대한수학교육학회지 <학교수학> 제 14 권 제 1 호
Journal of Korea Society of Educational Studies in Mathematics
School Mathematics Vol.14, No.1, 45-64. Mar 2012

Exploring Teachers' Knowledge of Partitive Fraction Division[1)]

Lee, Soo Jin*

The purpose of the present study was to investigate middle grades (Grade 5-7)[1)] mathematics teachers' knowledge of partitive fraction division. The data were derived from a part of 40-hour professional development course on fractions, decimals, and proportions with 13 in-service teachers. In this study, I attempted to develop a model of teachers' way of knowing partitive fraction division in terms of two knowledge components: knowledge of units and partitioning operations. As a result, teachers' capacities to deal with a sharing division problem situation where the dividend and the divisor were relatively prime differed with regard to the two components. Teachers who reasoned with only two levels of units were limited in that the two-level structure they used did not show how much of one unit one person would get whereas teachers with three levels of units indicated more flexibilities in solving processes.

I. Background

Most mathematics educators have accepted and supported that mathematics teachers need to have a qualitatively different and significantly richer understanding of mathematics than most teachers currently possess in order to support students' ways of knowing mathematics. A number of studies have just demonstrated that supporting teachers to meet the visions of such mathematics reform is difficult, but it is not as clear how different and how much richer their understanding of mathematics needs to be. (e.g., Borko et al., 1992; Jaworski, 1994; Kazemi & Franke, 2004; Shifter, 1998). Since Shulman (1986) announced the 'missing paradigm' in educational research by emphasizing the importance of teachers' subject matter knowledge, research on teacher knowledge parsed teacher knowledge into subject matter content knowledge, pedagogical knowledge, pedagogical content knowledge, and curricular knowledge. Many studies of mathematics teaching have investigated teacher knowledge under this frame (e.g., Ball, 1990; Ball, Lubienski, & Mewborn, 2001; Ball, Thames, & Phelps, 2008; Borko & Putnam, 1996; Ma, 1999). There are some variations in the way which various

* Montclair State University (leesoo@mail.montclair.edu)

1) In U.S. although fraction division is first introduced in grade 5, the only problem type students need to learn is division of a whole number by a unit fraction and vice versa. Even the Common Core State Standards for Mathematics, which is the new curriculum standards in U.S, states that division of a fraction by a fraction is not a requirement at grade 5. It is in grade 6 and 7 where students are exposed to all different kinds of fraction division problems with the emphasis on meanings of division. Thus, considering the educational situations in both U.S. and Korea, the participating middle grade teachers in this study may be compatible with the teachers at upper elementary levels in Korea. As a matter of fact, in U.S., 'upper elementary level' usually indicates grade 5, 6 and 7.

components of teacher knowledge are described and delineated in these studies, most researchers agree that teachers' content knowledge and their understanding of students' learning and thinking are critical aspects of good teaching. In mathematics education, Ball and colleagues (Ball, Thames, & Phelps, 2008) have used the phrase mathematical knowledge for teaching that is related to pedagogical content knowledge to emphasize knowledge that teachers use when solving problems that arise in practice—for instance, using curricular materials judiciously, choosing and using representations, skillfully interpreting and responding to students' work, and designing assessments. While research on teacher knowledge (e.g., Hill, Schilling, & Ball, 2004) in this line proved that there is a positive correlation between teacher knowledge and student achievement, research is needed to elaborate mathematical knowledge for teaching particular topics (Izsák, 2008).

Likewise, in the field of teaching 'rational numbers', even though researchers have provided considerable insight into students' conceptual understanding of rational numbers, the literature has yet to provide comparable insight into teachers' knowledge of these concepts. Recently several studies were done to examine teachers' knowledge in a deeper level and to attempt to understand teacher knowledge by coordinating with research on student knowledge. Izsák and his colleagues (Izsák, 2008; Izsák, Tillema, & Tunç-Pekkan, 2008) examined teacher knowledge based upon the literatures that decomposed mathematical concepts into numerous smaller cognitive structures (e.g., Behr, Harel, Post, & Lesh, 1994; Steffe, 1992, 1994). They examined teachers' knowledge of fraction addition and fraction multiplication at a grain size using Steffe's unit structure. In brief, Izsak et al. determined if teachers use two levels of units (e.g., one unit decomposed into a unit of units like 5 fifths) or three levels of units (e.g., one unit decomposed into a unit of unit of units like 5 fifths and each of fifths contains three thirds) by looking at their operating strategies of iterating and recursive partitioning. As a result of the study, they revealed that teachers' attention to the fixed whole and reasoning with three levels of units might be necessary for mathematical knowledge for teaching in fraction multiplication and fraction addition.

Traditionally, division of fractions has often been taught by emphasizing the algorithmic procedure "invert and multiply" without considering students' ways of constructing fraction division knowledge. While teachers do see the value of students' invention of the algorithm, it is difficult for them to provide the situation in which students can bring forth various operations that would eventually help students' construction of the knowledge because the teachers rarely understand the conceptual underpinnings of fraction division themselves. Although previous studies (e.g., Ball, 1990; Borko, 1992; Simon, 1993; Ma, 1999) have stressed errors and constraints on teachers' knowledge of fraction division, they have not considered teachers' capacities to reason in a sequence of fraction division situations. Further, even though the studies have revealed various knowledge components that are critical in developing teachers' knowledge of fraction, very few studies analyzed teachers' knowledge at a finer grain size.

In the present study I investigated middle grade (Grade 5-7) teachers' capacities to reason with fractional quantities in partitive division situations[2)] during a professional development program, which will be explained later. Through this research, I attempted to develop a model of the fractional knowledge of elementary and middle school teachers to see whether the model concurs with or deviates from existing models of children's fractional knowledge. This would be the first step toward building a learning trajectory[3] of teachers' ways of thinking, which can be extremely useful for thinking about how to build an effective professional development program and a teacher education program. Especially, I focused on one approach for supporting teachers' development of mathematical knowledge for teaching fraction division by emphasizing the relationship between the units and operations associated with partitive fraction division. Based on previous research findings (e.g. Izsák, 2008; Izsák et al., 2008), teacher's operations may appear different in terms of two knowledge components: (1) knowledge of units (referent units and levels of units), (2) partitioning operations (common partitioning, cross partitioning, and distributive partitioning). Thus, the research questions that guided this study were:

How did the participating teachers make sense of and provide solutions for partitive fraction division problems? Especially, what sort of knowledge components (knowledge of units and partitioning operations) did emerge and interrelate each other in the process of solving the partitive fraction division problems?

II. Theoretical Orientations

In this chapter, I will explain some of the terms by which the present study was guided. The premise that underpins my study is that teachers will reorganize their mathematical operations for a sharing situation through experiencing more complicated problems so that the ways could be applicable to more sharing problem situations and generalizable to a broader spectrum of problems. As teachers reorganize their ways of solving sharing problems, I assumed they would associate more knowledge elements, and my focus was to identify these knowledge elements and to describe to what extent the teachers coordinated the elements. It is by no means my assumption that teachers have never used the knowledge elements previously in their lives. Teachers may have already constructed such mathematical knowledge but may begin to be aware of it through revisiting mathematics with quantitative reasoning that has often been neglected by the curriculum.

1. Knowledge of Units

A. Referent Units

The concept of referent unit in this study is similar to the term adjectival quantities that Schwartz (1988) defined. According to Schwartz, "All quantities that arise in the course of counting or measuring or in the subsequent computation with counted

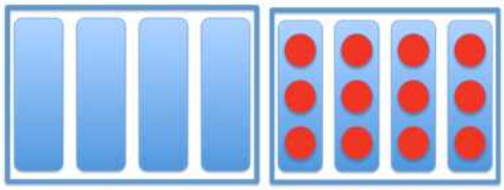
2) Division is typically thought of as having two different cases - quotitive and partitive. In partitive division, the dividend is shared into groups of the size of the divisor to figure out how much or how many one person or one thing gets.

3) I adapted the meaning of a learning trajectory of teachers' knowledge from Simon's hypothetical learning trajectory (1995). A *hypothetical learning trajectory* (HLT) is the sequential development and enrichment offered to learners as a result of working on a sequence of tasks and interacting with their teacher.

and/or measured quantities have referents and will be referred to as adjectival quantities." (p. 41) He further stated that all quantities have referents and that the "composing of two mathematical quantities to yield a third derived quantity can take either of two forms, referent preserving composition or referent transforming composition."(p. 41). To be more specifically, the referent unit of the third quantity remains the same in addition and subtraction, but it is different from either of the two original quantities in multiplication and division. For instance, the referent unit of the product of the following multiplication problem situation, "3 children have 4 apples each. How many apples do they have altogether?" is the total number of apples, which is different from the referent units of the multiplier (the number of children) and of the multiplicand (the number of apples per child). However, in the following addition context, the referent unit of the quotient for "4 apples and 3 oranges are in the basket. How many fruit are in the basket?" is the total number of fruit, which is same as the referent unit of the two addends.

B. Levels of Units[4)]

A student who forms two levels of units can understand a whole number, say, seven not only as seven separate units (one level of unit) but also as one group of seven conceived as a single entity by unitizing (the whole group is a second level of unit). Steffe (2003) states that such a student has formed composite units. He further elaborates that a student produces three levels of units by nesting composite units within composite units through interiorization and coordination of those composite units. For instance, a child who constructed a three-levels-of-units structure can assimilate an array of 12 candies as a single group of 12 (the whole group is one level), being composed of four units (a second level), where each of the four units is also a composite unit composed of three separate units (a third level) (See Figure II-1).



[Figure II-1] (a) The number '4' understood with two levels of units.
(b) The number '12' understood with three levels of units.

2. Partitioning Operations

A. Partitioning Operations to Produce a Common Multiple

In terms of partitioning operations, the problem situation in which the divisor and the dividend are relatively prime entails more complex operations than the situation in which the divisor or the dividend is a factor of another. Consider the more complicated problem of sharing two candy bars among five people. Here, using length quantities to determine the answer requires coordinating two three-level structures. One can start again by constructing a two-level structure for 2 as in Figure II-2a. The new challenge is that fifths do

4) Note that this section is extracted from Lee&Shin (2011). For more explanation and examples for 'levels-of-units structures', refer to Lee&Shin (2011).

not partition two units of one evenly. Thus, one needs to anticipate a finer partition that simultaneously subdivides twos and fifths. One way to accomplish this goal is to use one's knowledge of whole number factor-product combinations – 2 and 5 are factors of 10. Figure II-2a shows 2 as two units of five-fifths, and Figure II-2b shows 2 as five units of two-fifths. One now has constructed two three-level structures from which to find one-fifth of each candy bar, that is two-fifths of one. I called this a partitioning operation to produce a common multiple, which is similar to a common partitioning operation in that both partitioning operations produce two three-level structures. However, they are different because the former operation produces common multiples[5)] while the latter operation produces a common denominator between the dividend and the divisor quantities. Thus, the smallest unit from the partitioning operation for producing a common multiple is ten units of one-fifth.

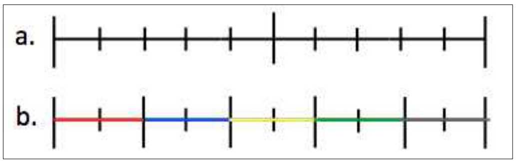


Figure II-2. Determining 2÷5. a. The first three-level structure for 2. b. The second three-level structure for 2.

One may use a cross partitioning operation to determine the answer to the candy bar problem by partitioning a bar vertically and horizontally by coordinating two composite units of 2 and 5 as in Figure II-3. The cross partitioning operation is different from the common partitioning operation in that the former provides one with a simultaneous repartitioning of each part of an existing partition without having to insert a partition into each of the individual parts (Olive, 1999). If one is aware of intervals of 1, then the one is reasoning with a cross partitioning operation. In other words, if one is not aware of the initial mid-level unit (two units of 1), one is not using a cross partitioning operation but a cross partitioning procedural strategy. Thus, using a cross partitioning operation implies one's reasoning with two three-level structures, whereas a cross partitioning strategy merely entails one's reasoning with two-level structures.

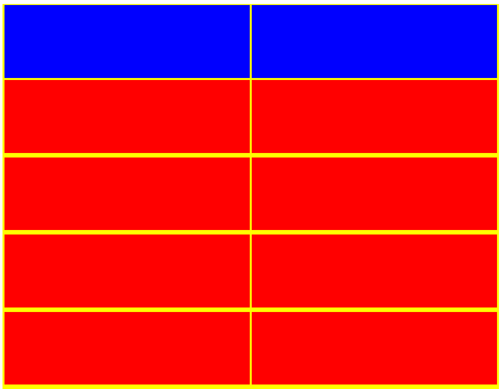


Figure II-3. Determining 2÷5 using a cross partitioning.

B. Distributive Partitioning Operations

According to Steffe and Olive (2010), a distributive partitioning operation is used when one partitions the whole quantity into the number of pieces in the divisor by partitioning each bar into the number of pieces in the divisor and assembles one piece from each bar (See Figure II-4). It

5) I am using plural because any common multiple between the 2 and 5 can be produced. For instance, one could subdivide each of the 2-part bars into 10 parts instead of 5 parts.

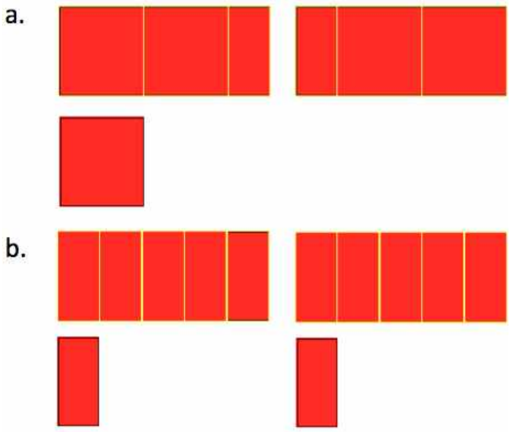


Figure II-4. Determining 2÷5 pulling one of.
a. Two 5-part bars. b. 5-subpart from two bars.

requires one's flexibility to forming three-level structures but does not necessarily entail distributive reasoning.

If a distributive partitioning operation is used with distributive reasoning, when one partitions each bar into five parts, one knows that the result of pulling one part of the bar shown in Figure II-4a. is the same quantity as taking one of the subparts from each bar as shown in Figure II-4b. If one cannot use the result of recursive partitioning[6) as given material, one may not think of 1 of the five subparts in each bar as one-fifth of 1 and assemble two of those results into two-fifths of 1. Unless one's distributive reasoning supported the distributive partitioning operation, one might lose track of the whole and focus on the number of pieces (10 pieces for the candy bar problem) without considering the fact that increasing the number of pieces in the whole decreases the size of the pieces. As a result, one may get 2/10 as an answer to the candy bar problem.

III. Methodology

1. Context and Participants

The present study was conducted within the activities of a project, Does it Work?: Building Methods for Understanding Effects of Professional Development (DiW), funded by the National Science Foundation (NSF). The data came from a part of 40-hour professional development course on fractions, decimals, and proportions offered in Fall 2008. The course was an InterMath course developed for the DiW research and was designed by the DiW team to support teachers in learning about the mathematics necessary for teaching the new standards in Georgia, which are similar to standards by the National Council of Teachers of Mathematics (2000). It requires teachers not only to compute efficiently and accurately with fractions, decimals, and proportions, but also to reason about them embedded in problem situations by engaging them in solving technology-enhanced, task-based investigations and by exploring a variety of drawn representations.

The course participants included 12 sixth and seventh grade mathematics teachers, 1 fifth grade teacher. Half of the teachers had teaching experience of more than 10 years. Among the 12 (grade 6-7) teachers, two were special education teachers at the middle school level. Ten participants had Master's degrees, and one had an Education Specialist degree. Teachers came from seven schools in the district. A few of the teachers knew each other from a workshop they took together in the summer of 2008.

The three themes the research project emphasized

6) Recursive partitioning operation is the first operation to be genuine fraction multiplication. It is also based on the construction of three levels of units. For more explanations, refer to Steffe & Tzur (1994)

for the InterMath were 'referent unit', 'drawn representations', and 'proportionality'. Referent unit referred to the whole for a given quantity. Drawn representations included array models, area models, single number lines, double number lines, tables, and graphs. These representations were intended to be used to reason about a given problem, not just as a picture of the solution. Proportionality referred to multiplicative reasoning in fraction and decimal operations as well as situations involving direct and inverse proportions. The content of the course was directly related to the state standards for fifth, sixth, and seventh grade mathematics. Each class meeting focused on a specific topic of rational numbers, for example fraction partitive division. For all 12 class meetings, teachers examined open-ended tasks pertaining to issues on multiplication and division with fractions and decimals, direct and inverse proportions, and worked with various technologies including Fraction Bars (Orrill, 2003). They met in a computer lab in the district's central offices. The lab contained 20 computers organized in five rows of two pairs in each row. A typical class consisted of the instructor greeting the class and either posing a warm-up problem or a task for the class to consider as a whole. A warm-up was either a portion of a task, an open-ended mathematical problem, or a mathematical question that we decided would provide a good start to our class. After some time for the teachers to consider the problem or task individually or with their partner, they had a mathematical discussion as a whole group. There were generally two whole group tasks that followed this format. Then teachers were asked to work on an individual task, which they needed to complete by the following week.

2. Data Collection & Analysis

The class met 3 hours per week for 14 weeks from September to December 2008. All class sessions were videotaped using two cameras: one recorded the class view, which included the instructor and overall classroom discussion, and another camera recorded the written work view, which captured a much closer look at participants' facial expressions, hand gestures, and their working with paper or computers. These two sources were then mixed to create a restored view of the event (Hall, 2000) so that both the class view and written work view could be analyzed simultaneously. As a research staff of the project, I participated in all the class sessions.

Data analysis occurred in stages. The first stage was ongoing analysis throughout the implementation of the professional development course. The DiW principal investigator, the instructor of a course, and I debriefed at the end of each session. Our discussion focused on how the participating teachers were making sense of the content and on planning for future sessions. Immediately after each session, the instructor created annotated timelines of each session using a lesson graph format. These summaries provided a written description of teachers' mathematical activities and interactions with the instructor. From the lesson graph, I added another column in each of the lesson graphs and wrote down emerging key points in teachers' reasoning that were taken into account for the next session. Teachers' partitioning operations and flexibilities with units were the two key knowledge elements that emerged at the time. I highlighted the time line and descriptions whenever I saw instances where teachers used the two knowledge components. Comments that were made by both the instructor and the co-principle investigator were fruitful.

The second stage of analysis was retrospective analysis. I worked from our initial lesson graphs that outlined the key incidents of the class meetings to identify key episodes and developed the lesson graphs that only contained the episodes related to my research interests. I generated a hypothesis that teachers seemed to use more sophisticated operations. Then, I went back and forth between the video data and my lesson graphs several times to ensure that my descriptions were reliable.

Ⅳ. Analysis

With regard to exploring participating teachers' partitive fraction division knowledge, I limited myself to examining one problem that teachers approached with a sharing goal. It was the 'licorice problem' where teachers were to share 11 inches of licorice equally among 12 people, and to answer, "How much licorice is there for one person?" The reason I chose the licorice problem was that when the dividend and the divisor were relatively prime, it opened the possibilities of teachers' using various partitioning operations with their available units and thus the differences between teachers with two levels of units and those with three levels of units clearly came into view. All of the teachers had their own computer to construct bars (they could also use paper and pencil to draw), and the instructor gave the teachers 10 or 15 minutes to work in small groups prior to the whole group discussion. The problem was given to the teachers to acquaint them with the Fraction Bar software, which allowed to create and enact various operations (i.e., partitioning, disembedding, iterating, pulling-out, breaking, etc.) on various geometric figures. Asking the problem not only helped the teachers to utilize various functions of the Fractions Bar software but also provided a good deal of data for answering my research question "How did the participating teachers make sense of and provide solutions for the partitive fraction division problem?" Depending on levels of units afforded to the teachers, they indicated different mathematical reasoning with different partitioning operations.

1. Teachers Who Reasoned with Three Levels of Units[7]

A. Partitioning through Finding a Common Multiple

The following protocols in this section shows how teachers' partitioning operations with coordination of two three-levels-of-units structures enabled them to find a common multiple and eventually led them to figure out the one person's share using distributive reasoning.

Protocol 1: Claire explaining to Carrie how she came up with her solution during whole-group discussion.[8]

CA: Say it one more time Claire please?

7) I would like to emphasize that my analysis is only based on those teachers who were captured on camera. By no means am I saying that all teachers could or did use the knowledge components.

8) For convenience, I will write the first two letters of speaker's name (e.g., Carrie=CA, Claire=CL, Diane=DI, Keith=KE, Donna=DO, Sharlene=SH, Rose=RO, Walt=WA). Rachael (RA) was an instructor of the class and the others were participants. All names are pseudonyms.

CL: Okay, there is 11 bar that represents 11 inches.
CA: Eleven bars going down.
CL: No across.
CA: Okay.
CL: Eleven horizontal bars each one represents one inch.
CA: Okay.
CL: and I split each bar into 12 equal parts. So the first person would get 11/12 [of the first bar], second person would get 11/12 of the second bar, etc, and then the twelfth person would get the blue pieces [purple in the screen] that ends the every little bar which comes out to 11/12 again.

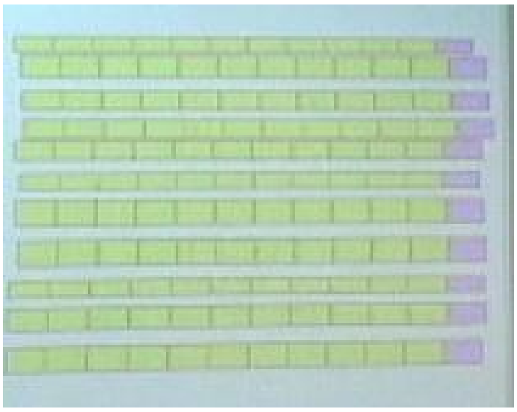


Figure IV-1. Claire's model of the licorice problem.[1]

Claire used a partitioning operation for a common multiple to solve the licorice problem. Claire conceived of the 11-inch licorice bar as 11 groups of twelve-part one inch bars and laid each bar horizontally as in Figure IV-1 (i.e. Claire made 11 12-part[9] bars). Then she colored the 11 12-part bars to show the distribution to 12 people. She colored eleven-twelfths of each 12-part bar in green to show a portion for 11 people and then colored the remaining one-twelfth of each 12-part bar in purple to show a portion of the 12th person. Thus, her partitioning operation was supported by distributive reasoning in that she knew that the rightmost column in Figure IV-1 was not just one-twelfth of each 12-part bar but eleven-twelfths of one bar. She was also explicitly aware of the two three-level structures with different mid-level units (one unit of 11 [a unit of units], 11 units of one [singleton units], 12 units of 11/12 [unit of units of units], and 132 units of 1/12 [unit of units]) by coordinating the two three level units structures. In other words, when she coordinated the second three level units structure (one unit of 11, 12 units of 11/12, and 132 units of 1/12) with the first three level units structure (one unit of 11, 11 units one, 132 units of 1/12), she was aware of the initial mid-level unit (11 units of one) and got 11/12 of one as an answer to the licorice bar problem.

Diane was another teacher who reasoned with three levels of units in making sense of the licorice bar problem. Unlike Claire who used common partitioning driven by her goal to pull out one-twelfth from the entire 11-inch licorice bar, Diane's partitioning which is illustrated in the protocol 2 was enacted by her goal to share each of the 11 inch-bars among 12 people.

Protocol 2: Diane focuses on the number of pieces to solve the licorice bar problem.

DI: I actually did the exact same thing as she (Claire) did. I just instead of making individual bars, I split it up into 11 pieces, the whole bar into 11 pieces
RA: So hang on, you just made one big bar
DI: square.
RA: Okay.

9) N-part refers to one bar partitioned into n parts.

DI: and then I split it into 11 and that is 11 inches, and here is my 11 inches going this way (she points each inch bar from the bottom in Figure IV-2). So basically I think mine is reversed from your [Claire] model. I think you [Claire] did eleven this way (moves her hand from left to right)? And then I took each bar of one inch and divided it [each bar] into 12, and I was going to give each person one piece [1/12 of 1] from all eleven bars (she slides her hand from the bottom to the top) so that is one whole piece [the leftmost column] for everybody they get 12 [whole piece]. Two, four, six, eleven (she counts the pieces in the leftmost column) pieces out of [each inch] divided into 12. So that's what I did. It's the same concept [as Claire], just a little different.

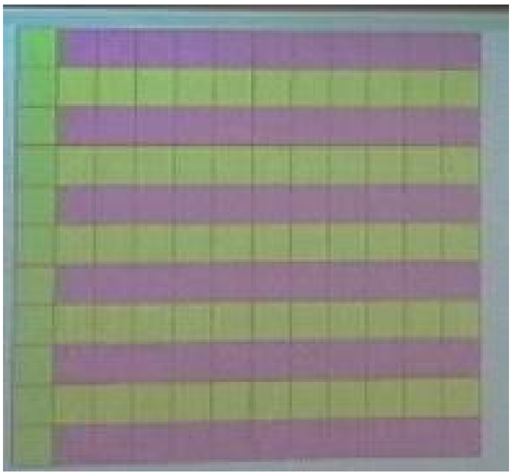


Figure IV-2. Diane's model of the licorice problem.

RA: How did you know I mean (pause) so how much does one person get?
DI: for mine, they get 1/12 because they get um one out of twelve pieces that were divided up.
RA: so you are saying this (the leftmost column) is one piece?
DI: that's out of all 132 pieces. They get 11 pieces out of 132.
CL: It works.
RA: 11 pieces out of 132. (Silent for about 2 minutes) so you are saying one person, if you answer the question one person, are you saying one person gets 1/12? What's the referent unit for that?
DI: Eleven inches whereas her [referent unit to the Claire's answer] was one inch and she had 11/12.
RA: Okay, so you got a different answer depending on what your referent unit is kind of like the Sam and Morgan [the problem discussed in the previous week] right? Which one are you saying is the whole? And then when you say 11 out of
DI: 132?
RA: What is the referent unit there?
DI: Um that was eleven pieces divided up into twelve. 11-inch is divided up into twelve each so each inch is divided up into twelve pieces. So the referent unit is the whole thing all eleven inches.
RA: Okay.

It may be argued that Diane used a distributive partitioning operation, but her partitioning was limited in the sense that it lacked distributive reasoning. After Diane made a bar, she said she partitioned the bar horizontally into 11 parts and then she subdivided each inch bar into 12 parts so that she could share the inch bar among 12 people as in Figure IV-2. Then she said one person got one piece from all eleven bars, which amounted to the leftmost column that she had colored in green.[10)]

While Claire used common partitioning by coordinating two three levels of units, Diane's partitioning was based on a distributive partitioning

10) Note that each of the even rows is colored in yellow not green.

operation in that her partitioning operation was motivated by her goal to share each of the 11 inches of bar among 12 people instead of sharing an entire bar. Her attention to 11 units of one inch was induced by her goal to share each inch bar among 12 people. Diane said she subdivided each inch bar into 12 parts because it was easier for her to share each inch among 12 people. Diane was reasoning flexibly with her whole number three levels of units. Diane said each person gets one piece from each of the 11-inch bars that were divided into 12 parts, and that these pieces summed up to 11 out of 132, which was 1/12. She conceived of 132 pieces as 11 units of 12 and 12 units of 11 and was aware of the equivalent relationship between 11/132 and 1/12. Such flexibility requires having at least constructed the ability of working with three levels of units. The smallest unit that she used was 1/132, and 11 of those units constituted each column in Figure IV-2. Hence, each person got 1/12 and the referent unit for her answer was the 11-inch bar. She said her answer was different from Claire's because their referent units were different. She further stated that Claire used one inch as the referent unit for her answer 11/12 whereas she used 11-inches as the referent unit for her answer of 1/12. Diane used her whole number three levels of units to solve the problem correctly. Also, it allowed her to see the distinction between hers and Claire's answers to the problem. However, her distributive partitioning was not based on distributive reasoning because she conceived the smallest unit as 1/132 as opposed to 1/12. Her meaning of 1/12 switched during the discussion of the licorice problem in the class.

To illustrate, whenever Rachael asked Diane how much one person's portion was, Diane said it was either 1/12 of each inch bar that was divided up into 12 pieces or 11 out of 132. Her meaning of 1/12 changed. She thought that 11/132 was 1/12 of 11 and thus each person would get each of the twelve columns. Even though she said the referent unit for 1/12 was 11-inches (or sometimes she used the term 'whole' or 11 inches), she was not thinking of the whole 11-inches as 12 units of 11/12 but as 12 units of 11. For her, it was unnecessary to view the smallest unit as 1/12 because her goal was different from Claire's. She was answering the question "How much of a whole (11-inch) licorice bar would one person get?" as opposed to "How much of a one-inch licorice bar one person would get?" Nevertheless, I still claim that her meaning of 1/12 switched a few times. Her switching does not necessarily mean that she was inflexible with referent units but more likely means that her interpretation of the referent unit was different from others. To elaborate, when Rachael asked Diane what the referent was when Diane said each person got "11 out of 132", Diane responded to Rachael's question that the referent unit for "11 out of 132" was 11 inches where each inch was composed of twelve pieces. Furthermore she said, "So the referent unit is the whole thing-all eleven inches." The fact that she used pieces to indicate twelve sub-parts and inches to indicate each inch and 11 inches or the 11-inch bar shows her conflation of the unit. I did not hear her say that one of the 12 sub-divided units was worth 1/12-inch. Diane was aware that the whole was the 11-inch bar and 11 units of one inch comprised the whole; however, the third-level unit for her was 132 units of 1/132, while the third-level unit for the 11-inch bar was 132 units of 1/12. Had Diane set the goal to pull out 1/12 of each bar as she knows a priori that 1/12 of 11 equals (1/12 of 1) 11 times, namely, had Diane used distributive reasoning, she could have solved the problem with coordination of two three levels of units structures without any conflation of units.

B. 11/12 of 11-inch bar = 1/12?

When Rachael asked "What is 11/12 of the 11-inch bar?", Diane quickly responded to her that 11/12 of the 11-inch bar was 1/12, whereas Claire used distributive reasoning to get 10 and 1/12. Claire even indicated from her model (See Figure IV-1) that 11/12 of the 11-inch bar was the green ones in the picture, which were all but one column. Even though Rachael explained to the whole class how Claire's reasoning made sense, most of the teachers were confused, and Diane was also confused considering that she had repeatedly mentioned that 11/12 of 11-inch was 1/12.

When Keith explained to the whole class that he could not understand why the teachers shared 11 inches of bars when they could just pull out 1/12 of the 11-inch bar, Diane said

It is because my unit is one inch and your unit is eleven inches so I have to show that there are 11 units of mine divided into 12 because I was using one inch as my unit as opposed to using the whole eleven inches as my unit. So I think that's why there is a difference.

If "my unit" was referring to the referent unit, she was definitely conflating as protocol 2 shows that she had repeatedly stated that the referent unit for her answer was 11-inches of the whole bar. One might think that "my unit" referred to the smallest unit; still she was conflating because it clearly shows that she was not attending to the 1/12 unit, which is related to my argument of her unit conflation that she was not conceiving 132 units of 1/12 but 132 units of 1/132. I hypothesize that if one uses a distributive partitioning operation that is given rise to by one's distributive reasoning, one could have accomplished the ideal coordination of two three levels of units in which the units are not conflated. It would entail one's awareness of the fact that the mid-level unit from the first three-level structure was shifted when it was coordinated with the second three-level structure. The two three levels of units in the coordination are as follows:

1st three-level structure: One unit of 11, 11 units of one, and 132 units of 1/12

2nd three-level structure: One unit of 11,12 units of 11/12,and 132 units of 1/12

As I have mentioned before, this coordination is supported by one's construction of the ability to recognize the multiplicative relationship between 11 units of 12 and 12 units of 11. Diane seemed to have coordinated the following structure instead of the former two three-level structures to find an answer to the licorice problem:

1st three-level structure: One unit of 11, 11 units of one, and 132 units of 1/132

2nd three-level structure: One Unit of 11, 12 units of 11, and 132 units of 1/132

Because she used her whole-number knowledge to count 11 pieces in the first column and saw that it was 1/12 of an entire rectangle instead of using 1/12. She colored in the entire first column green and said the green column was 1/12 of '12

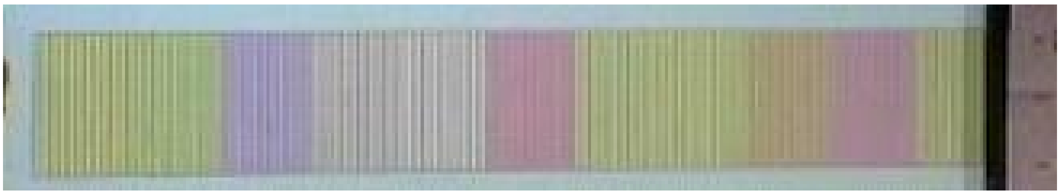


Figure IV-3. Donna's model of the licorice problem.

by 11' [12x11=132] pieces of licorice bar; thus one person's portion was 11/132. For her goal, reasoning with such unit structures was enough.

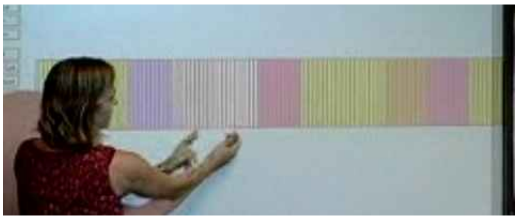


Figure IV-4. Rachael bracketing 11/12 inches from Donna's bar.

C. Shifting Mid-Level Units

When one uses partitioning operations to find a common multiple to figure out one person's share in sharing division situations, it is significant to strategically shift mid-level units when forming three-level structures. The following protocol illustrates the case in which one teacher flexibly changes the mid-level units depending on situations.

Protocol 3: Donna explains her strategy to the whole class.

DO: I was just literally thinking of an 11-inch strip of licorice. So I made this [Figure IV-3] long strip and divided it into 11 inches and then I divided each inch into 12 pieces and shaded 11 of each 12 [pieces].

RA: So this [11/12 of the leftmost portion in Figure IV-3] is 11 little pieces [of 1/12] that are yellow.

DO: Uhuh [yes]. So I have each person's portion together instead of the little left over [like what Claire did].

RA: So there should be 12 colored blocks if you went [all up to the 11 inches] (she points her fingers to the right in Figure IV-4) this is grey, this is white, it went all the way to the end.

CL: Okay.

Donna first divided the bar into 11 inches, then subdivided each inch into twelve pieces, and it gave the initial three level unit structure where one unit of 11 was the biggest unit, the length of one was the mid-level unit, and the length of 1/12 was the smallest unit. After that, she colored in 11 consecutive pieces to show each person's portion together. In other words, her initial three level unit structure was coordinated with the second three level unit structure of 11 in which the mid-level unit of the first three level unit structure (11 groups of one) was in the background and the second mid-level unit of the length of 11/12 was in the foreground. The ability to acknowledge a relationship between the two mid-level units is to know that the initial mid-level unit structure of 11 groups of one is actually 11 groups of 12(1/12) and the second mid-level unit of 12 groups of 11/12 are 12 groups of 11(1/12). In other words, the quantity remains the same whether you iterate the quantity 11 twelve times or the quantity 12 eleven times. Note that the whole is not 132 but still 11 because the smallest unit that one needs to attend to is not one but 1/12.

2. Teachers Who Reasoned with Two Levels of Units

Some of our teachers (Keith, Sharlene, Carrie, Linda) reasoned with only two levels of units while others (Claire, Donna, Diane, Mike, Walt) reasoned with three levels of units. Teachers who reasoned with only two levels of units were limited in that the two-level structure they used did not show how much of one (inch licorice bar) one person would get. In other words, teachers did not have the goal of measuring how much of one inch does one person get for the licorice problem.

A. Ignoring a Unit in Partitioning Operation

Figure IV-5. Reconstruction of Keith's model.

Unlike teachers who reasoned with three levels of units, Keith's reasoning was based on two levels of units to solve the licorice problem - the whole and 12 units of 1/12. When Rachael asked him how much one person gets, he pulled out one of the 12 pieces and told her it was one person's portion like in Figure IV-5. As Rachael continued to ask him how much the portion was, he clicked 'MEASURE' option of the computer program and told her it was 1/12 of the 11-inch bar but did not mention how much it was in terms of one. Even though his solution method was based on reasoning with two levels of units, he was able to understand other teachers' models that required him to reason with three levels of units because he explicitly stated that he knew 11/12 of one was equal to 1/12 of 11. He and Sharlene said they just could not understand why the others like Claire, Donna, Diane, and Mike needed to break a part (one inch) into finer pieces. This seems to have an implication to teachers' mathematical knowledge for teaching because it looks hard for teachers to reason with quantitative units when they have already abstracted school mathematics by teaching mathematics for several years. In particular, Keith was a high school teacher for a long time and it was his first teaching in the middle school.

B. Working with Partitioned Bar versus Non-Partitioned Bar

The following protocol contains more evidence to support the importance of reasoning with the partitioned 11-inch bar instead of the 11-inch bar with no partition for the sake of students' learning. After Claire, Diane, Donna, and Mike explained their strategies to the whole class, Rachael asked Keith to share his strategy because she knew Keith used a different but simpler one.

Protocol 4: Keith and Sharlene question the purpose of starting with 11 inches instead of 11-inch Licorice Bar

KE: I am seeing how most people were dividing whatever piece they have in the eleven pieces and then taking twelve and then dividing that each of those eleven pieces in the twelve little pieces. Well I personally did it in a totally different way [compares to Claire, Diane, Donna, and Mike]. I just took the 11 inches as a whole, and then I cut it into 12 pieces. So I am not giving you a little piece, you a little piece, you a little piece, and then doing that 12 times [he is explaining distributive partitioning]. I am just saying cut here is your portion cut here is your portion.

SH: I did mine the same way, so that's why when I see all these representations. I am like (she shakes her head side by side to express no) why so many itzie bitzie pieces when you can just take that 1/12 of it [11 whole].

RO: Well, the itzie bitzie pieces are showing that they are equally divided I think more so.

WA: Well, no, the simple cut of 12 cuts is easy to see, but the problem is you can't (stopped as Diane was speaking simultaneously).

DI: I was saying that it is because my [referent] unit is one inch and your [referent] unit is eleven inches so I have to show that there are 11 units of mine divided into 12 because I was using one inch as my [referent] unit as opposed to using the whole eleven inches as my [referent] unit. So I think that's why there is a difference.

RO: But then how do you know it is originally eleven units if you don't represent …

DI: I think that's what they are saying that it is not eleven units the whole eleven inches is one unit and they are dividing it into twelfths whereas we are saying that each inch is one unit.

WA: the only problem is you cannot answer exactly what it [1/12 of 11 whole] is compared to an inch something a little less than an inch. (Claire also nodded and agreed with his thinking.)

CA: Okay whatever this is just very annoying.

Keith asked why some teachers attempted to start with 11 inches of bar or partition the bar into 11 inches prior to dividing each of them into 12 parts when they could just divide the whole 11-inch bar into 12 parts and take 1/12 of the whole bar. Sharlene supported Keith's point, which explains why she was so confused by Claire's model. Keith and Sharlene both emphasized that they understood other teachers'models of partitioning into finer pieces, but they thought that their methodology was more efficient in terms of the number of pieces they needed to deal with. Rose sort of touched the surface but did not explicitly get to the point of addressing the importance of working with multi-level unit structures. If one do not have the goal of measuring how much of one inch one person's portion is, reasoning with two levels of units is enough. But one needs to form a three-level structure to accomplish the goal.

Walt's italicized comment in the protocol 4 in response to Keith's question shows that he was aware of a drawback the model of Keith and Sharlene could have. Also, Walt was aware of the intervals of one in the 11-inch bar, which facilitated him to deduce the fact that 1/12 of an 11-inch bar was a little less than an inch. Despite the fact that Walt could see such a relationship, it may not be an easy task for students (assuming that Keith uses the model in teaching) to think of 1/12 of 11 as a little less than an inch. The model used by Keith merely displays the amount (that was not yet measured) one person receives from the 11-inch bar. He also stated during the small group discussion that the answer was 1/12 of the 11-inch bar, but the model did not show it nor did he answer the question "How much of one candy bar does one person get?" Walt's comment did not provoke them to restate the answer in this way. When Walt commented on the drawback of Keith's method, only a few teachers, such as Claire, agreed with his point, while some of them, such as Carrie, were more confused. Unfortunately, Walt's comment got lost in the discussion because of the emphasis other teachers were placing on the referent unit issue. Rachael wrapped up the discussion of the problem by emphasizing that the answer to the Licorice Bar problem depended on the referent unit one chose.

V. Discussion

Improving teachers' knowledge of mathematics is crucial for improving the quality of instruction (Ball, Lubienski, & Mewborn, 2001; Ma, 1999), and efforts to improve the quality of classroom instruction have led to increased attention to promoting the development of teachers' mathematical knowledge for teaching. In Korea, Pang & Li (2008) reported that 20% of the 291 participating pre-service teachers in their study could not understand basic 'meaning' of fraction division and argued that content-specific pedagogical knowledge needed to be emphasized in teacher preparation programs. This implies teachers need to have opportunities to improve their profound understanding of elementary mathematics to improve the quality of teaching practices in classrooms (Oh, 2004). However, previous studies (e.g., Ball, 1990; Borko et al., 1992; Simon, 1993; Ma, 1999; Tirosh & Graeber, 1989) have merely stressed errors and constraints on teachers' knowledge of fraction division, few studies have been conducted to explore teachers' knowledge of fraction division at a fine-grained level (Izsák, 2008). Also, because little research has been done to conduct fine-grained analysis of teacher knowledge, I adapted ideas that appeared from the Fractions Project, which has studied children, in order to study teachers' knowledge of fraction division. In contrast to most research on teacher knowledge, this allowed me to concentrate on teachers' operations and flexibilities with conceptual units and to study teacher knowledge at a fine-grained size in a partitive fraction division situation.

The present study indicated that the knowledge components found in the previous research literature about children's fractional knowledge appeared in the participating teachers' mathematical activities with fraction problems and further turned out to be essential for their mathematical thinking in the context of a partitive fraction division problem. The teachers' coordination of two three level unit structures activated more sophisticated[11] partitioning operations. For instance, common partitioning operations produce two three-level unit structures. The teachers with three level unit structures could keep track of the referent unit of the quotient by using partitioning operations for a common multiple or distributive partitioning operations. Reasoning with three-level structures was necessary if the teachers' goal was to measure how much one person's portion was.

Although applying the results from research with children could be a viable way to start and there were some compatibilities between children's and teachers' ways of knowing, we are sure to remember that the development of teachers' knowledge differs from that observed in children because the teachers are already well equipped with procedural knowledge and they are likely to have more sophisticated number sequences already developed. Some participating teachers' common partitioning operations were evoked by their strategy of finding the common denominator between the two fractions. While the teachers brought forth common partitioning

11) Sophisticated in the sense that they are the partitioning operations, which produce multi-level unit structures.

operations by themselves, they believed they used an algorithm. They were referring to the algorithm in finding a common denominator for two fractions, which was a procedural strategy that they usually used in fraction addition or subtraction problems. Even though the common denominator algorithm was associated with common partitioning operations for the teachers, some of them thought that they used the algorithm. This result coincides with the fact that Korean elementary teachers in pre-service were familiar to 'inclusive algorithm' but were not good at dealing with the fraction divisor. They misunderstood 'divide with 1/2' to 'divide to 2' by the Korean linguistic structure (Park, Song, & Yim, 2004). I would argue this may cause serious problems when the teachers go back to their classrooms because they may teach common partitioning operations as an algorithm to their students. This conclusion also has an implication for designing effective professional development program in Korea as well as in U.S. In the program, teachers need to be able to explicitly become aware of the associations they make between the operations and the procedural algorithms. Having such awareness will constitute the "conceptualized understanding" (von Glasersfeld, 1995) of mathematics that teachers need to have as a part of mathematical knowledge for teaching.

While the present study explored teachers' partitive fraction division knowledge in one problem, future studies should construct teachers' knowledge of fraction division across various sequences of tasks. In addition, the studies need to expand on teachers' knowledge of fraction division by including whole numbers, decimals and beyond. One possible research issue relevant to this study would be to look into teachers' operations for solving partitive division problems and comparably their operations for solving linear equations in algebra. My conjecture is that teachers who could use common partitioning operations to solve the licorice bar problem may use such operations to make sense of problems such as 12x=11. While many of our teachers paid too much attention to the semantics of the licorice problem, which caused them to have a hard time conceptualizing the quantities as units, the problem teachers could pose themselves to solve the problem may be "twelve of what will give me eleven?" and this is actually the same problem situation for 12x=11.

References

Ball, D. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education*, 21(2), 132-144.

Ball, D., Lubienski, S., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 433-456). New York: Macmillan.

Ball, D., Thames, M., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389-407.

Behr, M. J., Harel, G., Post, T. R., & Lesh, R. (1994). Units of quantity: a conceptual basis common to additive and multiplicative structures. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in* *the learning of mathematics*. (pp.121-176). Albany: NY: SUNY Press.

Borko, H., Eisenhart, M., Brown, C. A., Underhill, R. G., Jones, D., & Agard, P. C. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education*, 23, 194-222.

Borko, H., & Putnam, R. (1996). Learning to teach. In D. Berliner & R. Calfee (Eds.), *Handbook of educational psychology* (pp. 673-708). New York: Macmillan.

Hall, R. (2000). Videorecording as theory. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp.647-664). Mahwah, NJ: Erlbaum.

Hill, H. C., Schilling, S. G., & Ball, D. (2004). Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal*, 105(1), 11-30.

Izsák, A. (2008). Mathematical knowledge for teaching fraction multiplication. *Cognition and Instruction*, 26(1), 95-143.

Izsák, A., Tillema, E., & Tunç-Pekkan, Z. (2008). Teaching and learning fraction addition on number lines. *Journal for Research in Mathematics Education*, 39(1), 33-62.

Jaworski, B. (1994). *Investigating mathematics teaching: A constructivist enquiry*. London: Falmer Press.

Kazemi, E., & Franke, M. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education*, 7, 203-235.

Lee, S. J., & Shin, J. (2011). Preservice teacher's key developmental understandings (KDUs) for fraction multiplication. *Journal of the Korean School Mathematics Society*, 14(4), 477-490.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Oh, Y. Y. (2004). Preservice teachers' understanding about elementary mathematics: Focused on multiplication with fractions. *Journal of Korea Society of Educational Studies in Mathematics*, 6(3), 267-281.

Olive, J. (1999). From fractions to rational numbers of arithmetic: a reorganization hypothesis. *Mathematical Thinking and Learning*, 1(4), 279-314.

Orill, J. (2003). *Fraction Bars [Computer software]*. Athens, GA: Transparent Media

Pang, J., & Li, Y. (2008). An analysis on the preservice elementary teachers' knowledge in the case of division of fractions. *Journal of Korea Society of Mathematical Education Series A: The Mathematical Education*, 47(3), 291-310.

Park, K. S., Song, S. H., & Yim, J. H. (2004). A study on understanding of the elementary teachers in pre-service with respect to fractional division. *Journal of Korea Society of Educational Studies in Mathematics*, 6(3), 235-249.

Schwartz, J. L. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (Vol. 2, pp.41-52). Reston, VA: Erlbaum.

Shifter, D. (1998). Learning mathematics for teaching: From teacher's seminar to the classroom. *Journal of Mathematics Teacher Education*, 1, 55-87.

Shulman, L. S. (1986). Those who understand: knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.

Simon, M. A. (1993). Prospective elementary *teachers' knowledge of division. Journal for Research in Mathematics Education*, 24(3), 233-254.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26(2), 114-145.

Steffe, L. P. (1992). Schemes and action and operation involving composite units. *Learning and Individual Differences*, 4(3), 259-309.

Steffe, L. P. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics*. New York: SUNY Press.

Steffe, L. P. (2003). Fractional commensurate, composition, and adding scheme Learning trajectories of Jason and Laura: Grade 5. *Journal of Mathematical Behavior*, 22, 237-295.

Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. New York: Springer.

Steffe, L. P., & Tzur, R. (1994). Interaction and children's mathematics. *Journal of Research in Childhood Education*, 8(2), 99-116.

Tirosh, D., & Graeber, A. (1989). Preservice elementary teachers' explicit beliefs about multiplication and division. *Educational Studies in Mathematics*, 20(1), 79-96.

von Glaserfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. New York, NY: Routledge Falmer.

교사들의 등분제 분수 나눗셈 지식에 관한 연구

이 수 진 (몽클레어주립대학교)

본 정성 연구에서는 교사들의 등분제 분수 나눗셈에 대한 지식을 분석하였다. 자료 수집은 13명의 교사들이 참여한 분수, 소수, 비례 등에 대한 주제를 다룬 40시간의 교사교육 프로그램으로부터 수집되어 일부분이 활용되었으며, 교사들의 등분제 분수 나눗셈 지식을 세밀하게 분석하기 위해 두 가지 지식 요소들(단위에 대한 지식, 분할 조작)을 분석틀로 사용하였다. 그 결과, 제수와 피제수가 서로소인 등분제 나눗셈 문제 상황을 다루는 능력이 두 지식 요소의 사용여부와 수준에 따라 다르게 나타났다. 두 단계의 단위 구조만을 가지고 추론한 교사의 경우 한 사람의 몫을 주어진 단위로 정확하게 나타낼 수 없었다는 점에서 제한점을 보였으며, 세 단계의 단위 구조를 가지고 추론한 교사는 다양한 분할 조작과 참조 단위의 활용으로 보다 유연하게 문제 상황에 대처할 수 있음을 보여주었다.

*key words : partitive fraction division (등분제 분수 나눗셈), referent units (참조 단위), levels of units (단위의 단계), partitioning operations (분할 조작)

논문접수 : 2012. 1. 24
논문수정 : 2012. 2. 23
심사완료 : 2012. 3. 8