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An Analysis of Fraction Operation Sense to Enhance Early Algebraic Thinking[1]

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While many studies on early algebra have been conducted, there have been only a few studies on the operation sense as the fundamental element of algebraic thinking, especially the fraction operation sense. This study explored the dimensions of fraction operation sense and then investigated students' fraction operation sense. A total of 183 of sixth graders were surveyed and 5 students who showed high operation sense were clinically interviewed in order to analyze their algebraic thinking in detail. The results showed that students had a tendency to use direct calculation or employ inappropriate operation sense rather than to use the structure of operation or the relation between operations on the basis of algebraic thinking. This study implies that explicit instruction on early algebra is necessary from the elementary school years.

Keywords: early algebra, algebraic thinking, fraction operation sense, dimensions of fraction operation sense, structure and relationship of fraction operations
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1. INTRODUCTION

Operations are the fundamental elements in elementary mathematics so that early experience of operations plays a critical role in subsequent mathematics learning. In ele-

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mentary school, however, operations have been recognized mainly as procedures and the emphasis has been massively on the improvement on calculation skills. This caused many middle-grade students to face difficulties in studying algebra focused on operations as objects, variables, functions, and invariable relations or structures.

Many researchers advocate the 'early algebra' as an alternative for the current problem that strictly separates arithmetic in elementary school curriculum from algebra in secondary curriculum (e.g., Carraher & Schliemann, 2007; Kaput, 2008; Schifter, Monk, Russell & Bastable, 2008; Smith & Thompson, 2008). The early algebra emphasizes algebraic thinking even from the arithmetic context, which can emerge when students use operation sense (Schifter, 1997; Slavit, 1999). In particular, the exploration of the structure and relation of fraction operations should be emphasized, because rich experience on fraction as quantity as well as fraction operation is a vital prerequisite to understand linear functions, equations, and symbolic notation (Wu, 2001).

However, the studies on the early algebra are only at the beginning and their topics are limited mainly to the patterns, functions, or arithmetic of whole numbers curriculum (e.g., Boester & Lehrer, 2008; Kaput, 2008; Schifter, Monk, Russell & Bastable, 2008; Schoenfeld, 2008; Smith & Thompson, 2008). Studies on the fraction operation sense from the perspective on the early algebra have been hardly conducted.

Given this, this study first extracts the dimensions of the fraction operation sense related to algebraic thinking. It then explores 6th students' overall understandings on the fraction operations, especially structure and relationship of fraction operations, which are most directly related to algebraic thinking in elementary grades. This study additionally conducts in-depth interviews with students to explore their algebraic thinking emergent while justifying or implicitly generalizing their solution methods. These are expected to suggest instructional implications on applying the early algebra to elementary school grades.

2. THEORETICAL FRAMEWORK

2.1. Early Algebra

The content of early algebra for elementary students is not entirely new. Rather, it appears across various topics already taught in elementary school (e.g., operation, ratio and proportion, or measurement.). As Kaput (2008) put it, "[B]uilding generalizations from arithmetic and quantitative reasoning is taken by many educators and researchers as the primary route into algebra." (p. 12). Carraher, Schliemann & Schwartz (2008) highlighted that "early algebra builds heavily on background contexts of problems and only gradually introduces formal notation" (p. 236), emphasizing that "early algebra is not the same as algebra early" (p. 235).

Early algebra emphasizes algebraic thinking, which naturally emerges during the process of recognizing and generalizing the invariable structure and relationship from the arithmetic contexts. Many researchers assert that children are able to think about algebraic thinking such as invariable structure and relationships of operations without learning algebra (e. g., Smith & Thompson, 2008; Verschaffel, Greer & De Corte, 2007). When asked to explain why the sum of two odd numbers is always even, for example, even lower grade students were able to justify with their own words, focusing on the algebraic properties of numbers (Bastable & Schifter, 2008). For these reasons, studies on the early algebra suggest that even primary students have rich experience to develop their algebraic thinking. In order to foster the algebraic thinking from the arithmetic contexts, teachers should guide students to pay more attention to the structures, relationships, or reasoning of operations and their algorithms beyond simple arithmetic proficiency

Taken together, early algebra should be differentiated from typical algebra in terms of contents, subjects, and teaching methods. It focuses on algebraic thinking of lower graders in the process of generalizing the structure and relationship in the arithmetic contexts.

2.2. Operation sense to foster early algebraic thinking

Algebraic thinking can be fostered by emphasizing operation sense which includes the conceptual understanding of operations, properties of operations, and relations among them. However, previous studies that examine arithmetic foundation for algebra still focus mainly on the interpretations of the equal sign and whole number arithmetic (e.g., Bastable & Schifter, 2008; Carpenter, Franke & Levi, 2003; Verschaffel, Greer & De Corte, 2007).

In an exceptional study, Slavit (1999) presented the overall notion of operation sense as the following ten aspects, highlighting the role of operation sense in transitions from arithmetic to algebraic thinking:

1. A conceptualization of the base components of the process.
2. Familiarity with properties which the operation is able to possess.
3. Relationships with other operations.
4. Facility with the various symbol systems associated with the operation.
5. Familiarity with operation contexts.
6. Familiarity with operation facts.
7. Ability to use the operation without concrete or situational referents.
8. Ability to use the operation on unknown or arbitrary inputs.
9. An ability to relate the use of the operation across different mathematical objects.
10. An ability to move back and forth between the above conceptions (pp. 254–258).

In the same vein, Schifter (1997) stressed students' development of operation sense as a foundation for algebra, describing various situations in which students might be engaged in the widest array of activities with regard to four basic operations. Given this, the idea of operation sense plays a critical role in algebraic thinking emphasized in early algebra.

2.3. Dimensions of fraction operation sense

While fractions play a significant role in algebraic reasoning, "little attention has been given to the role that reasoning about fractional quantities can play in learning to reason with algebraic expressions." (Kilpatrick & Izsák, 2008, p. 13). As the common characteristics of operations in fraction and algebra can't be shared with those of whole numbers, rich experience on fraction operations should be fostered from the perspective of early algebra. For example, '$a+b$' can't be simplified any more, but if both 'a' and 'b' are substituted by the same unit such as $a=3c$, $b=2c$, '$a+b$' can be calculated further like $a+b=3c+2c=5c$. This is important in understanding equivalent expressions in algebraic operations, which can't be explained by operations with whole numbers. However, this can be connected to fraction operations in which equivalent fractions with the same unit are employed for the computation of fractions with unlike denominators (e.g., $\frac{1}{2}=\frac{3}{6}$, $\frac{1}{3}=\frac{2}{6}$, therefore $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$).

Given the importance of fraction operation in early algebra, the three dimensions of fraction operation sense in Table 1 were drawn on the basis of previous studies (e.g., Huinker, 2002; Markovits & Pang, 2007; Schifter, 1997; Slavit, 1999). The first dimension involves understanding of fraction operations and their meaning. Algebraic thinking is built on overall conceptual understanding of basic four operations (Bastable & Schifter, 2008; Mason, 2008). Mason (2008) asserted that students have already possessed powers for making sense of arithmetic. For this reason, it is important for students to use these powers and to develop them. Algebraic thinking emerges when students make use of those powers in the context of arithmetic.

The second dimension involves understanding of structure and relationship of fraction operations. Students' implicit awareness of arithmetic structure and relationship is connected directly to early algebra as generalized arithmetic (Carraher & Schliemann, 2007; Schifter, Monk, Russell & Bastable, 2008; Verschaffel, Greer & De Corte, 2007). To be more specific, students can identify regularities in arithmetic by investigating the commutative, associative, and distributive properties of operations and recognize the relationship of operation by comparing the difference between quantities and looking for multiplicative relations between them. These activities give opportunities for students to justify or implicitly generalize their solution methods.

The third dimension involves understanding of symbol system with regard to fraction operations. Kaput, Blanton & Moreno (2008) regarded the heart of algebraic thinking as generalization and symbolization. Students have to use specialized systems of symbols in the process of generalization and expressing such generalization.

In particular, as the understanding of structure and relationship of fraction operations is directly related to algebraic thinking in upper grades in elementary school, we focus on the results of this dimension in next section.

Table 1. Dimensions of fraction operation sense

Dimensions of fraction operation sense	Components
Understanding of fraction operations and their meaning	four basic fraction operations
	meanings of fraction operations
Understanding of structure and relationship of fraction operations	properties of fraction operations
	relationships among fraction operations
	results of fraction operations
Understanding of symbol system with regard to fraction operations	meaning of fraction operation algorithm
	letters and symbols in fraction operations

3. METHODOLOGY

3.1. Subjects

In order to identity students' overall fraction operation sense, a survey was conducted with 183 of sixth graders who learned the operations of fractions from 6 typical elementary schools in Korea. In addition, 5 students with high operation sense were clinically interviewed 3 times to closely analyze their algebraic thinking.

3.2. Questionnaire

A written questionnaire was designed on the basis of previous studies on operation sense and algebraic thinking (*e. g.*, Carpenter, Franke & Levi, 2003; Markovits & Pang, 2007; Schifter, Monk, Russell & Bastable, 2008). The questionnaire included 20 tasks with regard to the understanding of structure and relationship of fraction operation. First, 8 tasks are related to the properties of operations dealing with identity and inverse as well as commutative, associative, and distributive law (e.g., $5\frac{1}{3}+7\frac{2}{5}=\square+5\frac{1}{3}$, $3\frac{1}{4}+5\frac{3}{4}-5\frac{1}{4}=\square$, $\frac{1}{8}\times 8\frac{8}{11}=\square$,

$5\frac{11}{15}\times\frac{9}{7}\times\square=5\frac{11}{15}$, etc.).

Second, 7 tasks are related to the relationships among operations dealing with equal sign, reasoning of difference between quantities, and multiplicative reasoning (e.g., $2\frac{1}{4}+3\frac{7}{9}=\square+1\frac{1}{4}$, $55\times\frac{1}{4}=110\times\square$, etc.).

Third, 5 tasks are related to the results of operations dealing with estimating the results of operations (e.g., $\frac{2}{3}+\frac{1}{4}\ \ 1$, $\frac{2}{3}\div\frac{1}{6}\times5\frac{2}{9}\ \ \frac{2}{3}\times\frac{1}{6}\div5\frac{2}{9}$, etc.).

Mixed fractions, instead of simple fractions, were used to encourage students to recognize the structure and relationship of operations rather than to conduct direct calculation. Students had to write a correct number or the correct sign among “<” , “=”, “>” in each task and were encouraged to show how they solved it.

4. RESULTS

4.1. Understanding the properties of operations

Students’ performance varied according to the properties of operations as shown in Table 2. As many as 79% and 81.5% of the students recognized the property of commutative law, while only 35% and 16% students did for associative law and 17% and 24.5% did for distributive law. Note that more students found either the correct answers by direct calculation or the incorrect answers by using inappropriate fraction operation sense.

Most of the students tended to use their operation sense without direct calculation in solving a task of the commutative law in multiplication, as compared with other properties of operation. The following episode shows the process of one student’s justification about Task 2 (see Table 2):

Interviewer: (pointing to Task 2) Why is that?

Hyuk-min : In case of the rectangles (drawing two rectangles), multiplying this by this (pointing to the length and the width of the rectangle A) and multiplying that by that (pointing to the length and the width of the rectangle B) have the same area [see Figure 1: Note that the letters a and b in the figure, not from the student, were included for the reader].

Interviewer: What is the meaning of the same here?

Hyuk-min : It means that the product of multiplying these (pointing to the length and the width of the rectangle A) is equal to the product of multiplying those (pointing to the length and the width of the rectangle B).

In this episode, the student used an arbitrary rectangle, without specifying its length and width, to explain the commutative law in multiplication. He noticed that the area remains the same regardless of the arrangement of the rectangles. For this reason, he was able to explain with his words that changing the order of two factors in multiplication does not change the result. Given that he did not mark any numbers in his rectangles, the student recognized that the commutative law in multiplication can apply to all numbers. This made him solve the given task effectively by focusing on the relationship of numbers and quantities on the basis of algebraic thinking.

Table 2. Answers for tasks related to properties of operation (N=183)

Properties of Operation	Tasks		Correct		Incorrect		No answer
			M1	M2	M3	M4	
Commutative law	1	$5\frac{1}{3}+7\frac{2}{5}=\square+5\frac{1}{3}$	144 (79%)	8 (4%)	6 (3%)	25 (14%)	0 (0%)
	2	$5\frac{2}{3}\times\square=6\frac{3}{4}\times5\frac{2}{3}$	149 (81.5%)	6 (3%)	5 (3%)	22 (12%)	1 (0.5%)
Associative law	3	$3\frac{1}{4}+5\frac{3}{4}-5\frac{3}{4}=\square$	63 (35%)	44 (24%)	46 (25%)	27 (15%)	3 (1%)
	4	$2\frac{1}{5}-\frac{1}{2}-\frac{1}{2}=\square$	29 (16%)	38 (21%)	68 (37%)	45 (25%)	3 (1%)
Distributive law	5	$12\frac{6}{11}\div6=\square$	31 (17%)	38 (20%)	49 (27%)	65 (36%)	0 (0%)
	6	$\frac{1}{8}\times8\frac{8}{11}=\square$	45 (24.5%)	52 (28%)	36 (20%)	49 (27%)	1 (0.5%)
identity element, inverse element	7	$3\frac{8}{9}+2\frac{7}{8}-\square=3\frac{8}{9}$	99 (54%)	12 (7%)	24 (13%)	43 (23%)	5 (3%)
	8	$5\frac{11}{15}\times\frac{9}{7}\times\square=5\frac{11}{15}$	39 (21%)	11 (6%)	80 (44%)	47 (26%)	6 (3%)

M1 : Use of properties of operations M2: Direct calculation
M3: Incorrect use of properties of operations M4: Incorrect calculation

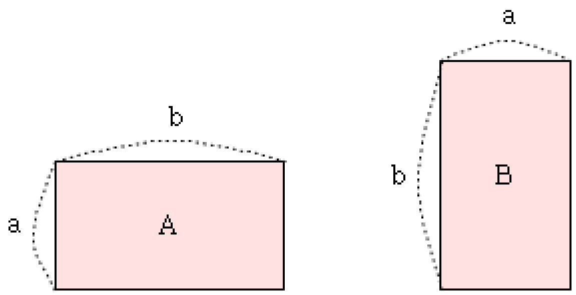

Figure 1. Hyuk-min’s rectangles in order to explain the commutative law in multiplication

A noticeable remark was that 37% of the students used incorrectly the property of associative law. In fact, 24.5% of the students (part of M3) solved the task as $\left(2\frac{1}{5}-\frac{1}{2}\right)-\frac{1}{2}=2\frac{1}{5}-\left(\frac{1}{2}-\frac{1}{2}\right)=2\frac{1}{5}-0=2\frac{1}{5}$, overgeneralizing the associative law of addition to that of subtraction. In contrast, 16% students (M1) found the correct answer by solving as $\left(2\frac{1}{5}-\frac{1}{2}\right)-\frac{1}{2}=2\frac{1}{5}-\left(\frac{1}{2}+\frac{1}{2}\right)=2\frac{1}{5}-1=1\frac{1}{5}$. The following episode illustrates how a student came up with this idea:

Uichan: [explains how he solved the Task 4] $\frac{1}{2}$ plus $\frac{1}{2}$, and then subtract all together.

Interviewer: Is $2\frac{1}{5}-\frac{1}{2}-\frac{1}{2}$ same as $2\frac{1}{5}-1$?

Uichan: Yes.

Interviewer: Why is that?

Uichan: Because it is to subtract and then subtract again. For example, this (pointing to $2\frac{1}{5}$) is alone on this side and these two (pointing to $\frac{1}{2}$ and $\frac{1}{2}$) are on the same side. So add these two (pointing to each $\frac{1}{2}$), and then kill this (pointing to $2\frac{1}{5}$).

This episode shows that even elementary school students can use the properties of operations effectively and explain it with their own words. The student in this episode recognized that subtracting $\frac{1}{2}$ twice is equal to subtracting 1 once. Such efficient thinking was connected to his idea that each $\frac{1}{2}$ would be on the same side of the battle so that they had to cooperate (i.e. add) to kill the opponent, $2\frac{1}{5}$.

With regard to the Task 6, 24.5% of the students (M1) recognized that a mixed fraction is comprised of a natural number and a proper fraction, and then used effectively the distributive law for multiplication over addition. In contrast, slightly more numbers of students used direct calculation of converting the mixed fraction into the improper fraction and then multiplying the given multiplicand: 28% of them produced the correct answer, while 27% did not.

The following episode is a typical example related to the distributive law for multiplication over addition:

Interviewer: (pointing to the result Eun-ju found) You found $1\frac{1}{11}$ directly?

Eun-ju: Oh, yeah. I just left the denominator because it cannot be divided. It is $\times\frac{1}{8}$ so the numerator and the natural number can be divided by it. 8 (pointing to the natural number of $8\frac{8}{11}$) is divided by 8, and then this becomes 1. Likewise, 8 (pointing to the numerator of $8\frac{8}{11}$) is also divided by 8, then that becomes 1. So I got $1\frac{1}{11}$.

Interviewer: Then, why did you calculate differently with the natural number and the numerator of $8\frac{8}{11}$? Is it correct not to change the mixed number into the improper fraction?

Eun-ju: Yes, anyway if I want to change $8\frac{8}{11}$ into an improper fraction, then I need to calculate 8×11 and then to add 8. This (pointing to the 8×11) can be divided by 8. Likewise, that (pointing to the addend 8) can be done by 8. It doesn't matter.

This episode shows the algebraic thinking essentially inherent in the process of explaining the distributive law for multiplication over addition. Eun-ju recognized implicitly that a mixed fraction is comprised of a natural number and a proper fraction. According to this, she could solve the task easily by using the distributive law for multiplication over addition. In order to probe how she understood the distributive law for multiplication over addition, when the interviewer asked why she calculated separately the natural number and the numerator of the mixed fraction, she explained that if she changed $8\frac{8}{11}$ into an improper fraction, then it would be $\frac{8\times11+8}{11}$ in which the natural number part '8×11' could be divided by 8 and the numerator part '+8' could be divided by 8 as well. So she didn't have to change it. This thinking process can be written as $\frac{1}{8}\times8\frac{8}{11}=\left(\frac{1}{8}\times8\right)+\left(\frac{1}{8}\times\frac{8}{11}\right)=1+\frac{1}{11}=1\frac{1}{11}$.

There was a remarkable difference in solving the Task 7 and 8. While 54% of the students found easily the inverse element in addition, only 21% did for multiplication. Instead, 44% of the students recognized incorrectly the inverse element in multiplication. In particular, 38% (part of M3) of them wrote the preceding number ($\frac{9}{7}$) as it was, of which strategy was applicable to the Task 7.

4.2. Understanding the relations of operations

The students who easily solved the tasks by relational thinking between operations were rare. They instead misunderstood the relationship between operations and the meaning of the equal sign. Some illustrative results were shown in Table 3.

Task 9 (see Table 3) can be solved by understanding the quantitative relationship between the fractions on each side of the equal sign and the commutative law of addition. This can be written as $2\frac{1}{4}+3\frac{7}{9}=\left(3\frac{7}{9}+1\right)+\left(2\frac{1}{4}-1\right)=4\frac{7}{9}+1\frac{1}{4}$. While 33% of the students used such difference comparison, a total of 34% (M2 and M4) tried direct calculation. A noticeable result was that 26% of the students (part of M3) recognized the equal sign as a command to add the two numbers on the left side, answering $2\frac{1}{4}+3\frac{7}{9}=6\frac{1}{36}$. This tendency was more evident in Task 13 in which students were asked to indicate whether $\frac{3}{5}\times\frac{5}{8}\times\frac{2}{3}=\frac{3}{\cancel{5}_{1}}\times\frac{\cancel{5}^{1}}{8}=\frac{\cancel{2}^{1}}{\cancel{8}_{4}}\times\frac{\cancel{2}^{1}}{\cancel{3}_{1}}=\frac{1}{4}$ is true or not and explain why. While 62% of the students accepted the expressions as correct, only 4% pointed out the incorrect use of the equal sign.

Table 3. Answers for tasks related to the relation of operations (N=183)

Relation of operations	Tasks		Correct		Incorrect		No answer
			M1	M2	M3	M4	
Difference comparison	9	$2\frac{1}{4}+3\frac{7}{9}=\square+1\frac{1}{4}$	60 (33%)	17 (9%)	55 (30%)	46 (25%)	5 (3%)
	10	$5\frac{4}{7}-\square=7\frac{6}{7}-3\frac{3}{7}$	48 (26%)	14 (8%)	67 (36.5%)	45 (24.5%)	9 (5%)
Multiplicative comparison	11	$55\times\frac{1}{4}=110\times\square$	34 (19%)	11 (6%)	80 (44%)	45 (25%)	13 (6%)
	12	$8\div\frac{3}{7}=\square\div\frac{1}{7}$	19 (10.5%)	15 (8%)	80 (44%)	59 (32%)	10 (5.5%)
The meaning of equal sign	13	$\frac{3}{5}\times\frac{5}{8}\times\frac{2}{3}=\frac{3}{\cancel{5}_1}\times\frac{\cancel{5}^1}{8}=\frac{\cancel{3}^1}{\cancel{8}_4}\times\frac{\cancel{2}^1}{\cancel{3}_1}=\frac{1}{4}$	8 (4%)	34 (19%)	114 (62%)	25 (14%)	2 (1%)

M1 : Use of the relations of operations M2: Direct calculation
M3: Incorrect use of the relations of operations M4: Incorrect calculation

Task 11 can be solved by understanding multiplicative comparison as well as the equality between left and right side: if a number becomes c times, then the other quantity must be $\frac{1}{c}$ times. While 19% of the students (M1) used such relational thinking, 44% of them (M3) employed in the wrong way: $55\times\frac{1}{4}=\left(55\times2\right)\times\left(\frac{1}{4}\times2\right)=110\times\frac{1}{2}$. The following episode illustrates how Task 11 was solved by relational thinking:

Uichan: Here (pointing to 55 on the left side and then 110 on the right side in the Task 11), it is multiplied by 2. Here (pointing to $\frac{1}{4}$ on the left side), it needs to be divided by 2.

Interviewer: Why is it the same?

Uichan: Because it became twice as before. To make it the same, the denominator should be bigger likewise.

Interviewer: What if the number 55 becomes 4 times bigger? What happens to this number (pointing to $\frac{1}{4}$ on the left side of the equation)?

Uichan: You need to divide it by 4.

Interviewer: How about 10 times?

Uichan: You need to divide it by 10.

Interviewer: Why is the result same when we divide by the quantity we multiplied?

Uichan: If you divide as much as you multiplied by, it is the same thing as nothing doing.

In this episode, Uichan recognized the multiplicative relationship of operation, $a \times b=(a \times c) \times (b \div c)$, and explained that '$55 \times \frac{1}{4} = (55 \times 2) \times \left(\frac{1}{4} \div 2\right) = 110 \times \frac{1}{8}$'. He even explained that such a relationship remains the same, even though we change the numbers in the equation differently. Lastly, when asked to generalize, the student was able to explain with his words. We can summarize his thinking in an algebraic way as $55 \times \frac{1}{4} = (55 \times x) \times \left(\frac{1}{4} \div x\right) = 55 \times \frac{1}{4} \times x \div x = 55 \times \frac{1}{4} \times 1 = 55 \times \frac{1}{4}$.

In the case of division problem such as Task 12, in contrast with multiplication, the results of operation on the left side and the right side become equal when the dividend multiplies c times, and then the divisor multiplies c times. In other words, the dividend multiplies $\frac{1}{3}$ times, and then the divisor should multiply $\frac{1}{3}$ times. While 10.5% students solved the problem by using such relationship between quantities, 44% of the students did not. The following episode represents a student's understanding about multiplicative comparison between fractional quantities. As the interviewer asked the student to justify why the result of the problem $8 \div \frac{3}{7} = \square \div \frac{1}{7}$ is $\frac{8}{3}$, the student explained:

Hyuk-min : This (pointing to $\frac{3}{7}$ on the left side) becomes 3 times smaller. So, this (pointing to 8 on the left side) should also become 3 times smaller.

Interviewer: Okay. $\frac{3}{7}$ is decreased by 3 times so that 8 should be decreased by 3 times. right?

Hyuk-min : Yes.

Interviewer: That was not what you said earlier (pointing to the Task 11 about multiplication). You said if the number [representing a multiplicand] becomes 2 times bigger, then the other number [representing a multiplier] should be divided by 2. Why the division is different from the multiplication?

Hyuk-min: When we divide 8 by $\frac{3}{7}$, 8 is divided by the larger number than $\frac{1}{7}$. So the result on the left side becomes smaller. As such, this result on the right side should become smaller. So 8 should become $\frac{8}{3}$.

In this episode, it is still early to conclude that the student completely understood the structure and relationship such as $a \div b=(a \times c) \div (b \times c)$. However, we can see that he had the relational thinking appropriate for fraction division. He recognized equality between the left and the right side of equal sign and grasped the multiplicative relationship of fraction quantities. When dividing 8 by $\frac{3}{7}$, 8 is divided by the larger number than $\frac{1}{7}$ so that the result of the operation should be smaller than $8 \div \frac{1}{7}$. For this justification, he recognized correctly with regard to quantitative relationship within the operation by using relational understanding of equal sign.

4.3. Understanding the result of operation

Students had a tendency in conducting direct calculation to estimate the result of an operation. The illustrative results were shown in Table 4. Students had to fill in the correct sign among "<" , "=", or ">" and explain why.

Table 4. Answers for task related to results of operation (N=183)

	Tasks		Correct			Incorrect		No answer
			M1	M3	M2	M3	M4	
Estimating the results of operation	16	$\frac{2}{3}+\frac{1}{4} \circ 1$	3 (1.5%)	21 (11.5%)	131 (72%)	6 (3%)	22 (12%)	0 (0%)
	17	$\frac{2}{3}\div\frac{1}{6}\times 5\frac{2}{9}$ $\frac{2}{3}\times\frac{1}{6}\div 5\frac{2}{9}$	43 (23.5%)	0 (0%)	81 (44%)	18 (10%)	38 (21%)	3 (1.5%)
	18	$4\frac{2}{3}\div\square > 4\frac{2}{3}$	31 (17%)	0 (0%)	24 (13%)	32 (17.5%)	91 (50%)	5 (2.5%)
	19	$3\frac{3}{5}\div\frac{13}{14}$, $3\frac{3}{5}\div\frac{9}{19}$ $3\frac{3}{5}\div 1\frac{1}{11}$, $3\frac{3}{5}\times\frac{24}{25}$	21 (11.5%)	3 (1.5%)	13 (7%)	80 (44%)	66 (36%)	0 (0%)
	20	$\frac{12}{13}+\frac{7}{8}$	14 (8%)	3 (1.5%)	22 (12%)	45 (24.5%)	99 (54%)	0 (0%)

M1 : Use of the result of an operation M2: Direct calculation
M3: Incorrect use of the result of an operation M4: Incorrect calculation

It seemed encouraging that 85% of the students chose the correct sign, but 72% of them solved the Task 16 by direct calculation. The given addition might look so easy for the students only to jump into computation, instead of utilizing the fraction operation sense. A noticeable remark was that 11.5% of the students (M3 with correct answer) misunderstood the result of addition of fraction. For instance, they thought that the sum of two proper fractions cannot exceed 1 or that natural numbers are larger than factions. Only 1.5% of the students applied proper estimation based on the understanding of the quantitative relationship of fractions: $\frac{1}{3}$ should be added to $\frac{2}{3}$ in order to make 1, but only $\frac{1}{4}$ which is less than $\frac{1}{3}$ was added, so the result must be less than 1.

When the task included rather complicated computations, students tended to use their operation sense (see Task 17 and 19 in Table 4). For instance, in case of Task 19, when asked to find which would have the greatest result among $3\frac{3}{5}\div\frac{13}{14}$, $3\frac{3}{5}\div\frac{9}{19}$, $3\frac{3}{5}\div 1\frac{1}{11}$, and $3\frac{3}{5}\times\frac{24}{25}$, 55.5% of the students (M1 and M3) estimated the results by operation sense in place of direct computation. However, only 11.5% of the students understood the fact that the result would be greater, as the multiplier would be bigger in the multiplication of fraction or the divisor would be smaller in the division of fraction. In particular, 50% of the students chose $3\frac{3}{5}\times\frac{24}{25}$, explaining that multiplication would make the result bigger whereas division would be the opposite.

Task 20 can be easily solved by estimating the result of operation in the following way: both $\frac{12}{13}$ and $\frac{7}{8}$ are less than 1 but these are closed to 1 so that the sum of $\frac{12}{13}$ and $\frac{7}{8}$ is approximately 2. While 8% of the students solved the task in this way, others who employed inappropriate operation sense thought that a proper fraction is smaller than 1 so that the sum of proper fractions should become smaller than 1.

5. DISCUSSION

This study analyzed the fraction operation sense of 6th grade students in terms of the understanding of the structure and relationship of fraction operations. Firstly, it is noticeable that students who solved the tasks by understanding associative law and distributive law were very rare in comparison with those who did by recognizing the commutative law in addition and multiplication. This might result from the fact that the commutative law can be easily recognized by its visual symmetry, whereas other properties were visually hidden so that further complicated procedures such as grouping or regrouping would be required (Schifter, Monk, Russell & Bastable, 2008). As the understanding of the properties of fraction operations is important to develop algebraic thinking (Wu, 2001), more emphasis should be given on such activity, beyond simply calculating various fraction operations. It seems encouraging that some students in this study used effectively such properties as needed and justified the process by their own words.

Secondly, students experienced difficulties in using relational thinking of operations. They showed the tendency to solve the tasks through calculations even when the tasks were easily solved by focusing on difference or multiplicative comparison. This tendency seemed to be more aggravated as students recognized the equal sign as a command to calculate the left side and to input the result on the right side (Smith & Thompson, 2008). As both the relational thinking between fraction operations and the understanding of equal sign are closely connected with the important algebraic concepts such as equations or equivalent expressions (Carpenter, Franke & Levi, 2003; Kilpatrick & Izsák, 2008), they should be continually highlighted from low grades.

Thirdly, many students conducted direct calculation in place of fraction operation sense. As algebraic thinking is based on operation sense (Slavit, 1999), students should have many opportunities to enrich their operation sense. Such opportunities should be intentionally provided in school mathematics curriculum, as they would not emerge naturally.

Lastly, students had a tendency to over-generalize either the property of a certain operation or the result in a certain number system. For instance, many students thought that the associative law of addition would be also applicable to subtraction (see Task 4 in Table 2). In the same vein, they thought that the characteristics of multiplication and division with whole numbers would be transferrable to those with fractions. This inappropriate operation sense can be a main barrier to developing algebraic thinking. The properties and results of operations should be re-examined by students in detail as the number system is extended. In particular, students of upper grades in elementary schools should have rich opportunities to compare and contrast the structure and relationship of whole number operations and fraction operations.

The focus on the structure and relationship of fraction operations, beyond their simple calculations, could be worthwhile to connect arithmetic in elementary school with algebra in secondary school. In this respect, this study which analyzed students' fraction operation sense relating to their algebraic thinking is intended to suggest instructional implications on what more attention needs to be paid in the upper grades in elementary schools.

REFERENCES

Bastable, V. & Schifter, D. (2008). Classroom stories: Examples of elementary students engaged in early algebra. In: J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 165–184). New York: Lawrence Erlbaum Associates.

Boester, T. & Lehrer, R. (2008). Visualizing algebraic reasoning. In: J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 211–234). New York: Lawrence Erlbaum Associates.

Carpenter, T. P.; Franke, M. L. & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school.* Portsmouth, NH: Heinemann. ERIC ED474452

Carraher, D. W. & Schliemann, A. D. (2007). Early algebra and algebraic reasoning. In: F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 669–705). Charlotte, NC: Information Age Publishing.

Carraher, D. W.; Schliemann, A. D. & Schwartz, J. L. (2008). Early algebra is not the same as algebra early. In: J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 235–272). New York: Lawrence Erlbaum Associates.

Huinker, D. (2002). Examining dimensions of fraction operation sense. In: B. Litwiller & G. Bright (Eds.), *Making sense of fractions, ratios, and proportions* (pp. 72–78). Reston, VA: NCTM.

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5–17). New York: Lawrence Erlbaum Associates.

Kaput, J. J.; Blanton, M. L. & Moreno, L. (2008). Algebra from a symbolization point of view. In: J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 19–56). New York: Lawrence Erlbaum Associates.

Kilpatrick, J. & Izsák, A. (2008). A history of algebra in the school curriculum. In: C. E. Greenes & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics* (pp. 3–18). Reston, VA: NCTM.

Lee, J. & Pang, J. (2012). *An analysis of fraction operation sense to enhance early algebraic thinking.* A paper presented at Topic Study Group 9 (Teaching and Learning of Algebra) of the 12th International Congress on Mathematical Education (ICME-12) held at COEX, Seoul, Korea; July 8–15, 2012.

Markovits, Z. & Pang, J. S. (2007). The ability of sixth grade students in Korea and Israel to cope with number sense tasks. In: J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proc. of the 31st Conf. of the Int. Group for the Psychology of Mathematics Education (PME-31)* held at COEX, Seoul, Korea; July 8–13, 2007, Vol. 3, (pp. 241–248). Seoul, Korea: PME. ERIC ED499416 Available from:
http://www.eric.ed.gov/PDFS/ED499416.pdf

Mason, J. (2008). Making use of children's powers to produce algebraic thinking. In: J. J. Kaput, D. W. Carraher & M. Blanton (Eds.), *Algebra in the early grades* (pp. 19–41). New York: Lawrence Erlbaum Associates.

Schifter, D. (1997). *Developing operation sense as a foundation for algebra.* Paper presented at the annual meeting of the American Educational Research Association, Chicago. ERIC ED408152 Available from:
http://www.eric.ed.gov/PDFS/ED408152.pdf

Schifter, D.; Monk, S.; Russell, S. J. & Bastable, V. (2008). Early algebra: What does understanding the laws of arithmetic mean in the elementary grades? In: J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 413–447). New York: Lawrence Erlbaum Associates.

Schoenfeld, A. H. (2008). Early algebra as mathematical sense making. In: J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 479–510). New York: Lawrence Erlbaum Associates.

Slavit, D. (1999). The role of operation sense in transitions from arithmetic to algebraic thought. *Educational Studies in Mathematics* **37(3)**, 251–274. Available from:
http://www.scribd.com/doc/208018/1999-the-Role-of-Operation-Sense-in-Transitions-From

Smith, J. P. & Thompson, P. W. (2008). Quantitative reasoning and the development of algebraic reasoning. In: J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95–132). New York: Lawrence Erlbaum Associates.

Verschaffel. L.; Greer. B. & De Corte. E. (2007). Whole number concepts and operations. In: F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 557–

628). Charlotte, NC: Information Age Publishing.

Wu. H. H. (2001). How to prepare students for algebra. *American Educator* 25(2), 10–17. ERIC EJ630397