SPECIAL WEAK PROPERTIES OF
GENERALIZED POWER SERIES RINGS

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Abstract. Let $R$ be a ring and $\text{nil}(R)$ the set of all nilpotent elements of $R$. For a subset $X$ of a ring $R$, we define $N_R(X) = \{a \in R \mid xa \in \text{nil}(R)$ for all $x \in X\}$, which is called a weak annihilator of $X$ in $R$. A ring $R$ is called weak zip provided that for any subset $X$ of $R$, if $N_R(X) \subseteq \text{nil}(R)$, then there exists a finite subset $Y \subseteq X$ such that $N_R(Y) \subseteq \text{nil}(R)$, and $R$ is called weak symmetric if $abc \in \text{nil}(R) \Rightarrow acb \in \text{nil}(R)$ for all $a, b, c \in R$. It is shown that a generalized power series ring $[R^S, \leq]$ is weak zip (resp. weak symmetric) if and only if $R$ is weak zip (resp. weak symmetric) under some additional conditions. Also we describe all weak associated primes of the generalized power series ring $[R^S, \leq]$ in terms of all weak associated primes of $R$ in a very straightforward way.

1. Introduction

All rings considered here are associative with identity. Any concept and notation not defined here can be founded in Ribenboim [17-19], Elliott and Ribenboim [6], and L. Ouyang [15-16].

Let $(S, \leq)$ be an ordered set. Recall that $(S, \leq)$ is Artinian if every strictly decreasing sequence of elements of $S$ is finite, and that $(S, \leq)$ is narrow if every subset of pairwise order-incomparable elements of $S$ is finite. Let $S$ be a commutative monoid. Unless stated otherwise, the operation of $S$ shall be denoted additively, and the neutral element by 0. The following definition is due to Elliott and Ribenboim [6].

Let $(S, \leq)$ be a strictly ordered monoid (that is, $(S, \leq)$ is an ordered monoid satisfying the condition that, if $s, s', t \in S$ and $s < s'$, then $s + t < s' + t$), and $R$ a ring. Let $[R^S, \leq]$ be the set of all maps $f : S \rightarrow R$ such that $\text{supp}(f) = \{s \in S \mid f(s) \neq 0\}$ is Artinian and narrow. With pointwise addition,
$[[R^S_\leq]]$ is an abelian additive group. For every $s \in S$ and $f$, $g \in [[R^S_\leq]]$, let $X_s(f, g) = \{(u, v) \in S \times S \mid u + v = s, f(u) \neq 0, g(v) \neq 0\}$. It follows from [18, Section 4.1] that $X_s(f, g)$ is finite. This fact allows to define the operation of convolution:

$$(fg)(s) = \sum_{(u, v) \in X_s(f, g)} f(u)g(v).$$

With this operation of convolution, and pointwise addition, $[[R^S_\leq]]$ becomes a ring (see [11-13] or [17-19]), which is called the generalized power series ring. The elements of $[[R^S_\leq]]$ are called generalized power series with coefficients in $R$ and exponents in $S$. 

For example, let $\mathbb{N}$ denote the set of positive integers. If $S = \mathbb{N} \cup \{0\}$ and $\leq$ is the usual order, then $[[R^{\mathbb{N} \cup \{0\}, \leq}]] \cong R[[x]]$, the usual ring of power series. If $S$ is a commutative monoid and $\leq$ is the trivial order, then $[[R^S_\leq]] \cong R[S]$, the monoid-ring of $S$ over $R$. Let $(S, \leq)$ be a strictly totally ordered monoid, which is also Artinian. For any $s \in S$, set $X_s =\{(u, v) \mid u + v = s, u, v \in S\}$. Then from [18, Section 4.1], it follows that $X_s$ is a finite set. Let $V$ be a free abelian additive group with the base consisting of elements of $S$. Then $V$ is a coalgebra over $\mathbb{Z}$ with the comultiplication map and the counit map as following:

$$\Delta(s) = \sum_{(u, v) \in X_s} u \otimes v, \quad \varepsilon(s) = \begin{cases} 1, & s = 0 \\ 0, & s \neq 0. \end{cases}$$

Clearly $[[R^S_\leq]] \cong \text{Hom}(V, R)$-the dual algebra.

Further examples and some properties of $[[R^S_\leq]]$ are given in [11-13] and [17-19].

Let $s \in S$, $r \in R$. We define $C^s_r \in [[R^S_\leq]]$ as follows:

$$C^s_r(s) = r, \quad C^s_r(t) = 0 \quad (s \neq t \in S).$$

Let $[[R^S_\leq]]$ be the generalized power series ring over $R$. Then $R$ is canonically embedded as a subring of $[[R^S_\leq]]$, and for each $f \in [[R^S_\leq]]$, and $r \in R$, $f \cdot r = f \cdot C^0_r$.

Given a ring $R$ we use $\text{nil}(R)$ to denote the set of all nilpotent elements of $R$. For a subset $X$ of $R$, $r_X(X) = \{a \in R \mid Xa = 0\}$ and $l_X(X) = \{a \in R \mid aX = 0\}$ will stand for the right and left annihilator of $X$ in $R$ respectively. Due to Marks [14], a ring $R$ is called NI if $\text{nil}(R)$ forms an ideal. A ring $R$ is called reduced if it has no nonzero nilpotent elements, and a ring $R$ is called semicommutative if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$. An ideal $I \subseteq R$ is said to be nilpotent if $I^n = 0$ for some natural number $n$.

In recent years, Ribenboim [17-19] and Zhongkui Liu [11-13] have carried out an extensive study of generalized power series rings. In this note we continue the study of generalized power series rings. Firstly, as a generalization of the right (left) annihilator, we introduce a notion of a weak annihilator of a subset in a ring. Next, we investigate various weak annihilator properties of the rings of generalized power series. Consequently, several known results
such as Ribenboim [17, 3.4] and Scott Annin [2, Theorem 5.2] and Hirano [8, Proposition 3.1] are generalized to a more general setting.

2. On weak annihilator

In this section, we first briefly develop the definition of the weak annihilator of a subset in a ring \( R \). Also we provide several basic results. Next we discuss some weak annihilator properties of generalized power series rings.

Definition 2.1. Let \( R \) be a ring. For a subset \( X \) of the ring \( R \), we define \( \mathit{N}_R(X) = \{ a \in R \mid xa \in \mathit{nil}(R) \text{ for all } x \in X \} \), which is called a weak annihilator of \( X \) in \( R \). If \( X \) is singleton, say \( X = \{ r \} \), we use \( \mathit{N}_R(r) \) in place of \( \mathit{N}_R(\{ r \}) \).

Obviously, for any subset \( X \) of a ring \( R \), \( \mathit{N}_R(X) = \{ a \in R \mid xa \in \mathit{nil}(R) \text{ for all } x \in X \} = \{ b \in R \mid bx \in \mathit{nil}(R) \text{ for all } x \in X \}, r_R(X) \subseteq \mathit{N}_R(X) \) and \( l_R(X) \subseteq \mathit{N}_R(X) \). For example, let \( \mathbb{Z} \) be the ring of integers and \( T_2(\mathbb{Z}) \) the \( 2 \times 2 \) upper triangular matrix ring over \( \mathbb{Z} \). We consider the subset \( X = \{(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}) \} \). Then \( r_{T_2(\mathbb{Z})}(X) = l_{T_2(\mathbb{Z})}(X) = \mathbb{Z} \), \( \mathit{N}_{T_2(\mathbb{Z})}(X) = \{ \begin{smallmatrix} 0 & 0 \\ 0 & m \end{smallmatrix} \mid m \in \mathbb{Z} \} \). Thus \( r_{T_2(\mathbb{Z})}(X) \subseteq \mathit{N}_{T_2(\mathbb{Z})}(X) \) and \( l_{T_2(\mathbb{Z})}(X) \subseteq \mathit{N}_{T_2(\mathbb{Z})}(X) \). If \( R \) is reduced, then \( r_R(X) = \mathit{N}_R(X) = l_R(X) \) for any subset \( X \) of \( R \). It is easy to see that for any subset \( X \subseteq R \), \( \mathit{N}_R(X) \) is an ideal of \( R \) in case \( \mathit{nil}(R) \) is an ideal.

Proposition 2.2. Let \( X, Y \) be subsets of \( R \). Then we have the following:

1. \( X \subseteq Y \) implies \( \mathit{N}_R(X) \supseteq \mathit{N}_R(Y) \).
2. \( X \subseteq \mathit{N}_R(N_R(X)) \).
3. \( \mathit{N}_R(X) = \mathit{N}_R(\mathit{N}_R(N_R(X))) \).

Proof. (1) and (2) are easy.

(3) Applying (2) to \( \mathit{N}_R(X) \), we obtain \( \mathit{N}_R(X) \subseteq \mathit{N}_R(\mathit{N}_R(N_R(X))) \). Since \( X \subseteq \mathit{N}_R(N_R(X)) \), we have \( \mathit{N}_R(X) \supseteq \mathit{N}_R(\mathit{N}_R(N_R(X))) \) by (1). Therefore we get \( \mathit{N}_R(X) = \mathit{N}_R(\mathit{N}_R(N_R(X))) \). \( \square \)

Proposition 2.3. Let \( R \) be a subring of \( S \). Then for any subset \( X \) of \( R \), we have \( \mathit{N}_R(X) = \mathit{N}_S(X) \cap R \).

Proof. Let \( r \in \mathit{N}_R(X) \). Then \( r \in R \) and \( xr \in \mathit{nil}(R) \) for each \( x \in X \), and so \( xr \in \mathit{nil}(S) \) for each \( x \in X \). Hence \( r \in \mathit{N}_S(X) \cap R \) and so \( \mathit{N}_R(X) \subseteq \mathit{N}_S(X) \cap R \). Assume that \( a \in \mathit{N}_S(X) \cap R \). Then \( a \in R \) and \( xa \in \mathit{nil}(S) \) for each \( x \in X \). Note that \( X \subseteq R \). We have \( xa \in \mathit{nil}(R) \) for each \( x \in X \). Thus \( a \in \mathit{N}_R(X) \) and so \( \mathit{N}_R(X) \supseteq \mathit{N}_S(X) \cap R \). Therefore \( \mathit{N}_R(X) = \mathit{N}_S(X) \cap R \). \( \square \)

Lemma 2.4. Let \( R \) be an \( NI \) ring and \( a, b \in R \). Then \( ab \in \mathit{nil}(R) \) implies \( arb \in \mathit{nil}(R) \) for every \( r \in R \).

Proof. Since \( \mathit{nil}(R) \) of an \( NI \) ring is an ideal, for every \( r \in R \), \( ab \in \mathit{nil}(R) \Rightarrow ba \in \mathit{nil}(R) \Rightarrow bar \in \mathit{nil}(R) \Rightarrow arb \in \mathit{nil}(R) \). \( \square \)
Proposition 2.5. Let $R$ be an NI ring and nil$(R)$ nilpotent, $S$ a cancellative torsion-free monoid, $(S, \leq)$ a strict order on $S$ and $f \in [[R^S, \leq]]$. Then $f \in \text{nil}([[R^S, \leq]])$ if and only if $f(s) \in \text{nil}(R)$ for every $s \in S$.

Proof. (⇒) Observe that $R/\text{nil}(R)$ is reduced and hence S-Armendariz in the sense of whenever $f, g \in [[R^S, \leq]]$ satisfy $fg = 0$, then $f(u)g(v) = 0$ for any $u, v \in S$ by [13, Lemma 3.1]. Suppose that $f^k = 0$ for some positive integer $k$. Then if we denote by $\overline{f}$ the corresponding generalized power series of $f$ in $[[R/\text{nil}(R)^S, \leq]]$, $\overline{f}^k = 0$. Since $R/\text{nil}(R)$ is S-Armendariz, $\overline{f(s)}^k = 0$ for any $s \in S$ by [13, Proposition 3.2]. Hence $f(s) \in \text{nil}(R)$ for any $s \in S$.

(⇐) Assume that $f(s) \in \text{nil}(R)$ for every $s \in S$. Then $f \in [[\text{nil}(R)^S, \leq]]$ where $[[\text{nil}(R)^S, \leq]] = \{f \in [[R^S, \leq]] \mid f(s) \in \text{nil}(R), s \in S\}$ is an ideal of $[[R^S, \leq]]$. Since nil$(R)$ is nilpotent, there exists some positive integer $k$ such that $(\text{nil}(R))^k = 0$. Then it is easy to see that $([[\text{nil}(R)^S, \leq]])^k = 0$. Hence we obtain $f^k = 0$. Therefore $f \in \text{nil}([[R^S, \leq]])$. □

Following Proposition 2.5, we obtain that if $R$ is an NI ring and nil$(R)$ nilpotent, $S$ a cancellative torsion-free monoid, $(S, \leq)$ a strict order on $S$, then the generalized power series ring $[[R^S, \leq]]$ is an NI ring and $\text{nil}([[R^S, \leq]]) = [[\text{nil}(R)^S, \leq]]$.

It was proved in Ribenboim [17, 3.3] that if $R$ is a Noetherian commutative ring, $(S, \leq)$ a cancellative torsion-free strictly ordered monoid and $f \in [[R^S, \leq]]$, then $f \in \text{nil}([[R^S, \leq]])$ if and only if $f(s) \in \text{nil}(R)$ for all $s \in S$. In the following, we show that the same is true even if $R$ is noncommutative.

Corollary 2.6. Let $R$ be a right Noetherian semicommutative ring, $(S, \leq)$ a cancellative torsion-free strictly ordered monoid and $f \in [[R^S, \leq]]$. Then $f \in \text{nil}([[R^S, \leq]])$ if and only if $f(s) \in \text{nil}(R)$ for all $s \in S$.

Proof. It suffices to show that nil$(R)$ is nilpotent. Since $R$ is a right Noetherian ring, we can find $a_1, a_2, \ldots, a_n \in \text{nil}(R)$ such that nil$(R)$ is generated by $a_1, a_2, \ldots, a_n$. Let $k \geq 1$ be such that $a_i^k = 0$ for all $1 \leq i \leq n$. We claim that nil$(R)^n = 0$. Consider a product

$$(a_1r_{11} + a_2r_{12} + \cdots + a_n r_{1n}) \cdots (a_1r_{nk+1,1} + a_2r_{nk+1,2} + \cdots + a_n r_{nk+1,n})$$

of $nk + 1$ elements in nil$(R)$. When this product is expanded, each term in it is a product of $2(nk + 1)$ elements, $nk + 1$ elements from the set $\{a_1, a_2, \ldots, a_n\}$, and $nk + 1$ elements from the set $\{r_{ij} \mid 1 \leq i \leq nk + 1, 1 \leq j \leq n\}$. Consider each term

$$a_{v_1}r_{v_1}a_{v_2}r_{v_2} \cdots a_{v_{nk+1}}r_{v_{nk+1}},$$

where $a_{v_1}, a_{v_2}, \ldots, a_{v_{nk+1}} \in \{a_1, a_2, \ldots, a_n\}$ and $r_{v_j} \in R$ for all $1 \leq j \leq nk + 1$. We will show that

$$a_{v_1}r_{v_1}a_{v_2}r_{v_2} \cdots a_{v_{nk+1}}r_{v_{nk+1}} = 0.$$

If the number of $a_1$ in $a_{v_1}r_{v_1}a_{v_2}r_{v_2} \cdots a_{v_{nk+1}}r_{v_{nk+1}}$ is greater than $k$, then we can write

$$a_{v_1}r_{v_1}a_{v_2}r_{v_2} \cdots a_{v_{nk+1}}r_{v_{nk+1}}.$$
as

\[ b_1a_1^{j_1}b_2a_2^{j_2} \cdots b_pa_p^{j_p}b_{p+1}, \]

where \( j_1 + j_2 + \cdots + j_p > k \) and \( b_i \in R \) for all \( 1 \leq q \leq p + 1 \). Since \( R \) is a semicommutative ring and \( a_1^{j_1+j_2+\cdots+j_p} = 0 \), it is easy to see that \( b_1a_1^{j_1}b_2a_2^{j_2} \cdots b_pa_p^{j_p}b_{p+1} = 0 \), and so \( a_1r_{v_1}a_2r_{v_2} \cdots a_{v_{n+k+1}}r_{v_{n+k+1}} = 0 \). If the number of \( a_i \) in \( a_1r_{v_1}a_2r_{v_2} \cdots a_{v_{n+k+1}}r_{v_{n+k+1}} \) is greater than \( k \), then similar discuss yields that \( a_1r_{v_1}a_2r_{v_2} \cdots a_{v_{n+k+1}}r_{v_{n+k+1}} = 0 \). Thus each term is zero, and so

\[(a_1r_1 + a_2r_2 + \cdots + a_n r_{1n}) \cdots (a_1r_{(nk+1)} + a_2r_{(nk+1)} + a_2r_{(nk+1)} + \cdots + a_n r_{(nk+1)n}) = 0.\]

Therefore nil(R) is nilpotent, as required. □

**Proposition 2.7.** Let \( R \) be an NI ring and nil(R) nilpotent, \((S, \leq)\) a cancellative torsion-free strictly ordered monoid, \( f, g, h \in [\langle R^S, \leq \rangle] \) and \( r \in R \). Then we have the following:

1. \( fg \in \text{nil}([\langle R^S, \leq \rangle]) \iff f(u)g(v) \in \text{nil}(R) \) for all \( u, v \in S \).
2. \( fgr = fgC_0^h \in \text{nil}([\langle R^S, \leq \rangle]) \iff f(u)g(v)r \in \text{nil}(R) \) for all \( u, v \in S \).
3. \( fgR \in \text{nil}([\langle R^S, \leq \rangle]) \iff f(u)g(v)h(w) \in \text{nil}(R) \) for all \( u, v, w \in S \).

**Proof.**

1. Suppose that \( fg \in \text{nil}([\langle R^S, \leq \rangle]) \). Then \( fg \in [\text{nil}(R)[S, \leq]] \) by Proposition 2.5. Thus \( f g = \bar{0} \) where \( f, g \) are the corresponding generalized power series of \( f, g \) in \([\langle R/\text{nil}(R)[S, \leq]\rangle] \). Since \( R/\text{nil}(R) \) is S-Armendariz, \( f(u)g(v) = 0 \) for any \( u, v \in S \). Hence \( f(u)g(v) \in \text{nil}(R) \) for any \( u, v \in S \). Conversely, let \( f, g \in [\langle R^S, \leq \rangle] \) be such that \( f(u)g(v) \in \text{nil}(R) \) for any \( u, v \in S \). Then \( (fg)(s) \in \text{nil}(R) \) since \( \text{nil}(R) \) is an ideal of \( R \). Hence \( fg \in \text{nil}([\langle R^S, \leq \rangle]) \) by Proposition 2.5.

2. (⇒) Suppose that \( fgC_0^h = f(gC_0^h) \in \text{nil}([\langle R^S, \leq \rangle]) \). Then for any \( u, v \in S \), by (1), we obtain \( f(u)(gC_0^h)(v) = f(u)g(v)r \in \text{nil}(R) \).

(⇐) Suppose that \( f(u)g(v)r \in \text{nil}(R) \) for all \( u, v \in S \). We show that \( fgr = f(gC_0^h) \in \text{nil}([\langle R^S, \leq \rangle]) \). For any \( s \in S \), we have

\[ (fgC_0^h)(s) = \sum_{(u,v) \in X_s(f,g)} f(u)g(v)r, \]

and \( f(u)g(v)r \in \text{nil}(R) \) for all \( u, v \in S \) implies that \( (fgC_0^h)(s) \in \text{nil}(R) \). Thus by Proposition 2.5, \( fgC_0^h \in \text{nil}([\langle R^S, \leq \rangle]) \).

3. It suffices to show (⇒). Suppose that \( fgh \in \text{nil}([\langle R^S, \leq \rangle]) \). Then from \( fgh = (fg)h \in \text{nil}([\langle R^S, \leq \rangle]) \), it follows that \( (fg)h \in \text{nil}(R) \) for each \( p, w \in S \). Now consider \( (fg)C_0^h \). Since \( \text{supp}(C_0^h) = \{0\} \) and \( C_0^h(0) = h(w) \), thus, by (1), we obtain \( (fg)C_0^h \in \text{nil}([\langle R^S, \leq \rangle]) \) for each \( w \in S \). Now by (2), we obtain \( f(u)g(v)h(w) \in \text{nil}(R) \) for all \( u, v, w \in S \). □

**Corollary 2.8.** Let \( R \) be a right Noetherian semicommutative ring, \((S, \leq)\) a cancellative torsion-free strictly ordered monoid, \( f, g, h \in [\langle R^S, \leq \rangle] \) and \( r \in R \). Then we have the following:

1. \( fg \in \text{nil}([\langle R^S, \leq \rangle]) \iff f(u)g(v) \in \text{nil}(R) \) for all \( u, v \in S \).
(2) \( fgr = fgC^0 \in \text{nil}([[R^{S \leq}]])) \iff f(u)g(v)r \in \text{nil}(R) \text{ for all } u, v \in S. \)

(3) \( fgh \in \text{nil}([[R^{S \leq}]])) \iff f(u)g(v)h(w) \in \text{nil}(R) \text{ for all } u, v, w \in S. \)

**Proof.** By analogy with the proof of Proposition 2.7, we can complete the proof. \( \square \)

Hirano observed relations between annihilators in a ring \( R \) and annihilators in \( R[x] \) (see [8]). In this note, we investigate the relations between weak annihilators in a ring \( R \) and weak annihilators in \( [[R^{S \leq}]] \). Given a ring \( R \), we define

\[
\text{NAnn}_R(2^R) = \{N_R(U) \mid U \subseteq R\},
\]

and

\[
\text{NAnn}_{[[R^{S \leq}]]}(2^{[[R^{S \leq}]]}) = \{N_{[[R^{S \leq}]]}(V) \mid V \subseteq [[R^{S \leq}]]\}.
\]

For a generalized power series \( f \in [[R^{S \leq}]] \), let \( C_f \) denote the set \( \{f(s) \mid s \in S\} \)
and for a subset \( V \) of \( [[R^{S \leq}]] \), let \( C_V \) denote the set \( \cup_{f \in V} C_f \).

Given a subset \( U \subseteq R \), let \( [[U^{S \leq}]] \) denote the set \( \{f \in [[R^{S \leq}]] \mid f(s) \in U, s \in S\} \). Then we can construct a map

\[
\phi : \text{NAnn}_R(2^R) \longrightarrow \text{NAnn}_{[[R^{S \leq}]]}(2^{[[R^{S \leq}]]})
\]

defined by \( \phi(N_R(U)) = N_{[[R^{S \leq}]]}([[U^{S \leq}]])) \text{ for any } N_R(U) \in \text{NAnn}_R(2^R). \)

**Proposition 2.9.** Let \( R \) be an NI ring and \( \text{nil}(R) \) nilpotent, \( (S, \leq) \) a cancellative torsion-free strictly ordered monoid. Then

\[
\phi : \text{NAnn}_R(2^R) \longrightarrow \text{NAnn}_{[[R^{S \leq}]]}(2^{[[R^{S \leq}]]})
\]

defined by \( \phi(N_R(U)) = N_{[[R^{S \leq}]]}([[U^{S \leq}]])) \text{ for any } N_R(U) \in \text{NAnn}_R(2^R) \) is bijective.

**Proof.** We show that \( \phi \) is injective. Suppose \( N_R(U) \in \text{NAnn}_R(2^R), N_R(U') \in \text{NAnn}_R(2^R) \) and \( N_R(U) \neq N_R(U') \). Without loss of generality, we may assume that there exists \( r \in R \) such that \( r \in N_R(U) \), and \( r \notin N_R(U') \). Then it is easy to see that \( C^0_r \in N_{[[R^{S \leq}]]}([[U^{S \leq}]])) \) and \( C^0_r \notin N_{[[R^{S \leq}]]}([[U'^{S \leq}]])) \), and so \( N_{[[R^{S \leq}]]}([[U^{S \leq}]])) \neq N_{[[R^{S \leq}]]}([[U'^{S \leq}]])) \). Hence \( \phi(N_R(U)) \neq \phi(N_R(U')). \)

Therefore \( \phi \) is injective.

Now we show that \( \phi \) is surjective. For any

\[
N_{[[R^{S \leq}]]}(V) \in \text{NAnn}_{[[R^{S \leq}]]}(2^{[[R^{S \leq}]]}), \quad V \subseteq [[R^{S \leq}]],
\]

then \( N_R(C_V) \in \text{NAnn}_R(2^R) \). To show \( \phi \) is surjective, it suffices to show

\[
\phi(N_R(C_V)) = N_{[[R^{S \leq}]]}([[C_V^{S \leq}]])) = N_{[[R^{S \leq}]]}(V).
\]

Since \( V \subseteq [[[C_V^{S \leq}]], N_{[[R^{S \leq}]]}([[C_V^{S \leq}]])) \subseteq N_{[[R^{S \leq}]]}(V) \text{ is clear. Now we show that } N_{[[R^{S \leq}]]}(V) \subseteq N_{[[R^{S \leq}]]}([[C_V^{S \leq}]]). \)

Assume that \( f \in N_{[[R^{S \leq}]]}(V) \). Then \( gf \in \text{nil}([[R^{S \leq}]])) \) for all \( g \in V \). By Proposition 2.7, we obtain \( g(u)f(v) \in \text{nil}(R) \text{ for all } u, v \in S \), and so \( f(v) \in N_R(C_V) \) for every \( v \in S \). Then for each \( h \in [[[C_V^{S \leq}]], \text{by Proposition 2.7, it is easy to see that } h(f) \in \text{nil}([[R^{S \leq}]])), \)
and so \( f \in N[R^S;\leq][[[C_V];S;\leq]]. \) Hence \( N[R^S;\leq][V] \subseteq N[R^S;\leq][[[C_V];S;\leq]]. \) Therefore \( N[R^S;\leq][V] = N[R^S;\leq][[[C_V];S;\leq]] = \phi(N_{R}(C_V)), \) as required. \( \square \)

A ring \( R \) is called right zip if \( rR(Y) = 0 \) is weak zip provided that the right annihilator \( rR(X) \) of a subset \( X \) of \( R \) is zero, then there exists a finite subset \( Y \) of \( R \), such that \( rR(Y) = 0 \). Beachy and Blair [4] showed that if \( R \) is a commutative zip ring, then the polynomial ring \( R[x] \) over \( R \) is a zip ring. Hong et al. [9, Theorem 11] proved that \( R \) is a right (left) zip ring if and only if \( R[x] \) is a right (left) zip ring when \( R \) is an Armendariz ring. As a generalization of zip rings, in [15], L. Ouyang introduced the notion of weak zip rings and showed that if \( R \) is an \((\alpha,\delta)\)-compatible and reversible ring, then \( R \) is weak zip if and only if the Ore extension \( R[x;\alpha,\delta] \) is weak zip. In the following, we investigate the weak zip property of rings of generalized power series.

**Definition 2.10.** A ring \( R \) is called a weak zip ring provided that for any subset \( X \) of \( R \), if \( N_{R}(X) \subseteq \text{nil}(R) \), then there exists a finite subset \( Y \subseteq X \) such that \( N_{R}(Y) \subseteq \text{nil}(R) \).

Obvioulsy, all reduced zip rings are weak zip, and if \( R \) is a weak zip ring, then so is the \( n \times n \) upper triangular matrix ring over \( R \). Further examples and some properties of weak zip rings are given in [15].

**Proposition 2.11.** Let \( R \) be an NI ring and \( \text{nil}(R) \) nilpotent, \((S,\leq)\) a cancellative torsion-free strictly ordered monoid. Then \( R \) is weak zip if and only if \( [[R^S;\leq]] \) is weak zip.

**Proof.** Assume that \( R \) is weak zip and \( V \) a subset of \( [[R^S;\leq]] \) with \( N[[R^S;\leq]][V] \subseteq \text{nil}([[R^S;\leq]]). \) Now we show that \( N_{R}(C_{V}) \subseteq \text{nil}(R) \). If \( r \in N_{R}(C_{V}) \), then \( ar \in \text{nil}(R) \) for all \( a \in C_{V} \). So for any \( f \in V \) and any \( s \in S, \) 

\[
(fr)(s) = (fC_{v}^{a})(s) = f(s)r \in \text{nil}(R),
\]

and so by Proposition 2.5, \( fr \in \text{nil}([[R^S;\leq]]). \) Hence

\[
r \in N[[R^S;\leq]][V] \subseteq \text{nil}([[R^S;\leq]]).
\]

Thus \( r \in \text{nil}(R) \) and this implies \( N_{R}(C_{V}) \subseteq \text{nil}(R) \). Since \( R \) is weak zip, there exists a finite subset \( Y_0 = \{ q_1, \ldots, q_m \} \subseteq C_V \), such that \( N_{R}(Y_0) \subseteq \text{nil}(R) \). Let \( f_i \) be an element of \( V \) such that \( f_i(q_i) = q_i \) for some \( q_i \in S, \) \( i = 1,2,\ldots,m \). Let \( V_0 = \{ f_1, f_2, \ldots, f_m \}. \) Then \( V_0 \) is a finite subset of \( V \), and \( C_{V_0} \supseteq Y_0 \). So \( N_{R}(C_{V_0}) \subseteq N_{R}(Y_0) \subseteq \text{nil}(R) \). Now we show that \( N[[R^S;\leq]][V_0] \subseteq \text{nil}([[R^S;\leq]]). \) Suppose \( g \in N[[R^S;\leq]][V_0] \). Then \( fg \in \text{nil}([[R^S;\leq]]) \) for all \( f \in V_0 \). By Proposition 2.7, we obtain \( f(u)g(v) \in \text{nil}(R) \) for all \( u, v \in S \). Hence \( g(v) \in N_{R}(C_{V_0}) \subseteq \text{nil}(R) \) for all \( v \in S \), and so by Proposition 2.5, \( g \in \text{nil}([[R^S;\leq]]). \) Hence \( N[[R^S;\leq]][V_0] \subseteq \text{nil}([[R^S;\leq]]). \) Therefore \( [[R^S;\leq]] \) is weak zip.

Conversely, let \( Y \subseteq R \) with \( N_{R}(Y) \subseteq \text{nil}(R) \). If \( f \in N[[R^S;\leq]][Y] \), then \( gf = C_{v}^{a}f \in \text{nil}([[R^S;\leq]]) \) for all \( g \in Y \), and so \( gf(s) \in \text{nil}(R) \) for all \( y \in Y \), and \( s \in S \). Thus \( f(s) \in N_{R}(Y) \subseteq \text{nil}(R) \) for all \( s \in S \). By Proposition 2.5,
f ∈ nil([[R^{S,≤}]])). Hence N_{[[R^{S,≤}]]}(Y) ⊆ nil([[R^{S,≤}]])). Since [[R^{S,≤}]] is weak 
zip, there exists a finite subset Y_0 ⊆ Y such that N_{[[R^{S,≤}]]}(Y_0) ⊆ nil([[R^{S,≤}]])). 
Hence N_R(Y_0) = N_{[[R^{S,≤}]]}(Y_0) ∩ R ⊆ nil([[R^{S,≤}]]) ∩ R = nil(R). Therefore R 
is weak zip.

Following Lambek [10], a ring R is called symmetric if abc = 0 implies 
acb = 0 for all a, b, c ∈ R. It is obvious that commutative rings are symmetric. 
Reduced rings are symmetric by the results of Anderson and Camillo [1], but 
there are many nonreduced commutative (so symmetric) rings. As a general-
ization of symmetric rings, L. Ouyang introduced the notion of weak symmetric 
rings and showed that if R is an (α, δ)-compatible and reversible ring, then R 
is weak symmetric if and only if the Ore extension R[x; α, δ] is weak symmetric 
[16]. In the following, we investigate the weak symmetric property of the rings 
of generalized power series.

Definition 2.12. A ring R is called a weak symmetric ring if abc ∈ nil(R) ⇒ 
acb ∈ nil(R) for all a, b, c ∈ R.

Proposition 2.13. Let R be an NI ring and nil(R) nilpotent, (S, ≤) a can-
cellative torsion-free strictly ordered monoid. Then R is weak symmetric if and 
only if [[R^{S,≤}]] is weak symmetric.

Proof. Since any subring of a weak symmetric ring is again a weak symmetric 
ring, it suffices to show that if R is a weak symmetric ring, then so is [[R^{S,≤}]]. 
Let f, g, h ∈ [[R^{S,≤}]] be such that fgh ∈ nil([[R^{S,≤}]]). By Proposition 2.7, we 
have f(ug)vbh(w) ∈ nil(R) for all u, v, w ∈ S, and so f(u)h(2w)g(v) ∈ nil(R) 
for all u, v, w ∈ S since R is weak symmetric. Hence fgh ∈ nil([[R^{S,≤}]]). By 
Proposition 2.7. Therefore [[R^{S,≤}]] is a weak symmetric ring.

The following corollary will give more examples of weak zip rings and weak 
symmetric rings.

Corollary 2.14. Let (S_1, ≤_1), (S_2, ≤_2), . . . , (S_n, ≤_n) be cancellative torsion-
free strictly ordered monoids. Denote by (lex ≤) and (revlex ≤) the lexi-
ographic order, the reverse lexicographic order, respectively, on the monoid 
S_1 × S_2 × · · · × S_n. If R is an NI ring and nil(R) nilpotent, then we have the 
following:

1. R is weak zip ⇔ [[R^{S_1 × S_2 × · · · × S_n,(lex ≤)}]] is weak zip.
2. R is weak zip ⇔ [[R^{S_1 × S_2 × · · · × S_n,(revlex ≤)}]] is weak zip.
3. R is weak symmetric ⇔ [[R^{S_1 × S_2 × · · · × S_n,(lex ≤)}]] is weak symmetric.
4. R is weak symmetric ⇔ [[R^{S_1 × S_2 × · · · × S_n,(revlex ≤)}]] is weak symmetric.

Proof. It is easy to see that (S_1 × S_2 × · · · × S_n,(lex ≤)) and (S_1 × S_2 × · · · × 
S_n,(revlex ≤)) are cancellative torsion-free strictly ordered monoids. There-
fore we complete the proofs of (1), (2) by Proposition 2.11, and (3), (4) by 
Proposition 2.13.
3. Weak associated primes

Given a right \( R \)-module \( N_R \), the right annihilator of \( N_R \) is denoted by 
\[ r_R(N_R) = \{ a \in R \mid Na = 0 \}. \]
We say that \( N_R \) is prime if \( N_R \neq 0 \) and 
\[ r_R(N_R) = r_R(N'_R) \]
for every nonzero submodule \( N'_R \subseteq N_R \) (see [2], [3]). Let 
\( M_R \) be a right \( R \)-module, an ideal \( \varphi \) of \( R \) is called an associated prime of \( M_R \) if there exists a prime submodule \( N_R \subseteq M_R \) such that 
\[ \varphi = r_R(N_R). \]
The set of associated primes of \( M_R \) is denoted by \( \text{Ass}(M_R) \) (see [2], [3]). Associated primes are well-known in commutative algebra for their important role in the primary decomposition, and has attracted a lot of attention in recent years. In [5], Brewer and Heinzer used localization theory to prove that, over a commutative ring \( R \), the associated primes of the polynomial ring \( R[x] \) (viewed as a module over itself) are all extended: that is, every \( \varphi \in \text{Ass}(R[x]) \) may be expressed as \( \varphi = \varphi_0[x] \), where \( \varphi_0 = \varphi \cap R \in \text{Ass}(R) \). Using results of R. C. Shock in [20] on good polynomials, C. Faith has provided a new proof in [7] of the same result which does not rely on localization or other tools from commutative algebra. In [3], Scott Annin showed that Brewer and Heinzer’s result still holds in the more general setting of a polynomial module \( M[x] \) over an Ore extension ring \( R[x; \alpha, \delta] \), with possibly noncommutative base \( R \). So the properties of associated primes over a commutative ring can be profitably generalized to a noncommutative setting as well.

Motivated by the results in [2], [3], [7], [20], in this section, we first introduce the notion of weak associated primes, which is a generalization of associated primes. We next describe all weak associated primes of the generalized power series ring \( R^S \) in terms of the weak associated primes of the ring \( R \).

**Definition 3.1.** Let \( I \) be a right ideal of a nonzero ring \( R \). We say that \( I \) is an \( R \)-prime ideal if \( I \not\subseteq \text{nil}(R) \) and \( N_R(I) = N_R(I') \) for every right ideal \( I' \subseteq I \) and \( I' \not\subseteq \text{nil}(R) \).

**Definition 3.2.** Let \( \text{nil}(R) \) be an ideal of a ring \( R \). An ideal \( \varphi \) of \( R \) is called a weak associated prime of \( R \) if there exists an \( R \)-prime ideal \( I \) such that 
\[ \varphi = N_R(I). \]
The set of weak associated primes of \( R \) is denoted by \( N\text{Ass}(R) \).

**Example 3.3.** Let \( R \) be a domain and let
\[
R_n = \begin{cases} 
a & a_{12} & \cdots & a_{1n} \\
0 & a & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a \\
\end{cases} \mid a, a_{ij} \in R 
\]
be the subring of \( n \times n \) upper triangular matrix ring. Then \( \text{nil}(R_n) \) is an ideal of \( R_n \) and
\[
\text{nil}(R_n) = \begin{cases} 
0 & x_{12} & \cdots & x_{1n} \\
0 & 0 & \cdots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
\end{cases} \mid x_{ij} \in R 
\].
Example 3.4. Let $k$ be any field, and consider the ring $R = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ of $2 \times 2$ lower triangular matrices over $k$. One easily checks that $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \supseteq \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} \supseteq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a composition series for $R_R$. In particular, $R_R$ has finite length.

Next we shall determine the set $\text{Ass}(R)$. By an easy ad hoc calculation, we can write down all of the proper nonzero ideals of $R$:

$$
\begin{align*}
\{m_1 = \begin{pmatrix} 0 & 0 \\ k & k \end{pmatrix}, m_2 = \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix}, \alpha = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}\}.
\end{align*}
$$

Now since $\alpha^2 = 0$, 0 is not a prime ideal. Moreover, since $m_1 R m_2 \subseteq \alpha$, $\alpha$ is not a prime ideal. So the only candidates for the associated primes of $R$ are the maximal ideals $m_1$ and $m_2$.

We claim that $m_2 \not\subseteq \text{Ass}(R)$. Otherwise, there would exist a right ideal $I \supseteq 0$ of $R$ with $m_2 = r_R(I)$. So $I \cdot m_2 = 0$. Now, given $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I$, we have $0 = (a k^2 c) \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so $a = b = 0$. Also, $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ implies that $c = 0$. Thus $I = 0$, a contradiction. Hence $m_2 \not\subseteq \text{Ass}(R)$.

By virtue of $R_R$ being Noetherian, we know that $\text{Ass}(R) \neq \emptyset$. Hence $\text{Ass}(R) = \{m_1\}$.

Finally, we should determine the set of $\text{NAss}(R)$. Clearly, $\text{nil}(R) = \alpha$. Thus $\text{nil}(R)$ is an ideal. Now we show that $m_1 = N_R(m_2)$ and $m_2$ is a right $R$-prime ideal. Clearly, $m_1 \subseteq N_R(m_2)$ since $m_2 m_1 = 0$. Given $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in N_R(m_2)$, we have $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in N_R(0)$. Then $\alpha = 0$ and so $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in m_1$. Hence $m_1 = N_R(m_2)$. Next we see that $m_2$ is a right $R$-prime ideal. Let $n \not\subseteq \text{nil}(R)$ and $n \subseteq m_2$. Since $N_R(n) \supseteq N_R(m_2)$ is clear, we now assume that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in N_R(n)$, and find $\begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix} \in n$ with $h \neq 0$. Then we have $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \cdot \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix} \in \text{nil}(R)$. Thus $\alpha = 0$ and so $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in N_R(m_2)$. Hence we obtain $N_R(n) = N_R(m_2)$ and so $m_2$ is a right $R$-prime ideal. Thus we obtain $m_1 \in \text{NAss}(R)$. Similarly, we have $m_2 \in \text{NAss}(R)$. Therefore $\text{NAss}(R) = \{m_1, m_2\} \neq \text{Ass}(R)$.

If $R$ is reduced, then $\varphi$ is a weak associated prime of $R$ if and only if $\varphi$ is an associated prime of $R$. So $\text{NAss}(R) = \text{Ass}(R)$ in case $R$ is reduced.

In the following, unless stated otherwise, we shall always assume that $R$ is a semicommutative right Noetherian ring, and $(S, \leq)$ a strictly totally ordered monoid.

Let $f \in [[R^S, \leq]]$. We denote by $N(f)$ the right ideal of $R$ generated by $C_f = \{f(s) \mid s \in S\}$. Since $R$ is a right Noetherian ring, we can find $s_i \in S$, $i = 1, 2, \ldots, n$, and $s_1 \leq s_2 \leq \cdots \leq s_n$, such that $N(f)$ is generated by $f(s_1), f(s_2), \ldots, f(s_n)$. Consider the $n$ elements $f(s_1), f(s_2), \ldots, f(s_{k-1}), f(s_k), f(s_{k+1}), \ldots, f(s_n)$. If $f(s_k) \not\in \text{nil}(R)$, and $f(s_i) \in \text{nil}(R)$ for all $k < i \leq n$, then we say that the weak degree of $f$ is $k$. To simplify notations, we write $N \deg(f)$ for the weak degree of $f$. If $f(s_i) \in \text{nil}(R)$ for all $1 \leq i \leq n$, then we define $N \deg(f) = -1$. 
Definition 3.5. Let $f \in [[R^{S, \leq}]])$, $N(f)$ is generated by $f(s_1), f(s_2), \ldots, f(s_n)$, $s_i \leq s_j$ if $i \leq j$, and $N \deg(f) = k$. If $N_R(f(s_k)) \subseteq N_R(f(s_i))$ for all $i \leq k$, then we say that $f$ is a weak good generalized power series.

Lemma 3.6. Let $R$ be a semicommutative right Noetherian ring, $(S, \leq)$ a strictly totally ordered monoid. For any $f \notin \text{nil}([[R^{S, \leq}]])$, there exists $r \in R$ such that $fr = fC^0_r$ is a weak good generalized power series.

Proof. Assume that the result is false, and let $f \notin \text{nil}([[R^{S, \leq}]])$ be a counterexample of minimal weak degree $N \deg(f) = k \geq 1$. In particular, $f$ is not a weak good generalized power series. Suppose that $N(f)$ is generated by $f(s_1), f(s_2), \ldots, f(s_n)$, where $s_i \leq s_j$ if $i \leq j$. Hence there exists $i < k$ such that $N_R(f(s_i)) \subseteq N_R(f(s_i))$. So we can find $b \in R$ with $f(s_i)b \notin \text{nil}(R)$, and $f(s_i)b \notin \text{nil}(R)$. Consider the generalized power series $fb = fC^0_b \in [[R^{S, \leq}]])$. Clearly, $N(fb)$ is generated by $f(s_1)b, f(s_2)b, \ldots, f(s_n)b$, where $s_i \leq s_j$ if $i \leq j$, and $f(s_i)b \notin \text{nil}(R)$ implies $fb \notin \text{nil}([[R^{S, \leq}]])$. It is easy to see that $fb$ has weak degree at most $k - 1$. By the minimality of $k$, we know that there exists $c \in R$ with $f \cdot b \cdot c = f \cdot (bc)$ weak good. But this contradicts the fact that $f$ is a counterexample to the statement. □

Proposition 3.7. Let $R$ be a semicommutative right Noetherian ring, $(S, \leq)$ a strictly totally ordered monoid. Then $\text{NAss}([[R^{S, \leq}]]) = \{[\varphi^{S, \leq}] \mid \varphi \in \text{NAss}(R)\}.$

Proof. We first prove $\supseteq$. Let $\varphi \in \text{NAss}(R)$. By definition, there exists a right ideal $I \subseteq \text{nil}(R)$ with $I$ an $R$-prime ideal and $\varphi = N_R(I)$. It suffices to prove

\begin{equation}
[[\varphi^{S, \leq}] = N[[R^{S, \leq}]([I^{S, \leq}]),
\end{equation}

and

\begin{equation}
[[I^{S, \leq}]] \text{ is } [[R^{S, \leq}]]-\text{prime}.
\end{equation}

For Eq.(1), let $f \in [[I^{S, \leq}]]$ and let $g \in [[\varphi^{S, \leq}]]$. Then for any $u, v \in S$, since $f(u) \in I$ and $g(v) \in \varphi$, we obtain $f(u)g(v) \in \text{nil}(R)$. Applying Corollary 2.8 yields that $fg \in \text{nil}([[R^{S, \leq}]])$. Hence $[[\varphi^{S, \leq}]] \subseteq N[[R^{S, \leq}]([I^{S, \leq}])].$

Conversely, if $g \in N[[R^{S, \leq}]([I^{S, \leq}])]$, then $fg \in \text{nil}([[R^{S, \leq}]]$ for all $f \in [[I^{S, \leq}]].$ In particular, for any $b \in I$, $C^0_b \in \text{nil}([[R^{S, \leq}])].$ Thus, by Corollary 2.8, $bg(s) \in \text{nil}(R)$ for any $s \in S$, and so $g(s) \in N_R(I) = \varphi$ for any $s \in S$. Hence $g \in [[\varphi^{S, \leq}]]$, and so $N[[R^{S, \leq}]([I^{S, \leq}])] \subseteq [[\varphi^{S, \leq}]].$ Therefore $[[\varphi^{S, \leq}]] = N[[R^{S, \leq}]([I^{S, \leq}])].$

Note that the right ideal $I$ is an $R$-prime ideal. Then we have $I \not\subseteq \text{nil}(R)$. Thus

$$[[I^{S, \leq}]] \not\subseteq [[\text{nil}(R)^{S, \leq}]] = \text{nil}([[R^{S, \leq}]).$$

To see (2), we must show that if a right ideal $\mathcal{U} \not\subseteq \text{nil}([[R^{S, \leq}])$ and $\mathcal{U} \subseteq [[I^{S, \leq}]]$, then

$$N[[R^{S, \leq}]([\mathcal{U}]) = N[[R^{S, \leq}]([I^{S, \leq}])].$$
To this end, let $C_0 = \bigcup_{f \in \mathcal{U}} C_f$, where $C_f = \{ f(x) \mid x \in S \}$, and let $\varphi_0$ denote the right ideal of $R$ generated by $C_0$. Since $\mathcal{U} \not\subseteq \text{nil}([R^{S, \leq}]) = \{ \text{nil}(R)^{S, \leq} \}$, $C_0 \not\subseteq \text{nil}(R)$, and hence $\varphi_0 \subseteq I$, $\varphi_0 \not\subseteq \text{nil}(R)$. So we have $N_R(\varphi_0) = N_R(I) = \varphi$ because $I$ is $R$-prime. Since $N([R^{S, \leq}](I) \supseteq N([R^{S, \leq}](I^{S, \leq}))$ is clear, it suffices to show that
\[ N([R^{S, \leq}](I) \supseteq N([R^{S, \leq}](I^{S, \leq}))). \]

We now assume that $g \in N([R^{S, \leq}](I))$, then $fg \in \text{nil}([R^{S, \leq}])$ for every $f \in \mathcal{U}$. By Corollary 2.8, we obtain $f(u)g(v) \in \text{nil}(R)$ for all $u$, $v \in S$. It follows from Lemma 2.4 that $f(u)g(v) \subseteq \text{nil}(R)$ for all $u$, $v \in S$. Thus $g(v) \in N_R(\varphi_0) = N_R(I) = \varphi$ for all $v \in S$, and so $g \in [\varphi^{S, \leq}] = N([R^{S, \leq}](I^{S, \leq}))$. Hence $N([R^{S, \leq}](I)) \subseteq N([R^{S, \leq}](I^{S, \leq}))$ is proved, and so is $\subseteq$ in Proposition 3.7.

Now we turn our attention to proving $\subseteq$ in Proposition 3.7. Let $I \in N_{\text{Ass}}([R^{S, \leq}])$. By definition, we have an $[R^{S, \leq}]$-prime ideal $L$ with $I = N([R^{S, \leq}](L))$. Pick any $\pi \in L$, and let $N(\pi)$ be the right ideal of $R$ generated by $C_\pi$. Since $R$ is a right Noetherian ring, we can find
\[ s_1 < s_2 < \cdots < s_n \]
such that $N(\pi)$ is generated by $n$ elements
\[ \pi(s_1), \pi(s_2), \ldots, \pi(s_n). \]
Since $N(\pi) = k$, we have $\pi(s_k) \not\in \text{nil}(R)$, and $N_R(\pi(s_k)) \subseteq N_R(\pi(s_j))$ if $i \leq k$, and $\pi(s_i) \not\in \text{nil}(R)$ if $i > k$. Considering the right ideal $\pi(s_k)R$, and assuming that $U = N_R(\pi(s_k)R)$, we wish to claim that $I = [U^{S, \leq}]$. Let $\alpha \in [U^{S, \leq}]$. Then for each $v \in S$, $\alpha(v) \in U = N_R(\pi(s_k)R)$, and so $\pi(s_k)R\alpha(v) \subseteq \text{nil}(R)$. Since $\pi$ is a weak good generalized power series, and $N(\pi) = k$, we have
\[ \pi(s_i)R\alpha(v) \subseteq \text{nil}(R) \quad \text{for all} \quad 1 \leq i \leq k. \]

On the other hand, for all $i > k$, $\pi(s_i) \in \text{nil}(R)$, thus we have
\[ \pi(s_i)R\alpha(v) \subseteq \text{nil}(R) \quad \text{for all} \quad 1 \leq i \leq n. \]
Since $N(\pi)$ is generated by $\pi(s_i)$, $1 \leq i \leq n$, for each $u \in U$, there exist $r_i \in R$, $1 \leq i \leq n$, such that
\[ \pi(u) = \pi(s_1)r_1 + \pi(s_2)r_2 + \cdots + \pi(s_n)r_n. \]
Thus we obtain
\[ \pi(u)R \alpha(v) = \left( \sum_{i=1}^{n} \pi(s_i)r_i \right) R \alpha(v) \subseteq \text{nil}(R). \]

Hence for any \( h \in [[R^S]] \) and any \( u, v \in S \), we have \( \pi(u)h(w) \alpha(v) \in \text{nil}(R) \), and so by Corollary 2.8, we have \( \pi \alpha \in \text{nil}([[R^S]]) \). Thus \( \alpha \in N([[R^S]])(\pi [[R^S]]) = I \). Hence \([U^S] \subseteq I\).

Conversely, let 
\[ \beta \in I = N([[R^S]])(\mathcal{L}) = N([[R^S]])(\pi [[R^S]]). \]

Then for any \( \mathcal{C}^0 \subseteq [[R^S]] \), we have \( \pi \mathcal{C}^0 \beta \in \text{nil}([[R^S]]) \). Then by Corollary 2.8, we get \( \pi(s_k)r \beta(v) \in \text{nil}(R) \) for all \( r \in R \) and \( v \in S \). Hence \( \beta(v) \in N_R(\pi(s_k)R) = U \) for each \( v \in S \), and so \( \beta \in [[U^S]] \). Hence \( I \subseteq [[U^S]] \).

Therefore \( I = [[U^S]] \).

We are now to check that the principally right ideal \( \pi(s_k)R \) is \( R \)-prime. Since \( \pi(s_k) \not\in \text{nil}(R) \), \( \pi(s_k)R \not\subseteq \text{nil}(R) \). Assume that a right ideal \( Q \subseteq \pi(s_k)R \), and \( Q \not\subseteq \text{nil}(R) \). Then \( N_R(Q) \geq N_R(\pi(s_k)R) \) is clear. Now we show that 
\[ N_R(Q) \subseteq N_R(\pi(s_k)R). \]

Set \( W = \{ \pi r = \pi \mathcal{C}^0 \mid r \in Q \} \), and let \( W([[R^S]]) \) be the right ideal of \([R^S] \) generated by \( W \). It is obvious that \( W([[R^S]]) \subseteq \pi([[R^S]]) \). Since \( Q \not\subseteq \text{nil}(R) \), there exists \( a \in R \) such that \( \pi(s_k)a \in Q \) and \( \pi(s_k)a \not\in \text{nil}(R) \). If 
\[ (\pi \cdot \pi(s_k)a)(s_k) = (\pi \mathcal{C}^0_{\pi(s_k)a})(s_k) = \pi(s_k)\pi(s_k)a \in \text{nil}(R), \]
then we have 
\[ \pi(s_k)\pi(s_k)a \in \text{nil}(R) \Rightarrow \pi(s_k)a\pi(s_k) \in \text{nil}(R) \Rightarrow \pi(s_k)a \in \text{nil}(R). \]
This contradicts to the fact that \( \pi(s_k)a \not\in \text{nil}(R) \). Thus \( \pi \cdot \pi(s_k)a(s_k) \not\in \text{nil}(R) \), and so by Corollary 2.6,
\[ \pi \cdot \pi(s_k)a \not\in \text{nil}([[R^S]]) \]
This implies that \( W([[R^S]]) \not\subseteq \text{nil}([[R^S]]) \). Since \( \mathcal{L} \) is \([[R^S]]\)-prime, we obtain 
\[ N([[R^S]])(W([[R^S]])) = N([[R^S]])(\pi([[R^S]])) = I = [[U^S]]. \]

Suppose \( q \in N_R(Q) \). Then \( rq \in \text{nil}(R) \) for each \( r \in Q \). Then for any \( r \in Q \), and any \( \pi r \alpha = \pi \mathcal{C}^0 \alpha \in W([[R^S]]) \) and any \( s \in S \),
\[ (\pi \mathcal{C}^0 \alpha \mathcal{C}^0_q)(s) = \sum_{(u,v) \in S} \pi(u) \alpha(v) q. \]
From \( rq \in \text{nil}(R) \) and Lemma 2.4, we obtain \( \pi(u) \alpha(v) q \in \text{nil}(R) \) for any \( u, v \in S \), and so \( \pi \mathcal{C}^0 \alpha \mathcal{C}^0_q \subseteq \text{nil}(R) \). Thus by Corollary 2.6, \( \mathcal{C}^0 \alpha \mathcal{C}^0_q \subseteq \text{nil}([[R^S]]) \). Hence 
\[ \mathcal{C}^0_q \subseteq N([[R^S]])(W([[R^S]])) = I = [[U^S]]. \]
and so \( q \in U = N_R(\pi(s_k)R) \). Hence \( N_R(Q) \subseteq N_R(\pi(s_k)R) \), and this implies that \( N_R(Q) = N_R(\pi(s_k)R) \). Thus we obtain \( \pi(s_k)R \) is \( R \)-prime.

Assembling the above results, we finish the proof of Proposition 3.7. \( \square \)

**Corollary 3.8.** Let \( R \) be a semicommutative Noetherian ring. Then

\[
N_{\text{Ass}}(R[[x]]) = \{ \varphi[[x]] \mid \varphi \in N_{\text{Ass}}(R) \}.
\]

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