerical experiments, let $\mathrm{E}_k(\mu)$ be the Erlang distribution of order k with parameter μ and denote by $\mathrm{CE}_{k,j}(p;\mu_1,\mu_2)$ the Coxian distribution with Erlang node that is the composition of the mixture of $E_k(\mu_1)$ and $E_j(\mu_2)$ (see Figure 1) whose Laplace transform $f^*(s)$ is given by

$$f^*(s) = p\left(\frac{\mu_1}{\mu_1 + s}\right)^k \left(\frac{\mu_2}{\mu_2 + s}\right)^j + (1-p)\left(\frac{\mu_2}{\mu_2 + s}\right)^j, \quad s \geq 0.$$

We also consider the hyper-exponential distribution of order 2, denoted by $H_2(p;\mu_1,\mu_2)$ or simply H_2, with the probability density function of the form

$$f(t) = p\mu_1 e^{-\mu_1 t} + (1-p)\mu_2 e^{-\mu_2 t}, \quad t \geq 0.$$

For investigation of the influence of the moment of service time, we choose the distributions as in Table 1 and the numerical results for the blocking probability $P_B = \mathbb{P}(X_0 = c)$, V_0, L_1 and V_1 for $m_1 = 1.0$ and three cases of squared coefficient of variation of the service time $C_s^2 = (m_2 - m_1^2)/m_1^2 = 0.5, 2.0, 5.0$ are listed in Table 2.

Table 1: The moments of the distributions used in Table 2

C_s^2	Service time	m_3	m_4
0.5	$E_2(2.0)$	3.0	7.5
	$\mathrm{CE}_{3,1}(0.727834; 5.63299, 1.63299)$	3.0	7.6
	$\mathrm{CE}_{1,3}(0.000416858; 0.0698932, 3.018)$	10.0	454.1
2.0	$\mathrm{CE}_{1,3}(0.235821; 0.465637, 6.07839)$	18.0	153.1
	$H_2(0.788675; 1.57735, 0.42265)$	18.0	162.0
	$H_2(0.975327; 1.126717, 0.183618)$	28.0	535.3
5.0	$\mathrm{CE}_{1,2}(0.200212; 0.269343, 7.79223)$	66.0	979.6
	$H_2(0.788675; 3.73205, 0.267949)$	66.0	984.0
	$H_2(0.988848; 1.17673, 0.0698478)$	200.0	11257.0

Table 2: Effects of moments of service time in $M/G/3$ retrial queue with $\mu = 1.0$

γ	m_3	m_4	$\rho = 0.4$				$\rho = 0.8$			
			P_B	V_0	L_1	V_1	P_B	V_0	L_1	V_1
			$C_s^2 = 0.5$							
0.1	3.0	7.5	0.1129	0.9326	1.5927	2.5330	0.5591	0.6029	32.17	134.1
	3.0	7.6	0.1129	0.9326	1.5926	2.5350	0.5591	0.6028	32.17	134.1
	10.0	454.1	0.1129	0.9325	1.5902	2.5140	0.5590	0.6026	32.16	137.7
1.0	3.0	7.5	0.1196	0.9596	0.2317	0.3918	0.5779	0.6574	5.069	22.47
	3.0	7.6	0.1195	0.9592	0.2317	0.3918	0.5778	0.6570	5.068	22.47
	10.0	454.1	0.1188	0.9567	0.2256	0.3713	0.5764	0.6530	5.029	24.95
5.0	3.0	7.5	0.1296	0.9956	0.1081	0.1956	0.6093	0.7465	2.620	12.36
	3.0	7.6	0.1294	0.9950	0.1082	0.1956	0.6090	0.7458	2.620	12.36
	10.0	454.1	0.1283	0.9916	0.1021	0.1781	0.6067	0.7393	2.569	14.35
			$C_s^2 = 2.0$							
0.1	18.0	153.1	0.9402	3.5949	1.6761	0.1147	0.5625	0.6128	34.33	256.9
	18.0	162.0	0.9397	3.6076	1.6755	0.1146	0.5625	0.6125	34.32	257.0
	28.0	535.3	0.9398	3.5771	1.6718	0.1146	0.5623	0.6122	34.30	262.5
1.0	18.0	153.1	0.1262	0.9851	0.2976	0.7352	0.5931	0.7016	7.068	60.66
	18.0	162.0	0.1257	0.9831	0.2983	0.7354	0.5926	0.6999	7.067	60.62
	28.0	535.3	0.1248	0.9796	0.2866	0.6890	0.5907	0.6941	6.998	64.65
5.0	18.0	153.1	0.1366	1.0194	0.1609	0.4468	0.6261	0.7946	4.479	42.12
	18.0	162.0	0.1361	1.0175	0.1633	0.4486	0.6256	0.7924	4.491	42.07
	28.0	535.3	0.1349	1.0139	0.1515	0.4038	0.6230	0.7853	4.401	45.37
			$C_s^2 = 5.0$							
0.1	66.0	979.6	0.1170	0.9496	1.8363	6.2738	0.5682	0.6292	38.54	551.7
	66.0	984.0	0.1170	0.9494	1.8362	6.2787	0.5682	0.6291	38.53	551.7
	200.0	11257.0	0.1170	0.9499	1.7939	5.8355	0.5673	0.6262	38.38	620.5
1.0	66.0	979.6	0.1323	1.0052	0.4338	1.8433	0.6102	0.7501	11.00	183.6
	66.0	984.0	0.1321	1.0046	0.4343	1.8428	0.6099	0.7493	11.00	183.5
	200.0	11257.0	0.1279	0.9914	0.3419	1.3613	0.6004	0.7217	10.40	232.8
5.0	66.0	979.6	0.1415	1.0331	0.2864	1.3811	0.6404	0.8325	8.276	148.1
	66.0	984.0	0.1413	1.0324	0.2876	1.3807	0.6400	0.8312	8.284	148.1
	200.0	11257.0	0.1373	1.0219	0.1928	0.9144	0.6306	0.8070	7.504	189.7

Table 2 shows that P_B and V_0 are affected weakly by the second or the higher moments of the service time that is expected from the results for the system with $c = 1$. We can also see from Table 2 that L_1 and V_1 seem to depend on the third moment m_3 as well as the second moment m_2. We conclude from Table 2 that for an accurate approximation of the mean and variance of X_0 and X_1 in $M/G/c$ retrial queue using another system, the first three moments of service time should be consistent with those of the original system.

3. Approximations

There are some moment matching methods for fitting the general distribution by the PH distributions. In this section we briefly introduce moment matching methods to be used for approximation.

Hyper-exponential distribution: If a positive random variable X with the first three moments $m_i = \mathbb{E}(X^i)$, $i = 1, 2, 3$ and the squared coefficient of variation satisfy $C_s^2 > 1$ and

$$m_1 m_3 > \frac{3}{2} m_2^2, \tag{3.1}$$

then the distribution $\mathrm{H}_2(p; \mu_1, \mu_2)$ with the preassigned moments m_i, $i = 1, 2, 3$ is uniquely determined by the parameters (see Whitt, 1982 or Tijms, 2003)

$$\mu_{1,2} = \frac{1}{2}\left(a_1 \pm \sqrt{a_1^2 - 4a_2}\right), \qquad p = \frac{\mu_1(1 - \mu_2 m_1)}{\mu_1 - \mu_2}, \tag{3.2}$$

where

$$a_2 = \frac{6m_1^2 - 3m_2}{3/2\, m_2^2 - m_1 m_3}, \qquad a_1 = \frac{1}{m_1}\left(1 + \frac{1}{2} m_2 a_2\right).$$

The requirement (3.1) holds for the gamma distribution, lognormal distribution and Weibul distribution with $C_s^2 > 1$.

Coxian distribution with Erlang node: Bobbio *et al.* (2005) present explicit method to fit the first three moments of a positive random variable by $\mathrm{CE}_{1,j}(p; \mu_1, \mu_2)$ and $\mathrm{CE}_{k,1}(p; \mu_1, \mu_2)$; however, the formulae to determine the parameters are too complicated and are omitted here.

Mixture of Erlang distributions of common order: Johnson and Taaffe (1989) provide a method that a mixture $\mathrm{E}_{k,k}(p; \mu_1, \mu_2)$ of two Erlang distributions $\mathrm{E}_k(\mu_1)$ and $\mathrm{E}_k(\mu_2)$ with probability density function

$$f(t) = p\mu_1 \frac{(\mu_1 t)^{k-1}}{(k-1)!} e^{-\mu_1 t} + (1 - p)\mu_2 \frac{(\mu_2 t)^{k-1}}{(k-1)!} e^{-\mu_2 t}$$

can fit the first three moments m_1, m_2 and m_3 of a positive random variable X. The parameters are given by

$$\mu_{1,2}^{-1} = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right), \qquad p = \frac{\mu_1 - \mu_1\mu_2 m_1/k}{\mu_2 - \mu_1},$$

where

$$a = k(k+2)m_1 y, \qquad b = -\left(kx + \frac{k(k+2)}{k+1} y^2 + (k+2)m_1^2 y\right), \qquad c = m_1 x,$$

$$y = m_2 - \left(\frac{k+1}{k}\right) m_1^2, \qquad x = m_1 m_3 - \left(\frac{k+2}{k+1}\right) m_2^2.$$

Once the service time is approximated by a PH distribution, the $M/\mathrm{PH}/c$ retrial queue can be easily modeled by a level dependent quasi-birth-and-death process(LDQBD) (*e.g.* see Artalejo and Gómes-Correl, 2008) and one can use the algorithm in Latouche and Ramaswami (1999) for computing the stationary distribution of LDQBD process. There may be several PH distributions that match the first three moments of service time. It is recommended to use the PH distribution among them as small number of phases as possible to save the computer memory and computing time.

Table 3: Approximations of M/Weib(α,β)/3 retrial queue

γ		$\rho = 0.4$				$\rho = 0.8$			
		P_B	σ_0	L_1	σ_1	P_B	σ_0	L_1	σ_1
					$C_s^2 = 0.5$				
	App.	0.1129	0.9656	1.5925	1.5928	0.5591	0.7764	32.17	11.58
0.1	Sim.	0.1128	0.9656	1.5693	1.5838	0.5596	0.7770	32.22	11.43
	(c.i.)	±0.0009	±0.0012	±0.0264	±0.0200	±0.0039	±0.0043	± 0.59	±0.32
	App.	0.1194	0.9792	0.2319	0.6264	0.5778	0.8106	5.069	4.735
1.0	Sim.	0.1192	0.9788	0.2294	0.6235	0.5688	0.8112	5.090	4.778
	(c.i.)	±0.0008	±0.0010	±0.0043	±0.0115	±0.0039	±0.0041	±0.098	±0.125
	App.	0.1294	0.9974	0.1087	0.4433	0.6091	0.8636	2.622	3.510
5.0	Sim.	0.1290	0.9966	0.1074	0.4408	0.6099	0.8650	2.651	3.560
	(c.i.)	±0.0011	±0.0013	±0.0021	±0.0090	±0.0040	±0.0041	±0.064	±0.098
					$C_s^2 = 2.0$				
	App.	0.1146	0.9693	1.6757	1.9018	0.5624	0.7826	34.32	16.01
0.1	Sim.	0.1146	0.9691	1.6691	1.8971	0.5642	0.7805	34.48	15.98
	(c.i.)	±0.0013	±0.0016	±0.0286	±0.0420	±0.0057	±0.0066	±0.90	±0.74
	App.	0.1258	0.9916	0.3011	0.8628	0.5930	0.8372	7.080	7.739
1.0	Sim.	0.1253	0.9902	0.2973	0.8552	0.5945	0.8352	7.158	7.838
	(c.i.)	±0.0014	±0.0017	±0.0067	±0.0312	±0.0059	±0.0059	±0.225	±0.338
	App.	0.1363	1.0089	0.1668	0.6775	0.6260	0.8908	4.514	6.440
5.0	Sim.	0.1356	1.0073	0.1659	0.6735	0.6274	0.8887	4.590	6.521
	(c.i.)	±0.0015	±0.0019	±0.0050	±0.0323	±0.0057	±0.0060	±0.181	±0.312
					$C_s^2 = 5.0$				
	App.	0.1175	0.9755	1.8301	2.4826	0.5683	0.7934	38.53	23.76
0.1	Sim.	0.1169	0.9739	1.8110	2.4762	0.5722	0.7889	39.22	24.23
	(c.i.)	±0.0016	±0.0088	±0.0443	±0.1125	±0.0088	±0.0088	±1.64	±1.86
	App.	0.1318	1.0023	0.4045	1.3124	0.6082	0.8632	10.84	13.93
1.0	Sim.	0.1257	0.9993	0.4108	1.3087	0.6119	0.8581	11.20	14.36
	(c.i.)	±0.0018	±0.0023	±0.0153	±0.0859	±0.0085	±0.0090	±0.62	±1.17
	App.	0.1410	1.0164	0.2521	1.1176	0.6385	0.9106	8.042	12.53
5.0	Sim.	0.1397	1.0134	0.2669	1.1215	0.6395	0.9070	8.048	11.95
	(c.i.)	±0.0020	±0.0025	±0.0127	±0.0857	±0.0084	±0.0077	±0.470	±0.65

4. Numerical Examples

In this section we describe the approximation procedure and make some numerical comparisons for $M/G/3$ retrial queue. Two service time distributions, Weibul distribution Weib(α,β) with probability density function

$$f(x) = \frac{\beta}{\alpha}\left(\frac{x}{\beta}\right)^{\alpha-1} \exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right], \quad x > 0$$

and lognormal distribution LN(μ,σ^2) with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right), \quad x > 0$$

are considered. For an approximation, we first choose an appropriate PH distribution by fitting the first three moments of the service time and then compute the performance characteristics of the approximating system. In order to fit the first three moments of the service time distribution with $C_s^2 < 1$, we adopt the method in Bobbio *et al.* (2005) and the formula (3.2) is used for the case $C_s^2 > 1$.

In Table 3, the approximation results (App.) for M/Weib(α,β)/3 retrial queue with $m_1 = 1.0$ and $C_s^2 = 0.5, 2.0, 5.0$ are compared with the simulation results (Sim.). For fitting the distribution

Table 4: Approximations of $M/\mathrm{LN}(\mu,\sigma^2)/3$ retrial queue

γ		$\rho = 0.4$				$\rho = 0.8$			
		P_B	σ_0	L_1	σ_1	P_B	σ_0	L_1	σ_1
					$C_s^2 = 0.5$				
0.1	App.	0.1130	0.9658	1.5928	1.5899	0.5591	0.7765	32.18	11.59
	Sim.	0.1127	0.9651	1.5752	1.5796	0.5592	0.7764	32.13	11.53
	(c.i.)	±0.0010	±0.0011	±0.0295	±0.0255	±0.0043	±0.0043	±0.61	±0.35
1.0	App.	0.1196	0.9797	0.2308	0.6240	0.5778	0.8106	5.065	4.760
	Sim.	0.1193	0.9790	0.2279	0.6198	0.5783	0.8112	5.090	4.819
	(c.i.)	±0.0010	±0.0014	±0.0045	±0.0132	±0.0042	±0.0046	±0.103	±0.147
5.0	App.	0.1294	0.9977	0.1065	0.4384	0.6088	0.8633	2.611	3.538
	Sim.	0.1289	0.9967	0.1053	0.4361	0.6095	0.8642	2.639	3.594
	(c.i.)	±0.0011	±0.0014	±0.0021	±0.0106	±0.0041	±0.0045	±0.067	±0.117
					$C_s^2 = 1.0$				
0.1	App.	0.1136	0.9672	1.6207	1.6919	0.5603	0.7786	32.89	13.18
	Sim.	0.1131	0.9662	1.6069	1.6827	0.5614	0.7778	33.06	13.16
	(c.i.)	±0.0010	±0.0013	±0.0322	±0.0371	±0.0050	±0.0054	±0.81	±0.52
1.0	App.	0.1220	0.9846	0.2523	0.6975	0.5834	0.8205	5.733	5.811
	Sim.	0.1216	0.9837	0.2504	0.6939	0.5847	0.8202	5.793	5.915
	(c.i.)	±0.0012	±0.0014	±0.0062	±0.0215	±0.0050	±0.0051	±0.156	±0.238
5.0	App.	0.1322	1.0025	0.1234	0.5086	0.6156	0.8743	3.228	4.559
	Sim.	0.1319	1.0018	0.1226	0.5071	0.6171	0.8745	3.283	4.663
	(c.i.)	±0.0012	±0.0016	±0.0031	±0.0204	±0.0050	±0.0049	±0.109	±0.207
					$C_s^2 = 5.0$				
0.1	App.	0.1169	0.9744	1.7895	2.4064	0.5671	0.7911	38.36	25.06
	Sim.	0.1171	0.9745	1.8007	2.3932	0.5729	0.7875	39.59	26.33
	(c.i.)	±0.0019	±0.0023	±0.0503	±0.1217	±0.0094	±0.0088	±1.90	±2.46
1.0	App.	0.1275	0.9950	0.3370	1.1519	0.5997	0.8482	10.35	15.42
	Sim.	0.1300	0.9988	0.3698	1.1911	0.6101	0.8530	11.16	16.40
	(c.i.)	±0.0020	±0.0025	±0.0155	±0.0880	±0.0091	±0.0091	±0.80	±1.87
5.0	App.	0.1370	1.0104	0.1887	0.9412	0.6299	0.8972	7.460	13.94
	Sim.	0.1395	1.0134	0.2233	0.9947	0.6409	0.9021	8.290	14.88
	(c.i.)	±0.0021	±0.0020	±0.0132	±0.0904	±0.0088	±0.0091	±0.710	±1.85

Weib(α,β), we use the following distributions

$$\begin{array}{ll} \mathrm{CE}_{2,1}(0.751282; 2.88098, 2.09007), & \text{with } m_1 = 1.0,\ C_s^2 = 0.5,\ m_3 = 2.9078, \\ H_2(0.658726; 2.0365, 0.504441), & \text{with } m_1 = 1.0,\ C_s^2 = 2.0,\ m_3 = 16.4203, \\ H_2(0.908248; 1.8165, 0.183503), & \text{with } m_1 = 1.0,\ C_s^2 = 5.0,\ m_3 = 90.0. \end{array}$$

Simulation models are developed with ARENA. Simulation run time is set to 80,000 unit times including 20,000 unit times of warm-up period, where the expected value of service time is one unit time. Ten replications are conducted for each case and the average value; in addition, the half-length of 95% confidence interval(c.i.) are obtained.

Approximation results for $M/\mathrm{LN}(\mu,\sigma^2)/3$ retrial queue are listed in Table 4. The distributions $\mathrm{LN}(\mu,\sigma^2)$ with $m_1 = 1.0$ and $C_s^2 = 0.5, 1.0, 5.0$ are fitted by the following distributions

$$\begin{array}{ll} \mathrm{CE}_{1,3}(0.116747; 0.950128, 3.42026), & \text{with } m_1 = 1.0,\ C_s^2 = 0.5,\ m_3 = 3.375, \\ \mathrm{CE}_{1,2}(0.0896414; 0.509162, 2.242735), & \text{with } m_1 = 1.0,\ C_s^2 = 1.0,\ m_3 = 8.0, \\ H_2(0.99075; 1.15827, 0.06395), & \text{with } m_1 = 1.0,\ C_s^2 = 5.0,\ m_3 = 216.0. \end{array}$$

5. Conclusions

We have investigated numerically the effects of the moments of the service time to the performance measures related with the number X_0 of busy servers and the number X_1 of customers in orbit in $M/G/c$ retrial queue. Numerical experiments show that the effect of the third moment of the service time to L_1 is not negligible especially for ρ small and C_s^2 large and the variance V_1 of X_1 is strongly affected by the second moment as well as by the third moment of the service time; however, the distribution of X_0 is less sensitive to the second or higher moment of the service time than X_1. Based on these observations, we approximate the $M/G/c$ retrial queue by M/PH/c retrial queue where the PH distribution is determined by fitting the first three moments of the service time. Numerical experiments lead to approximations that are significantly accurate for a wide range of service times.

The most common technique to compute the stationary distribution of the number of customers in a multi-server Markovian queue with PH distribution of service time is the matrix analytic method. However, the matrix analytic method requires a long computation time and large memory capacity when both of the number of phases in PH distribution and the number c of servers are large. Therefore, the method is limited to small values of c and to PH distribution of a lower order. The method to approximate the multi-server queue by fitting the service time with PH distributions is not free from the restriction of the matrix analytic method and the application of the method proposed in this paper often restricts the number of servers and the number of phases of PH distribution. However, the many distributions with C_s^2 not close to 0 arising in practical situation can be fitted by the PH distribution with the moderate number of phases that reduces the computational problem. For example, the many distributions with $C_s^2 > 1$ can be fitted by the H_2 distribution and the size of the matrix components of the generator of the level dependent quasi-birth-and-death process corresponding to the system $M/H_2/c$ retrial queue is $c + 1$.

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