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The Type of Fractional Quotient and Consequential Development of Children's Quotient Subconcept of Rational Numbers[1)]

Kim, Ahyoung[*]

This paper investigated the conceptual schemes four children constructed as they related division number sentences to various types of fraction: Proper fractions, improper fractions, and mixed numbers in both contextual and abstract symbolic forms.

Methods followed those of the constructivist teaching experiment. Four fifth-grade students from an inner city school in the southwest United States were interviewed eight times: Pre-test clinical interview, six teaching / semi-structured interviews, and a final post-test clinical interview.

Results showed that for equal sharing situations, children conceptualized division in two ways: For mixed numbers, division generated a whole number portion of quotient and a fractional portion of quotient. This provided the conceptual basis to see improper fractions as quotients. For proper fractions, they tended to see the quotient as an instance of the multiplicative structure: $a\times b=c$; $a\div c=\frac{1}{b}$; $b\div c=\frac{1}{a}$.

Results suggest that first, facility in recall of multiplication and division fact families and understanding the multiplicative structure must be emphasized before learning fraction division. Second, to facilitate understanding of the multiplicative structure children must be fluent in representing division in the form of number sentences for equal sharing word problems. If not, their reliance on long division hampers their use of syntax and their understanding of divisor and dividend and their relation to the concepts of numerator and denominator.

I. Introduction

1. The necessity and the purpose of the research

Fractions represent a learning major obstacle in elementary school mathematics. Many children begin to think of mathematics as meaningless manipulation of symbols after learning fractions at school, lose their interest in mathematics, and decrease in achievement. These patterns coincide with instruction on fractions and rational numbers and proportions.

The results of NAEP(National Assessment of Educational Progress) testing show that students have a very weak understanding of fraction concepts (Wearne & Kouba, 2000). This is not a surprising phenomenon because rational numbers in fraction notation demand that children think fundamentally

* Soosong Junior High School (akim4@ewha.ac.kr)

different compared to the whole number domain. First, they need to think multiplicatively rather than additively(Vergnaud, 1983). Second, the unit in the whole number domain is always a whole or some easy factor, but units in fractions are often ambiguous parts of a whole(Hiebert, 1992). Third, many informal ways of thinking about operations that work well for whole numbers are not valid for fraction domains(Streefland, 1991).

Another reason for the difficulty of fraction concepts is their great variety. According to the literature, fractions as rational numbers can be interpreted in five ways in general: As a part-whole relationship, as a measurement, as a quotient, as a ratio, and as a operator(e.g., Behr, Lesh, Post, & Silver, 1983; Kieren, 1976; Lamon, 1999). Even though the various interpretations of fractions allow children to use fractions to describe and solve problems in diverse contextual situations, the complexity within and among these variations serve as a big obstacle children to coordinate each of those meanings into a coherent conceptual field.

Because of the importance of fractions to nearly all branches of higher mathematics, not to certain their direct utility, there have been many studies on each interpretation of fractions. However, the quotient interpretation has been less studied compared to other interpretations of fractions. In addition, in the area of quotient understanding, most previous studies' foci are restricted to children's partitioning strategies in equal sharing situations(e.g. Piaget, Inhelder, & Szeminska, 1960; Pothier & Sawada, 1983; Lamon, 1996; Streefland, 1991). However, if we think that the purpose of doing mathematics is not only to get an answer for a given question but also to develop a higher level of operative thinking about human action over given objects, it is necessary to study how children reflect on their concrete partitioning strategies to develop a more abstract, partitive, quotient fraction concept.

2. Research question

This paper investigated how children developed an abstract conception of quotient, and how they struggled overcoming the difficulty that their partitioning strategies hindered the direct mapping between fraction number representations and division representations of quotient situations. Therefore, the major research question is: What conceptual schemes do children construct as they relate division sentences to various fraction forms?

3. Definition of terms

A. Fractions

Fractions indicate bipartite symbols using a horizontal line or a slanted line, a certain form for writing numbers: $\frac{a}{b}$ or a/b. It is one of the notational systems representing rational numbers, but the fractions children use in the elementary school level can be thought of as a set of non-negative rational numbers which a is chosen from whole numbers and b is chosen from natural numbers (Lamon, 2007).

B. Division

In elementary school mathematics, division $x \div y$ is defined only when there is a whole number n such as $y \times n = x$. It is called division without a remainder. In the language of division, when x is divided by y, x and y, they are called dividend and divisor, respectively. The result of the dividing is called a quotient.

However, division of whole numbers is not enough to solve many real life problems. The equation $x \div y = q$ or $x = q \times y$ does not have a whole number solution when x is not a multiple of y. The situation is escaped by the introduction of division with remainder in whole number domain. However, when the solution is looked for without remainder, the result of such division has to be beyond the whole number domain. In order to distinguish this type of division with the division that has a whole number quotient, the former is called fraction division and latter is called whole number division(e.g. Sinicrope, 2002). In fraction division, the question asks "how much?" instead of "how many?"

In fraction division, the result of division (q) is symbolized as $\frac{x}{y}$. Its value is equal to $Q+r$ that include the whole number portion of the quotient (Q), which maybe zero, and the remainder (r). Using this, the former equation $x = y \times q$ can be transformed as $x = y \times (Q+r)$ and in this equation, r is a fractional portion of the quotient.

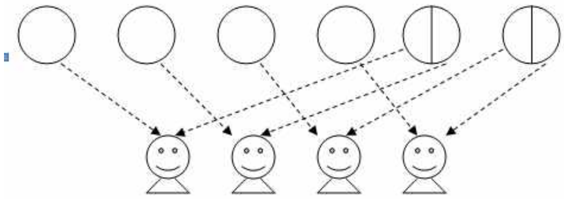
II. Review of the Literature

1. Preceding research

Children develop various partitioning strategies for equal sharing, depending on their previous social practice, the shapes of shared objects, the number of shared objects, and the number of people sharing (e.g. Pothier & Sawada, 1983; Lamon, 1996; Streefland, 1991). For example, children have a tendency to develop more economically efficient ways of partitioning - preserving wholes, cutting fewer pieces (Lamon, 1996). However, Charles and Nason(2000) argued children need three conditions to build partitive quotient as a general concept: construction of conceptual mapping through partitioning strategies for generating equal and quantifiable shares to abstract the partitive quotient notion of a fraction; direct mapping from the number of subjects to the unit fraction name of each share; direct mapping from the number of objects being shared to the number of the unit fraction in each share (i.e. as a dividend). Illustrated in [Figure II-1] is economically efficient, but it is hard for children to recognize the direct mapping between numerator of the fraction and the number of shared objects, and the direct mapping between the denominator of the fraction and the number of sharing people. However, their research on children's solutions was still limited to the results of partitioning concrete objects. Next, they only used small whole numbers from 1 through 6. These numbers were not large or varied to observe how children's knowledge of number facts affects their choice of strategy.

In addition, even though children may succeed in getting a fraction answer by partitioning given quantities equally, children still have difficulty recognizing the relationship between fractions and the division operation that signifies equal sharing. According to Toluk(1999), children do not interpret equal sharing situations with fraction results as a division problem at all. They tend to categorize problems where the solution is less than 1 as a fraction problem rather than "division". They initially

Q. If four friends share six cookies equally, how much cookies does each person get?



Symbolic expression of the question: $6 \div 4$

The answer from the partitioning of the objects: $1\frac{1}{2}$

[Figure II-1] The answer from the efficient partitioning strategy of equal sharing situation vs. symbolic expression of the situation

hold two separate and parallel conceptions pertaining to equal sharing problems. However, although she uncovered that children do plausible progression whereby more from contextual situations to give meaning to symbolically represented situations, she did not explain how different forms of children's fraction answers following different partitioning strategies were related this developmental progression. First, they did not consider the situations separately if number of shared objects and the number of sharers had a common factor or not. Second, they did not separate situations where the number of shared objects was bigger than the number of sharers where children preserved the whole number portion of quotient as mixed numbers, from situations where the number of shared objects was bigger than the number of sharers but where the shared objects could only be grouped if conceptualized at a higher level categorization (e.g. red markers and blue markers are both markers), so children dealt with each shared object and got an improper fraction answer.

2. Children's mathematical thinking about rational numbers written in fraction notation and different types of division problems

A. Quotient interpretations of rational numbers of fraction notation

Previous analyses show five different interpretations for fractions: part-whole relationship, measurement, quotient, ratio, and operator (e.g., Behr, Lesh, Post, & Silver, 1983; Kieren, 1976; Lamon, 1999; Ohlsson, 1988). Under the part-whole relationship interpretation, a fraction is used to represent a comparison between a whole broken up into various parts, and any number of those parts. Because children already bring primitive understandings of basic partitioning processes and some fraction words, for example, half, this part-whole interpretation is used first and most frequently to introduce fractions in elementary school. On the other hand, the symbol $\frac{a}{b}$ represents the quotient in division situations in which a whole number a is divided by b. It can be interpreted as partitive division which indicates the size of a group when a is put into b equal sized groups. The same fraction symbol can be interpreted as quotitive (i.e., measurement) division, too. In this case, the fraction means how many times a given quantity b is contained in another quantity a. Then, the fraction $\frac{a}{b}$ is interpreted as the number of groups to be made.

The problem is often that children conceptualize whole number division and fractional division as separate and distinct ideas. This makes it difficult to understand fractions as quotients. However, some recent researches report that children can develop this understanding by a sequence of specific instruction (e.g. Toluk, 1999; Middleton, de Silva, Toluk & Mitchell, 2001). According to them, at the beginning of developing the quotient fraction concept, many children think division always yields a whole number quotient with/without remainder and fractions cannot be the result of division. However, through equal sharing activities particularly those that partition a remainder in whole number division, they recognize the existence of a fraction quotient. By relating the division operation to equal sharing activities, they come to understand that division can produce a fraction quotient. However, the reverse of the operation does not come across their mind immediately. They develop this schema later and after that, they can grasp the concept of division as number.

B. Equivalent group division problems.

Division problems can be categorized as equivalent group problems, scalar or multiplicative comparison problems, and Cartesian product problems (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Ohlsson, 1988). Among them, equivalent group problems can be categorized into two different types of division problems: partitive division and quotitive division.

Partitive division is built on the notion of sharing or distributing a given quantity equally into a specified number of parts. Partitive division, therefore, is finding the number of objects in the group, in essence determining "what is the size of each group?"

Quotitive division arises from the notion of measurement. It asks finding how many groups are made with the specified number of objects: for example, "how many times is the extracting part put into the whole?" For the equation $3 \times 5 = 15$, each division problem can be posed like this:

Partitive division - Megan put 15 cookies into 5 bags with the same number of cookies in each bag. How many cookies are in each bag?

Quotitive division - Megan has 15 cookies. She puts 3 cookies in each bag. How many bags can she fill?

III. Method

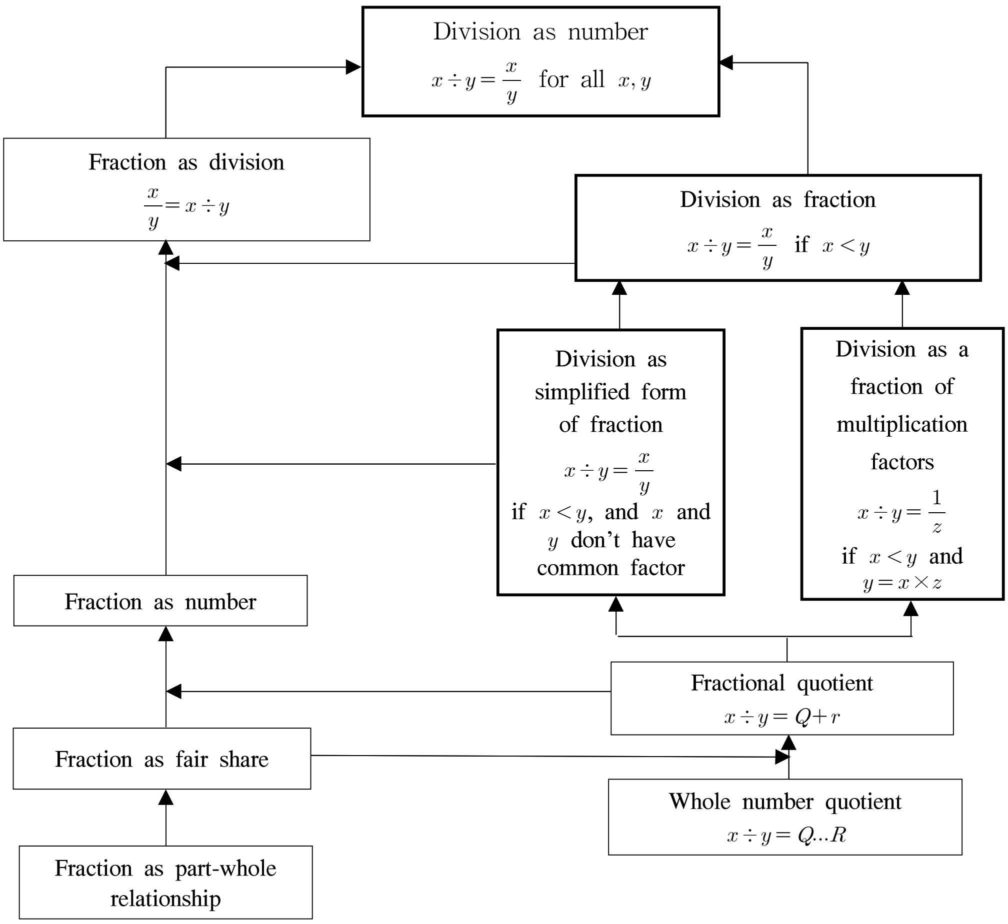
1. Participants

Four students from a fifth grade mathematics class in an inner city school in the southwest United States participated in the study. One boy and one girl were chosen from the high mathematics performance group and the other boy and the other girl were chosen from the low mathematics performance group each based on their mathematics teacher nomination (see <Table III-1>). A pseudonym was used to represent each participant.

2. Research Method and Research Design

The individual teaching experiment method was used. It consisted of a clinical interview phase, a teaching phase and an analysis phase (Steffe & Thompson, 2000). Each clinical interview and teaching interview was videotaped and transcribed by the researcher.

For this study, two clinical interviews were conducted with each student to assess his/her initial mathematical knowledge under investigation and to aim at the changes in their mathematical thinking each. Identical problem sets were used for both interviews. Six teaching episodes followed the initial clinical interview used to construct models of the children's mathematical thinking and guide them to develop more reflective ways of thinking about partitive quotient fraction that built on the results from the initial clinical interview. The analysis phases involved examining the data from the clinical interviews and the teaching episodes. The purpose of the analysis was to test research the hypotheses and to generate and test ad hoc hypotheses during the teaching episodes by planning, testing, and revising following teaching episodes (Steffe & Thompson, 2000). Therefore, both ongoing and retrospective



[Figure III-1] Hypothetical learning trajectory of partitive quotient fraction analyses were conducted.

A. Hypothesis testing

The Hypothetical learning trajectory of partitive quotient fraction is outlined in the following diagram([Figure III-1]). It is based on Toluk's work(1999) and is revised by the researcher based on pilot tests for a clinical interview.

The majority of emphasis in teaching episodes resided in the highlighted cells: Constructing a division interpretation of fractions depending on the forms in which the fractions were presented. The actual learning trajectory from the interaction with each particular child was different from the hypothetical learning trajectory and had some constraints. Therefore, these hypotheses were revised between teaching episodes to adjust them to the developmental level of the individual child.

B. Clinical interview

There were three groups of questions were used in the clinical interviews: partitive division questions in contextual problem type that had no provided diagrams, numeric symbol questions with division symbol, and fraction symbol's meaning questions(see Appendix 1). The researcher used probing questions(e.g., "Show me what you did"; "Why did you do that?"; "Show me how that works") in an attempt to draw out verbal, gestural, and written evidence of their thinking.

C. Teaching episodes

The question set used in each pair of episodes was categorized into three groups according to the forms of fractions that can be generated from each set of questions. The set of proper fraction questions was given children first. At this time, the effect of contemplating equivalent fractions was the major focus of a study. After that, to reduce the biased result caused by introducing fraction forms in different order, and to maximize the effect of the order of introducing fraction forms, the researcher divided children into different orders of presentation. In addition, the two levels of children's prior mathematical performance were considered as much as possible when the researcher distributed the children into two presentation orders. <Table III-1> shows the overall schedule used for clinical interviews and teaching episodes.

Each set of questions included two different types of questions similar to clinical interviews. First group of questions was contextual problems. After children solved each problem, the researcher asked them to represent their solutions with a

<Table III-1> Overall Schedule for Clinical Interviews and Teaching Episodes (Question Set 1(Q.S1): proper fraction questions, Question Set 2(Q.S. 2): mixed number questions, Question Set 3(Q.S. 3): improper fraction questions)

Subject	Initial Performance Level	Week 1 (interview 1)	Week 2 (interview 2, 3)	Week 3 (interview 4, 5)	Week 4 (interview 6, 7)	Week 5 (interview 8)
S1(girl: April)	High	Pre-test	Q.S.1	Q.S.2	Q.S.3	Post-test
S2(girl: Corey)	Low	Pre-test	Q.S.1	Q.S.2	Q.S.3	Post-test
S3(boy: Hunter)	High	Pre-test	Q.S.1	Q.S.3	Q.S.2	Post-test
S4(boy: Sam)	Low	Pre-test	Q.S.1	Q.S.3	Q.S.2	Post-test

mathematical expression. At the end of the group of questions, the researcher asked them if they had seen any kind of pattern between the numbers in the given questions and the numbers in their answers.

Second group of questions was division symbol questions. For the children who solved the problems with the manipulation of numbers, the researcher asked them to explain their solution with diagrams. For the children who solved the problems with diagrams, the researcher asked if they could solve the questions without drawing. For the children who could not solve the problems, the researcher asked them to explain the problems with diagrams or contextual situations.

During the interviews, the researcher read the questions aloud to the child and asked him/her to think aloud as he/she solved them. The researcher used probing questions which were similar to the things she used in the pre-test clinical interview.

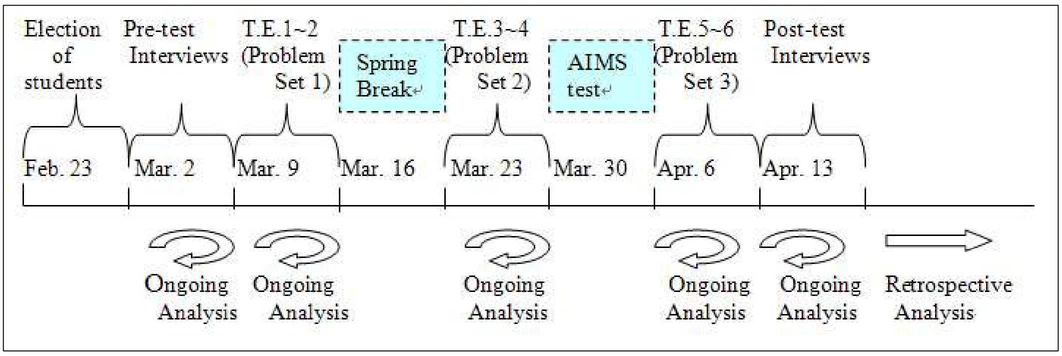
D. Analysis

Data analysis consisted of an ongoing analysis phase and a retrospective phase (Huberman & Miles, 1994). [Figure III-2] shows a time line of this teaching experiment.

IV. Results

At the beginning of the study, when equal sharing word problems were given to students to help them to move from whole number division to fraction division, all of them represented the situations as long divisions. To solve a given division number sentence, they rewrote it as long division. Then, they searched how many times the divisor (outside number of long division symbol) goes into the dividend (inside number of long division symbol). Two(Corey and Sam) of them solved the question as whole number division with remainder, others computed the quotient beyond the decimal point until there was no remainder. It showed they solved long division using a quotitive division interpretation. Later, this interpretation prevented students from solving division number sentences using a partitive division interpretation even though they could provide equal sharing situations for division number sentences.

On the other hand, the students' knowledge of multiplication fact families were very weak. They neither had understanding of the relationship among multiplication fact families nor had clear memory of multiplication fact families at the



[Figure III-2] Time line of data collection and analysis beginning of the study. In addition, they did not feel the necessity to writing a division number sentence based on multiplication fact families because they solved the problem using long division. Writing division number sentences was not a necessary step to solve division problems and students conceptualized any given problems in long division symbolic form.

Under the condition where students had lack of knowledge of multiplication fact families, when they tried to represent equal sharing word problems as division number sentences or rewrite long division symbolic forms as division number sentences, they conceptualized the given problem as long division and just changed the long division symbol to the division symbol and followed the direction of long division procedure.

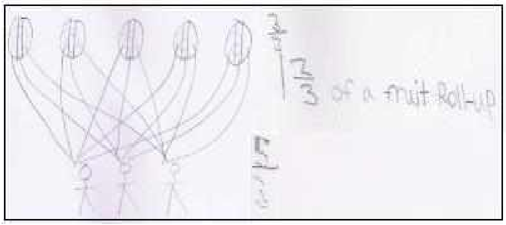
They conceptualized partitioning as yielding fractional quotient when the dividend was bigger than the divisor, but none of them could not conceptualize partitioning as fraction division itself.

In addition, their major understanding of fractions was a part-whole relationship. All students understood proper fractions as a part-whole relationship only. They understood mixed numbers as a combination of two different parts of numbers in a part-whole relationship : a whole number part and a proper fraction part. Two students (April and Corey) showed measurement understanding for fractions bigger than 1 and only one student (Corey) interpreted improper fractions as equal sharing situations. Except these two, other students did not understand what improper fractions meant even though they could read or convert mixed numbers to improper fractions.

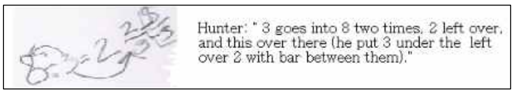
During the six teaching episodes, the students first conceptualized equal sharing word problems where the number of shared objects was bigger than the number of sharers as mixed fractions, and then they conceptualized equal sharing word problems in which the number of shared objects was smaller than the number of sharers as proper fractions. The reason that conceptualizing equal sharing as proper fraction came later than mixed numbers was the problems children had quantifying the result of their partitioning. If they had the whole number portion of quotient, they had a tendency to keep the unit a whole when they quantified pieces. If not, then they changed unit from a whole to part of a whole, or to the whole. For example, when Corey had to divide 3 pints of ice cream equally to 5 children and was asked that how much pint of ice cream would each got (Pre-Test Set, Q I-3, see Appendix 1) she changed 1 pint to 2 cups and drew three rectangles and cut each in half (cup) to each child in the picture then cut rest one half to fifth. She reported an answer as $1\frac{1}{5}$ cup of ice cream, but she quantified it as pint unit. Later, the problem that was designed to give each shared object with time interval reduced the burden of partitioning.

When they related measurement concepts to partitioning, students conceptualized equal sharing situations as improper fractions. In addition, equal sharing word problems specially designed such that each shared object had different qualities (e.g., five different flavored fruit roll-ups shared by three friends) helped students to understand the relationship between mixed numbers and improper fractions (see [Figure IV-1]).

However, moving from mixed fractions and improper fractions to equal sharing situations was



[Figure IV-1] Sam’s solution to the fruit roll-up question, 7th interview



[Figure IV-2] Hunter’s solution for division number sentence, 7th interview

different. The students conceptualized improper fractions as equal sharing at first. After that, two students conceptualized mixed fractions as equal sharing situations by converting them to improper fractions. For proper fractions, no one conceptualized them to equal sharing situations voluntarily because their part-whole relationship was too strong.

Throughout the study, students also conceptualized equal sharing situations as division number sentences. After they conceptualized equal sharing situations, in which the number of shared objects was a multiple of the number of sharers, as division number sentences, they conceptualized equal sharing situations in which the number of shared objects was bigger than the number of sharers as division number sentences. Then, they got mixed number quotients for the division number sentences. When students could convert the mixed numbers to improper fractions, they quickly conceptualized this group of division number sentences as improper fractions.

As an intermediate step, two of them(Hunter, Corey) discovered direct mapping from the remainder of the division number sentence to the numerator of the fractional portion of a mixed number quotient, and from the divisor of the division number sentence to the denominator of the mixed number quotient (see [Figure IV-2]).

After that, all students conceptualized improper fractions as a division number sentence by perceiving direct mapping of the dividend to the numerator of the quotient, and the divisor to the denominator (or fraction name) of the quotient regardless of quotient size (See [Figure IV-3]) and two students (April and Corey) predicted the generalization of the same direct mapping relationship regardless of the relative size of dividend to divisor. Then, all students conceptualize mixed numbers as division number sentences through its corresponding improper fractions.

Finally, students conceptualized equal sharing situations in which the number of shared objects was smaller than the number of sharers as a division number sentence. As an intermediate stage, students perceived the relationship that if the dividend was a factor of the divisor, then the quotient was a unit fraction and the denominator of the unit fraction was the second factor in the multiplication fact family. They explained the relationship such that: "If you do division following the order of divisor and dividend in the long division procedure, the answer is the number of pieces in one shared object." When they reflected on their solution by converting their quotient to the equivalent fraction with where the denominator the same as the divisor of the division number sentence, they perceived a direct mapping from the numerator to the dividend, and from the denominator to the divisor.

Among the four interviewed students, April solidly conceptualized division as a fraction and fraction

Q2. $22 \div 4 = 5\frac{1}{2} = 5\frac{2}{4} = \frac{22}{4}$ Q3. $5 \div 9 = \frac{5}{9}$ Q4. $8 \div 40 = \frac{8}{40} = \frac{1}{5}$

Corey: 22 divided by 4. 4 times 5 is 20. I did 22 minus 20 is 2, out of 4, so it would be 2 over 4, and then I simplified that, I got one half. (So the answer will be?) $5\frac{1}{2}$ or $5\frac{2}{4}$.

Researcher: Can you also combine these two (pointing 5 and $\frac{2}{4}$)?)

Corey: Improper fraction? (She wrote $\frac{22}{4}$.) Oh, so it's like that. (She wrote the dividend over the divisor and made the same improper fraction.)

Researcher: how about is this? (Showing a new question)

Corey: 5 divided by 9. I guess… (She wrote 5 over 9). It's hard to start with a small number and give it to big… (She started to solve it using a picture. Finally she got $\frac{5}{9}$ from the picture.) I will give each one person one ninth, and I have five of these, so five ninth. My prediction is right.

Researcher: how about this? (Showing a new question)

Corey: my prediction is 8 over 40.

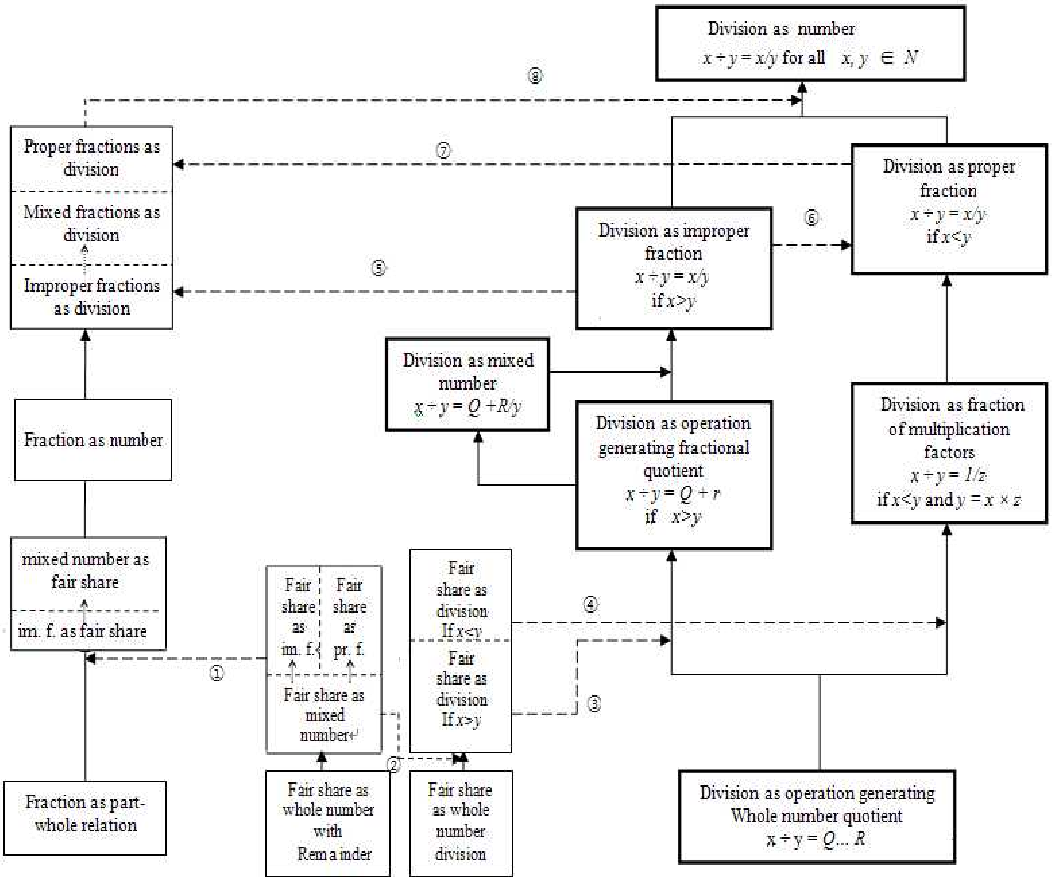
Researcher: is it your prediction or your answer?

Corey: I'm sure it's going to be my answer. But, I have to prove it.

[Figure IV-3] Corey solution for division number sentence at Post-Test

as division through the study. In Corey's case, she quickly conceptualized equal sharing situations as fractions and fractions as equal sharing situations from equal sharing as whole number quotient with remainder. However, she had a difficulty representing equal sharing situations as a division number sentence. So her conceptualization of the relationship between division and fraction retarded. In order, she started by conceiving of the method of writing division number sentences, she identified division with mixed number first, division with improper fractions next, and then division with proper fractions. In Hunter's case, he perceived the direct mapping from the division number sentence to fraction, but he did not have proper partitioning skill for equal sharing questions, so it was hard for him to check if the fraction generated from the direct mapping was correct or not. In addition, he was half way to conceptualize fractions as equal sharing situation at the post-test, so he could not yet conceptualize fractions as division. In Sam's case, his starting point was a lot behind than other students. Sam's partitioning strategy and his understanding of quantifying the result of partitioning did not progress enough, so he solved many problems incorrectly. As a result, even though he perceived the direct mapping from a division number sentence to an improper fraction answer on one occasion, he could not check it. Finally, he could not conceptualize division number sentence as any type of fraction or any type of fraction as division.

[Figure IV-4] shows this learning trajectory of the students' development of partitive quotient fraction which the highest level is division as



[Figure IV-4] Learning trajectory of partitive quotient fraction

number.

V. Discussion

The construction of the quotient interpretation for rational numbers in fraction form is important for children to be able understand both fractions and division as numbers. Later, it becomes the foundation for understanding of the algebraic form of division and its operation.

The result of previous Toluk's study(1999) was an initial model to develop this study, but the learning trajectory of partitive quotient fraction from this teaching experiment showed a quite different and more detailed learning trajectory. Contrary to her learning trajectory which showed vaguely conceptualizing division as a proper fraction preceded conceptualizing division as numbers, the order of the development of division as different forms of fractions in this study showed that students may begin to understand division as a mixed fraction first, and then begin to conceptualize division as improper fractions with division as proper fractions next. There appeared to be no order between the conceptualization of division as an improper fraction and as a proper fraction when equal sharing situations were used to relate division and fractions.

There were many variables to affect the students' conceptualization. To generate a proper fraction was usually faster and easier than generating improper fractions for equal sharing situations because the understanding of improper fractions was more related to the measurement concept than the part-whole concept. The order of conceptualization between equal sharing as proper fractions and equal sharing as improper fractions depended on how much this measurement concept was emphasized over the part-whole concept when they learned fractions. On the other hand, conceptualization of equal sharing as a division number sentence was easier when the number of shared objects was bigger than the number of sharers, so children generated division number sentences for quotient bigger than 1 earlier and easier than division number sentences for quotients smaller than 1.

Even though this teaching experiment revealed unknown facts about children's conceptualization of partitive quotient fractions, following this study with a bigger group is necessary to verify the result. In addition, this research mentioned little about the effect of other different forms of number notation-decimal vs. percent- because of the hugeness of the analysis. If any succeeding study includes those variables together, then we can fill out much of the map for conceptualization of quotient fractions.

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분수 몫의 형태에 따른 아동들의 분수꼴 몫 개념의 발달

김 아 영 (수송중학교)

본 연구는 아이들이 문장제 또는 수식 형태의 나눗셈의 결과를 여러 타입의 분수들-진분수, 가분수, 대분수-과 연관시키면서 분수가 가지는 여러 하위 개념 중 몫에 대한 개념 도식을 어떻게 구성해 가는지에 대하여 미국의 5학년 초등학생 네 명을 대상으로 이루어졌다. 실험 결과는 다음과 같았다. 균등분배 상황에서, 아이들은 나눗셈을 두 가지 방식으로 개념화하였다: 첫째, 아이들이 나눗셈을 통해 대분수 형태의 몫을 산출했을 경우, 이 대분수 형태의 몫은 진분수와 가분수 형태의 분수들을 부분-전체의 하위개념이 아니라 몫이라는 하위개념으로 이해하는데 개념적인 기초가 되었다. 둘째, 진분수 형태의 몫을 얻은 경우, 아이들은 그 몫을 곱셈구조의 예로 보려는 경향이 있었다. 즉, $a\times b=c$; $a\div c=\frac{1}{b}$; $b\div c=\frac{1}{a}$. 하지만, 장제법 계산은 소수 형태의 몫을 생산함으로써 아이들이 이 구조를 깨닫는 것을 어렵게 했다.

*key words : division(나눗셈), fractional quotient(분수꼴 몫), rational number(유리수), constructivism (구성주의)

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<Appendix 1> The Examples of Tasks Used in the Teaching Experiment

Pre-test/Post-test questionnaire

* Solve below questions thinking aloud.

Q I-1. With 96 pink roses, you can make 8 bouquets. How many pink roses will be in each bouquet?

Q I-2. If 8 children share 10 apples equally, then how much of an apple would each one get?

Q I-3. 5 children bought 3 pints of ice cream. If they share it equally, how much pint of ice cream would each child get?

Q I-4. 15 students want to share 5 yards of tape equally. How much of a yard does each one can get?

Q I-5. With 12 submarine sandwiches, 30 friends want to share equally. How much of a submarine sandwich does each one get?

Q I-6. 6 children want to share 9/10 of a gallon of lemonade. How much of a gallon does each one equally get?

* Solve each question thinking aloud.

Q II-1. $78 \div 6 =$ Q II-2. $22 \div 4 =$

Q II-3. $5 \div 9 =$ Q II-4. $8 \div 40 =$

Q II-5. $28 \div 42 =$ Q II-6. $\frac{2}{5} \div 4 =$

* Read each fraction. What does it mean to you?

Q III-1. $\frac{2}{9}$ Q III-2. $\frac{2}{4}$ Q III-3. 6

Q III-4. $1\frac{2}{3}$ Q III-5. $\frac{5}{2}$ Q III-6. $\frac{15}{6}$

Q III-7. $\dfrac{\frac{2}{3}}{\frac{1}{5}}$