T-STRUCTURE AND THE YAMABE INVARIANT

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Abstract. The Yamabe invariant is a topological invariant of a smooth closed manifold, which contains information about possible scalar curvature on it. It is well-known that a product manifold $T^m \times B$ where $T^m$ is the $m$-dimensional torus, and $B$ is a closed spin manifold with nonzero $A$-genus has zero Yamabe invariant.

We generalize this to various $T$-structured manifolds, for example $T^m$-bundles over such $B$ whose transition functions take values in $Sp(m, \mathbb{Z})$ (or $Sp(m - 1, \mathbb{Z}) \oplus \{\pm 1\}$ for odd $m$).

1. Introduction to Yamabe invariant

The Yamabe invariant is an invariant of a smooth closed manifold depending on its smooth topology.

Let $M$ be a smooth closed manifold of dimension $n$. Given a smooth Riemannian metric $g$ on it, the conformal class $[g]$ is defined as

$$[g] = \{ \varphi g \mid \varphi : M \to \mathbb{R}^+ \text{ is smooth} \}.$$  

The famous Yamabe problem ([13]) states that there exists a metric $\tilde{g}$ in $[g]$ which attains the minimum

$$\inf_{\tilde{g} \in [g]} \left( \frac{\int_M s_{\tilde{g}} \, dV_{\tilde{g}}}{\left( \int_M dV_{\tilde{g}} \right)^{\frac{n}{n-2}}} \right),$$

where $s_{\tilde{g}}$ and $dV_{\tilde{g}}$ respectively denote the scalar curvature and the volume element of $\tilde{g}$.

It turns out that when $n \geq 3$, a unit-volume minimizer $\tilde{g}$ in $[g]$ has constant scalar curvature, which is equal to the above minimum value called the Yamabe constant of $[g]$ and denoted by $Y(M, [g])$.

It is known that the Yamabe constant of any $n$-manifold is bounded above by $Y(S^n, [g_0])$ where $[g_0]$ denotes a standard round metric. Thus following a

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min-max procedure we define the Yamabe invariant
\[ Y(M) := \sup_{[g]} Y(M, [g]) \]
of \( M \).

The following facts are noteworthy.
- \( Y(M) > 0 \) if and only if \( M \) admits a metric of positive scalar curvature.
- If \( M \) is simply-connected and \( \dim M \geq 5 \), then \( Y(M) \geq 0 \). With the
  further assumption that \( M \) is spin, \( Y(M) > 0 \) if and only if the \( \alpha \)-genus
  of \( M \) is 0.
- For \( r \in \left[ \frac{\alpha}{\beta}, \infty \right] \),
  \[ |Y(M, [g])| = \inf_{\tilde{g} \in [g]} \left( \int_M |s_{\tilde{g}}|^r d\mu_{\tilde{g}} \right)^{\frac{1}{r}} \left( \text{Vol}_{\tilde{g}} \right)^{\frac{1}{r} - \frac{1}{2}}, \]
where the infimum is attained only by the Yamabe minimizers.
- When \( Y(M, [g]) \leq 0 \),
  \[ Y(M, [g]) = -\inf_{\tilde{g} \in [g]} \left( \int_M |s_{\tilde{g}}|^r d\mu_{\tilde{g}} \right)^{\frac{1}{r}} \left( \text{Vol}_{\tilde{g}} \right)^{\frac{1}{r} - \frac{1}{2}}, \]
where \( s_{\tilde{g}} \) is defined as \( \min\{s_g, 0\} \).

Therefore when \( Y(M) \leq 0 \),
\[ Y(M) = -\inf_{g} \left( \int_M |s_g|^r d\mu_g \right)^{\frac{1}{r}} \left( \text{Vol}_g \right)^{\frac{1}{r} - \frac{1}{2}}, \]
so that \( Y(M) \) measures how much negative scalar curvature is inevitable on \( M \).
- As an application of the above formula, if \( M \) has an \( F \)-structure which
  will be explained in a later section, \( M \) admits a sequence of metrics
  with volume form converging to zero while the sectional curvature are
  bounded below, so that \( Y(M) \geq 0 \) (See [14]).

2. Computation of Yamabe invariant

We now discuss how to compute the Yamabe invariant. When \( M \) is a closed
oriented surface, by the Gauss-Bonnet theorem
\[ Y(M) = 4\pi \chi(M), \]
where \( \chi \) denotes the Euler characteristic.

When \( M \) is a closed oriented 3-manifold, the Ricci flow gave many answers
by the proof of geometrization theorem due to G. Perelman (See [1]). For
example, \( Y(M) > 0 \) if and only if \( M \) is a connected sum of \( S^1 \times S^2 \)'s and finite
quotients of \( S^3 \), and \( Y(\mathbb{H}^3 / \Gamma) \) is realized by the hyperbolic metric.

When \( \dim M = 4 \), the Seiberg-Witten theory enables us to compute the
Yamabe invariant of Kähler surfaces through the Weitzenböck formula. LeBrun
[10, 11, 12] has shown that if $M$ is a compact Kähler surface whose Kodaira dimension is not equal to $-\infty$, then

$$Y(M) = -4\sqrt{2}\pi \sqrt{2\chi(M)} + 3\tau(M),$$

where $\tau$ denotes the signature and $\hat{M}$ is the minimal model of $M$, and for $\mathbb{C}P^2$,

$$Y(\mathbb{C}P^2) = 4\sqrt{2}\pi \sqrt{2\chi(\mathbb{C}P^2)} + 3\tau(\mathbb{C}P^2).$$

In higher dimensions, few examples have been computed so far, such as

$$Y(S^1 \times S^{n-1}) = Y(S^n) = n(n-1)(\text{vol}(S^n(1)))^{\frac{2}{n}},$$

where $S^n(1)$ is the unit sphere in $\mathbb{R}^{n+1}$, and

$$Y(T^n) = Y(T^n \times H) = Y(T^n \times B) = 0,$$

where $H$ is a closed Hadamard-Cartan manifold, i.e., one with a metric of non-positive sectional curvature, and $B$ is a closed spin manifold with nonzero $\hat{A}$-genus. These $T^n$-bundles have such property, because they admit a $T$-structure and never admit a metric of positive scalar curvature by Gromov-Lawson enlargeability method [5, 9].

We call a closed $n$-manifold $M$ enlargeable if the following holds: for any $\epsilon > 0$ and any Riemannian metric $g$ on $M$, there exists a Riemannian spin covering manifold $\hat{M}$ of $(M, g)$ and an $\epsilon$-contracting map $f : \hat{M} \to S^n(1)$, which is constant outside a compact subset of $\hat{M}$ and of nonzero $\hat{A}$-degree defined as $\hat{A}(f^{-1}(\text{any regular value of } f)).$

Here a smooth map $F$ is called $\epsilon$-contracting if the norm of $DF$ is less than $\epsilon$. By using the Weitzenböck formula for an appropriate twisted Dirac operator, they showed that such manifolds never admit a metric of positive scalar curvature.

They also generalized this to so-called weakly-enlargeable manifolds, where “$\epsilon$-contracting” is replaced by “$\epsilon$-contracting on 2-forms” meaning that the induced map of $DF$ on tangent bi-vectors, i.e., a section of $\Lambda^2(TM)$ has norm less than $\epsilon$.

$\hat{A}$-genus of a closed spin manifold $M$ is the integral over $M$ of

$$\hat{A}(TM) := 1 - \frac{p_1}{24} + \frac{-4p_2 + 7p_1^2}{5760} + \cdots ,$$

where $p_i \in H^{4i}(M, \mathbb{Z})$ is the $i$-th Pontryagin class of $TM$. An important fact is that a closed spin manifold with a metric of positive scalar curvature has zero $\hat{A}$-genus.

Then a natural question for us to explore is:

**Question 2.1.** Let $M$ be a $T^m$-bundle over a closed spin manifold $B$ with nonzero $\hat{A}$-genus. Is $Y(M)$ equal to zero?
3. **T-structure**

An *F-structure* which was introduced by Cheeger and Gromov [3, 4] generalizes an effective $T^m$-action for $m \in \mathbb{N}$.

**Definition 3.1.** An *F-structure* on a smooth manifold is given by data $(U_i, \hat{U}_i, T^{k_i})$ with the following conditions:

1. $(U_i)$ is a locally finite open cover.
2. Each $\pi_i : \hat{U}_i \mapsto U_i$ is a finite Galois covering with covering group $\Gamma_i$.
3. Each torus $T^{k_i}$ of dimension $k_i$ acts effectively on $\hat{U}_i$ in a $\Gamma_i$-equivariant way, i.e., $\Gamma_i$ also acts on $T^{k_i}$ as an automorphism so that
   \[ \gamma(gx) = \gamma(g)\gamma(x) \]
   for any $\gamma \in \Gamma_i$, $g \in T^{k_i}$, and $x \in \hat{U}_i$.
4. If $U_i \cap U_j \neq \emptyset$, then there is a common covering of $\pi_i^{-1}(U_i \cap U_j)$ and $\pi_j^{-1}(U_i \cap U_j)$ such that it is invariant under the lifted actions of $T^{k_i}$ and $T^{k_j}$, and they commute.

As a special case, a $T$-structure is an $F$-structure in which all the coverings $\pi_i$’s are trivial.

Typical examples of $T$-structure are torus bundles.

**Theorem 3.2.** Any $T^m$-bundle over a smooth manifold whose transition functions are $T^m \times GL(m, \mathbb{Z})$-valued has a $T$-structure. In particular, any $S^1$ or $T^2$-bundle has a $T$-structure.

**Proof.** Here $T^m$ acts by translation, and hence the transition functions are affine maps at each fiber direction. Obviously the local $T^m$ actions along the fiber are commutative on the intersections to give a global $T$-structure.

The second statement follows from the well-known fact that the diffeomorphism group of $T^m$ for $m = 1, 2$ is homotopically equivalent to $T^m \times GL(m, \mathbb{Z})$. Thus we may assume that the transition functions are $T^m \times GL(m, \mathbb{Z})$-valued. $\square$

Other typical examples are manifolds with a nontrivial smooth $S^1$ action. Such examples we will use are projective spaces such as $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$, and $CaP^2$ (For the case of the Cayley plane which actually has an $S^3$-action, see [2]). Or one can construct more examples by gluing $T$-structured manifolds. For example graph manifolds are obtained by gluing Seifert-fibred 3-manifolds along the toral boundaries, and also:

**Theorem 3.3** (Paternain and Petean [14]). Suppose $X$ and $Y$ are $n$-manifolds with $n > 2$, which admit a $T$-structure. Then $X \# Y$ also admits a $T$-structure.

4. **Main results**

Motivated by Gromov-Lawson enlargeability technique, we prove:
Theorem 4.1. Let $B$ be a closed spin manifold of dimension $4d$ with nonzero $\tilde{A}$-genus, and $M$ be a $T^{m}$-bundle over $B$ whose transition functions take values in $Sp(m, \mathbb{Z})$ (or $Sp(m-1, \mathbb{Z}) \oplus \{\pm 1\}$ for odd $m$). Then

$$Y(M) = 0.$$ 

Proof. By Theorem 3.2, $M$ has a $T^{m}$-structure so that $Y(M) \geq 0$.

We only have to show that $M$ never admits a metric of positive scalar curvature. To the contrary, suppose that it admits such a metric $h$, and we will derive a contradiction. The basic idea is to apply the Bochner-type method to a twisted Spin$^c$ bundle on $M$ whose topological index is nonzero.

First, we consider the case when $m$ is even, say $2k$. Let $\Lambda$ denote a lattice in $\mathbb{R}^{2k}$ so that $T^{2k} = \mathbb{R}^{2k}/\Lambda$. Take an integer $n \gg 1$. There is an obvious covering map from $\mathbb{R}^{2k}/n\Lambda$ onto $\mathbb{R}^{2k}/\Lambda$ of degree $n^{2k}$, and we claim that this covering map can be extended to all the fibers in $M$ to give a covering projection $p : M_n \to M$.

The following lemma justifies this:

Lemma 4.2. The same transition functions as $\mathbb{R}^{2k}/\Lambda$-bundle $M$ give $\mathbb{R}^{2k}/n\Lambda$-bundle $M_n$ with the covering projection $p$.

Proof. For a transition map $g_{\alpha\beta} \in Sp(2k, \mathbb{Z})$ downstairs, the same transition map $g_{\alpha\beta}$ upstairs is the unique lifting map which satisfies $p \circ g_{\alpha\beta} = g_{\alpha\beta} \circ p$ and sends 0 to 0.

It only needs to be proved that the transition maps satisfy the axioms for the bundle, in particular the axiom $g_{\beta\gamma} \circ g_{\alpha\beta} = g_{\alpha\gamma}$. This follows from the uniqueness of the lifting map sending 0 to 0 (In fact, this cocycle condition holds without modulo $\mathbb{Z}$). \qed

We endow $M_n$ with a metric $h_n := p^* h$.

Lemma 4.3. There exists a closed 2-form $\omega$ on $M_n$ such that $\omega^{k+1} = 0$ and it restricts to a generator of $H^2(T^{2k}, \mathbb{Z})$ at each fiber $T^{2k}$.

Proof. For each $U \times T^{2k}$ where $U$ is an open ball in $B$, take $\omega$ to be a standard symplectic form of $T^{2k}$ representing a generator of $H^2(T^{2k}, \mathbb{Z})$. Since $\omega$ is invariant under $Sp(2k, \mathbb{Z})$, it is globally defined on $M_n$ (Note that the transition functions are locally constant). Obviously $\omega^{k+1} = 0$ at each point. \qed

Let $E$ be the complex line bundle on $M_n$ whose first Chern class is $[\omega]$. Take a connection $A^E$ of $E$ whose curvature 2-form $R^E = dA^E$ is equal to $-2\pi i \omega$.

We claim that

$$|R^E|_{h_n} \to 0 \quad \text{as} \quad n \to \infty.$$ 

Lemma 4.4. $|R^E|_{h_n} = O(\frac{1}{n^2})$.

Proof. Take a local coordinate $(x_1, \ldots, x_{4d}) \times (y_1, \ldots, y_{2k})$ of $B \times T^{2k}$ so that

$$\omega = dy_1 \wedge dy_2 + \cdots + dy_{2k-1} \wedge dy_{2k}.$$
We will show that $|dy_\mu|_{h_n} = O(\frac{1}{n})$ for all $\mu$. First,
\[ h_n(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = h(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}), \]
\[ h_n(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_\mu}) = nh(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_\mu}), \]
\[ h_n(\frac{\partial}{\partial y_\mu}, \frac{\partial}{\partial y_\nu}) = n^2 h(\frac{\partial}{\partial y_\mu}, \frac{\partial}{\partial y_\nu}) \]
for all $i, j, \mu$, and $\nu$. Thus
\[ h_n = \begin{pmatrix} O(1) & O(n) \\ O(n) & O(n^2) \end{pmatrix} \]
where the block division is according to the division by $x$ and $y$ coordinates, and
\[
(h_n)^{-1} = \frac{1}{\det(h_n)} \adj(h_n)
= \frac{1}{O(n^{4k})} \begin{pmatrix} O(n^{4k}) & O(n^{4k-1}) \\ O(n^{4k-1}) & O(n^{4k-2}) \end{pmatrix}
= \begin{pmatrix} O(1) & O\left(\frac{1}{n}\right) \\ O\left(\frac{1}{n}\right) & O\left(\frac{1}{n^2}\right) \end{pmatrix},
\]
which means $|dx_i|_{h_n} = O(1)$ and $|dy_\mu|_{h_n} = O(\frac{1}{n})$ for all $i$ and $\mu$, completing the proof. \hfill \Box

In order to use the Bochner argument, we need to show that $M_n$ is spin$^c$. Using the orthogonal decomposition by $h_n$,
\[ TM_n = V \oplus H = V \oplus \pi^*(TB), \]
where $V$ and $H$ respectively denote the vertical and horizontal space, and $\pi : M_n \to B$ be the torus bundle projection. Obviously $H$ is spin, because $B$ is spin. Since $V$ is a symplectic $\mathbb{R}^{2k}$-vector bundle, it admits a compatible almost-complex structure, and hence it can be viewed as a $\mathbb{C}^k$-vector bundle. Thus
\[ w_2(M_n) = w_2(V) + w_2(H) \equiv c_1(V) \mod 2 \]
meaning that $M_n$ is spin$^c$. Let $S$ be the associated vector bundle to the Spin$^c$ bundle over $M_n$ obtained using $\sqrt{\det\mathcal{F}}$. Consider a twisted spin$^c$ Dirac operator $D^E$ on $S \otimes E$ where $E$ is equipped with a connection $A^E$. The Weitzenböck formula says that
\[ (D^E)^2 = \nabla^* \nabla + \frac{1}{4} h_n + \mathcal{R}^E. \]
Here $\mathcal{R}^E(\sigma \otimes v) = \sum_{i<j} (e_i, e_\sigma) \otimes (R^E_{e_i, e_j} v)$ where $\{e_i\}$ is an orthonormal frame for $(M_n, h_n)$. Note that
\[ |\mathcal{R}^E|_{h_n} \leq C |R^E|_{h_n}, \]
where $C$ is a positive constant depending on the dimension of $M$. By taking $n$ sufficiently large, we can ensure that $s_n > |\Re^E|\rho_n$ everywhere, and hence $\ker D^E = 0$. Thus the index of the operator

$$D^E_+: \Gamma(S_+ \otimes E) \to \Gamma(S_- \otimes E)$$

is

$$\dim \ker D^E|_{S_+ \otimes E} - \dim \ker D^E|_{S_- \otimes E} = 0,$$

where $S_\pm$ respectively denotes the plus and negative spinor bundle.

On the other hand, we can also compute the index using the Atiyah-Singer index theorem \cite{9}. Note that $V$ has locally constant transition functions, and hence can be given a flat connection. This implies that $\hat{A}(V) = 1$ so that

$$\text{index}(D^E_+) = \{\text{ch}(E) \cdot \hat{A}(TM_n)\}[M_n]$$

$$= \{(1 + [\omega] + \cdots + \frac{1}{k!}[\omega]^k) \cdot \hat{A}(V) \cdot \hat{A}(\pi^*(TB))\}[M_n]$$

$$= \{(1 + [\omega] + \cdots + \frac{1}{k!}[\omega]^k) \cdot \pi^*(\hat{A}(B))\}[M_n]$$

$$= \frac{1}{k!}[\omega]^k \cdot \hat{A}(B)[B] \int_{\pi^{-1}(pt)} \frac{1}{k!}[\omega]^k$$

$$\neq 0,$$

which yields a contradiction.

The odd $m$ case is reduced to the even case. If $m$ is odd, consider an $S^1$-bundle over $M$ with transition functions exactly equal to the transition functions $\{\pm 1\}$ of the last $S^1$-factor of $T^m$ in $M$ over $B$. Then $M'$ is a $T^{m+1}$-bundle over $B$ with transition functions taking values in $Sp(m + 1, \mathbb{Z})$. We put a locally product metric on $M'$. Then it also has positive scalar curvature, yielding a contradiction from the above even case. □

**Theorem 4.5.** Let $B$ be a closed spin manifold of dimension $4d$ with nonzero $\hat{A}$-genus, and $M$ be an $S^1$ or $T^2$-bundle over $B$ whose transition functions take values in $GL(1, \mathbb{Z})$ or $GL(2, \mathbb{Z})$ respectively. Then

$$Y(M) = 0.$$

**Proof.** Again by Theorem 3.2, $M$ has a $T$-structure so that $Y(M) \geq 0$. It remains to show $M$ never admits a metric of positive scalar curvature, and let’s assume it does.

First, the case of $S^1$ bundle can be reduced to the case of $T^2$ bundle by considering a Riemannian product $M \times S^1$ which also has positive scalar curvature. From now on, we consider the case of $T^2$ bundle.
Secondly, we may also assume that $M$ is orientable, i.e., the transition functions for the torus bundle are orientation-preserving. Otherwise, we consider $M$ from the lemma below, which also admits a metric of positive scalar curvature by lifting the metric of $M$.

**Lemma 4.6.** There exists a finite covering $\tilde{M}$ of $M$ such that $\tilde{M}$ is an orientable $T^2$-bundle over a closed spin manifold of nonzero $\hat{A}$-genus.

**Proof.** Let $\tilde{B}$ be the universal cover of $B$, and $\tilde{M}$ be the manifold obtained by lifting the torus bundle over $B$ to $\tilde{B}$. Since $\tilde{B}$ is simply-connected, $\tilde{M}$ is orientable, and $\pi_1(B)$ acts on $\tilde{M}$ to give $\tilde{M} = \tilde{M}/\pi_1(B)$.

Let $G$ be a subset of $\pi_1(B)$, which consists of elements preserving orientation of the fiber torus. Then $G$ is a subgroup of index 2. Thus $\tilde{M}/G$ is an orientable $T^2$-bundle over $\tilde{B}/G$ which is a double cover of $B$ so that it is also spin with nonzero $\hat{A}$-genus. □

Now if $M$ is orientable, its transition functions take values in $SL(2, \mathbb{Z}) = Sp(2, \mathbb{Z})$ so that the previous theorem can be applied to derive a contradiction. □

**Remark 4.7.** In fact, Theorem 4.1 holds for any $T^m$-bundle with $T^m \times GL(m, \mathbb{Z})$-valued transition functions, which has a finite covering diffeomorphic to $M$ as in Theorem 4.1.

Combining our results with the previous results in [16], we can compute more general $T$-structured manifolds:

**Corollary 4.8.** Let $M$ be a $T^m$-bundle in all the above so that $Y(M) = 0$. If dim $M = 4n$, then

$$Y(M_k \mathbb{P}^n \oplus \overline{\mathbb{P}^n}) = 0,$$

and if dim $M = 16$, then

$$Y(M_k \mathbb{P}^4 \oplus \overline{\mathbb{P}^4} \oplus k' \mathbb{C}a\mathbb{P}^2 \oplus l' \overline{\mathbb{C}a\mathbb{P}^2}) = 0,$$

where $k, l, k', l'$ are nonnegative integers, and the overline denotes the reversed orientation.

In low dimensions such as 2 and 3, we understand all $T$-structured manifolds with zero Yamabe invariant. In dimension 4, we can compute the Yamabe invariant of some torus bundles by using the Seiberg-Witten theory.

**Theorem 4.9.** Let $B$ be a closed oriented manifold of dimension $\leq 3$, and $X$ be an $S^1$ or $T^2$-bundle over $B$. Suppose that $X \times T^m$ for $m = 4 - \dim X$ has a finite cover $M$ with $b_2^+(M) > 1$ which is a $T^2$-bundle over an oriented surface whose transition functions take values in a discrete subset of $T^2 \times SL(2, \mathbb{Z})$. Then

$$Y(X) = 0.$$
Proof. It suffices to show that $M$ never admits a metric of positive scalar curvature.

Using the 2-form $\omega$ on $M$ which restricts to a standard symplectic form on each fiber, we have a symplectic form $\pi^*\sigma + \omega$ on $M$, where $\sigma$ is a symplectic form of $\tilde{B}$, and $\pi : M \to \tilde{B}$ is the $T^2$-bundle projection.

Then the Seiberg-Witten invariant of the canonical $\text{Spin}^c$ structure of $M$ is $\pm 1$ so that it never admits a metric of positive scalar curvature. 

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