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RINGS OVER WHICH POLYNOMIAL RINGS ARE ARMENDARIZ AND REVERSIBLE

Jung Ho Ahn, Min Jeong Choi, Si Ra Choi, Won Seok Jeong, Jung Soo Kim, Jeong Yeol Lee, Soon Ji Lee, Young Sun Lee, Dong Hyun Noh, Yu Seung Noh, Gyeong Hyeon Park, Chang Ik Lee* and Yang Lee

Abstract. A ring R is called *reversibly Armendariz* if $b_ja_i = 0$ for all i, j whenever $f(x)g(x) = 0$ for two polynomials $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j$ over R. It is proved that a ring R is reversibly Armendariz if and only if its polynomial ring is reversibly Armendariz if and only if its Laurent polynomial ring is reversibly Armendariz. Relations between reversibly Armendariz rings and related ring properties are examined in this note, observing the structures of many examples concerned. Various kinds of reversibly Armendariz rings are provided in the process. Especially it is shown to be possible to construct reversibly Armendariz rings from given any Armendariz rings.

1. Reversibly Armendariz rings

Throughout this section every ring is associative (possibly without identity). Let R be a ring. The polynomial ring with an indeterminate x over R is denoted by $R[x]$. $C_{f(x)}$ denotes the set of all coefficients of $f(x) \in R[x]$. The n by n full (resp. upper triangular) matrix ring over R is denoted by $Mat_n(R)$ (resp. $U_n(R)$). Use e_{ij} for the matrix with (i, j)-entry 1 and elsewhere 0. Let $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n).

A ring is usually called *reduced* if it has no nonzero nilpotent elements. Following Rege and Chhawchharia [19], a ring R is called *Armendariz*

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*Corresponding author.

if $ab = 0$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$ whenever any two polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$. This nomenclature was used by them since it was Armendariz [4, Lemma 1] who initially showed that a reduced ring always satisfies this condition. It is obvious that the class of Armendariz rings is closed under subrings.

A ring is usually called *Abelian* if every idempotent is central. Reduced rings are clearly Abelian.

According to Cohn [6], a ring R is called *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Anderson and Camillo [2], observing the rings whose zero products commute, used the term ZC_2 for what is called reversible; while Krempa and Niewieczerzal [13] took the term C_0 for it. It is obvious that the class of reversible rings is closed under subrings. Liu and Yang [17] called a ring R *strongly reversible* if $g(x)f(x) = 0$ whenever polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$. Note that R is strongly reversible if and only if $R[x]$ is reversible. Strongly reversible rings are clearly reversible but the converse need not hold by [11, Example 2.1].

Lambek introduced the concept of a symmetric right ideal, unifying the sheaf representation of commutative rings and reduced rings in [15]. Lambek called a right ideal I of a ring R *symmetric* if $rst \in I$ implies $rts \in I$ for all $r, s, t \in R$. If the zero ideal is symmetric then R is usually called *symmetric*; while Anderson and Camillo [1] used the term ZC_3 for this concept. It is proved by Lambek that a ring R is symmetric if and only if $r_1 r_2 \cdots r_n = 0$ implies $r_{\sigma(1)} r_{\sigma(2)} \cdots r_{\sigma(n)} = 0$ for any permutation σ of the set $\{1, 2, \ldots, n\}$, where $n \geq 1$ and $r_i \in R$ for all i in [15, Proposition 1]. Anderson-Camillo also obtained this result independently in [1, Theorem I.1]. It is evident that commutative rings are both symmetric and reversible. Reduced rings are both symmetric and reversible by [1, Theorem I.3], but there are many kinds of non-reduced commutative rings.

As we noted above, reduced rings are both Armendariz and reversible. We adapt this fact to the following new concept. In this note a ring R shall be called *reversibly Armendariz* if

$$b_j a_i = 0 \text{ for all } i, j \text{ whenever } f(x)g(x) = 0$$
$$\text{for } f(x) = \textstyle\sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x].$$

It is obvious that the class of reversibly Armendariz rings is closed under subrings. Anderson and Camillo [1, Theorem 2] showed that a ring R is Armendariz if and only if $R[x]$ is Armendariz. While, Kim and Lee [11, Example 2.1] showed that polynomial rings over reversible rings need not be reversible. But if R is reversibly Armendariz, then $R[x]$ is also reversibly Armendariz as we see in the following.

Proposition 1.1. *Given a ring R the following conditions are equivalent:*

(1) *R is reversibly Armendariz;*
(2) *R is both strongly reversible and Armendariz;*
(3) *R is both reversible and Armendariz;*
(4) *$R[x]$ is both reversible and Armendariz; and*
(5) *$R[x]$ is reversibly Armendariz.*

Proof. (2)⇒(3)⇒(1) and (4)⇒(3) are obvious.

(1)⇒(2): Let R be a reversibly Armendariz ring and $f(x)g(x) = 0$ where $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$. Then $b_j a_i = 0$ for any i and j. Thus,

$$g(x)f(x) = \sum_{k=0}^{m+n} \left(\sum_{i+j=k} b_j a_i \right) x^k = 0,$$

entailing that $R[x]$ is reversible (i.e., R is strongly reversible). Moreover, we have that $a_i b_j = 0$ for any i and j since R is reversible, entailing that R is Armendariz.

(3)⇒(4): Let R be both reversible and Armendariz. Then $R[x]$ is Armendariz by [1, Theorem 2]. Let $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$. Since R is Armendariz, $a_i b_j = 0$ for all i, j, but since R is reversible we moreover have $b_j a_i = 0$ for all i, j. This implies that $R[x]$ is reversible.

(4)⇔(5) is the same as (1)⇔(3). □

Reduced rings are clearly reversibly Armendariz, but the converse does not hold in general as can be seen by $S = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in R \right\}$ over a reduced ring R. Indeed, S is reversibly Armendariz by [11, Proposition 2], [12, Proposition 1.6], and Proposition 1.1. We will use Proposition 1.1 without mention.

Proposition 1.2. *Let R be a ring and I be a proper ideal of R satisfying that R/I is a reversibly Armendariz ring. If I is a reduced ring then R is reversibly Armendariz.*

Proof. Suppose that R/I is a reversibly Armendariz ring and I is a reduced ring. Then R is reversibly Armendariz by [9, Theorem 11] and [12, Proposition 1.12]. □

The ring $\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in R\right\}$ over a reduced ring R is reversibly Armendariz as we see above. Combining this and Proposition 1.2, one may conjecture that R is a reversibly Armendariz ring if R/I is reversibly Armendariz for any proper ideal I of R which is reversibly Armendariz as a ring. However the following example erases the possibility.

Example 1.3. The argument is essentially due to [12, Example 1.11] and [14, Example 2.9]. Let F be any field and consider

$$R = \left\{\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} | a, b, c \in F\right\}.$$

Then R is Armendariz by [11, Proposition 2], but R is not reversibly Armendariz since R is not reversible as can be seen by $me_{12}ne_{23} = mne_{13} \neq 0$ for $m, n \in F\backslash 0$ and $ne_{23}me_{12} = 0$. Note that the following are all nonzero proper ideals in R:

$$I_1 = \begin{pmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}, I_2 = \begin{pmatrix} 0 & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, I_3 = \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}, I_4 = \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$I_5 = \left\{\begin{pmatrix} 0 & \alpha b & c \\ 0 & 0 & \alpha d \\ 0 & 0 & 0 \end{pmatrix} \mid b, c, d, \alpha \in F \text{ and } b \neq 0, d \neq 0 \text{ are fixed}\right\}.$$

I_1 and I_5 are not reversible as can be seen by $(\alpha b)e_{12}(\alpha d)e_{23} \neq 0$ when $\alpha \neq 0$ and $(\alpha d)e_{23}(\alpha b)e_{12} = 0$. But I_2, I_3, I_4 are all reversibly Armendariz since they are nilpotent of index 2. R/I_2 and R/I_3 are both isomorphic to $\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in F\right\}$, so they are reversibly Armendariz. The ring R/I_4 is reversible by [12, Example 1.11]. Moreover R/I_4 is isomorphic to $\left\{\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} | a, b, c \in F\right\}$, entailing that R/I_4 is Armendariz.

Armendariz rings need not be reversible as we see in the preceding example. But we can always obtain reversibly Armendariz rings from given any Armendariz ring.

PROPOSITION 1.4. *Let R be an Armendariz ring and $Z(R)$ be the center of R. Let S be a subring of $Z(R)$, and N be a nilpotent ideal of R with $N^2 = 0$. Then $S + N$ is a reversibly Armendariz ring.*

Proof. Let $T = S + N$. Note that T forms a subring of R, and so T is clearly Armendariz. Let $(a + m)(b + n) = 0$ for $a, b \in S$ and $m, n \in N$. Then $0 = (a+m)(b+n) = ab+an+mb = ba+na+bm = ba+na+bm+nm = (b+n)(a+m)$; hence T is reversible. □

The ring R in Example 1.3 is Armendariz but not reversible. Let $S = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a \in Z(D) \right\} \subseteq Z(R)$ and $N = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} | b, c \in D \right\}$. Then $S + N$ is reversibly Armendariz by Proposition 1.4.

2. Examples of reversibly Armendariz rings

Throughout this section every ring is associative with identity unless otherwise stated. Given a ring R, $N_*(R)$, $N^*(R)$, and $N(R)$ denote the prime radical, the upper nilradical (i.e., sum of nil ideals), and the set of all nilpotent elements in R, respectively. Based on Artin and Wedderburn, the sum of all nilpotent ideals in R, written by $N_0(R)$, is called the Wedderburn radical of R (in spite of this sum being not a radical, it was given the name). It is well-known that $N_0(R) \subseteq N_*(R) \subseteq N^*(R) \subseteq N(R)$. We first observe nilradicals of reversibly Armendariz rings.

PROPOSITION 2.1. *If R is a reversibly Armendariz ring then $N(R[x]) = N^*(R[x]) = N_*(R[x]) = N_0(R[x]) = N(R)[x] = N^*(R)[x] = N_*(R)[x] = N_0(R)[x]$.*

Proof. Let R be a reversibly Armendariz ring. For any ring A we have $N_0(A)[x] = N_0(A[x])$ by [5, Corollary 5]. Since R is Armendariz, we get $N(A)[x] = N(A[x])$ and $N^*(R)[x] = N_0(R)[x] = N_*(R)[x]$ by [2, Corollary 5.2] and [10, Lemma 2.3(5)], respectively. Since R is reversible, we get $N_*(R) = N^*(R) = N(R)$ through a simple computation. Now combining these all, we finally obtain

$$\begin{aligned} &N(R[x]) = N^*(R[x]) = N_*(R[x]) = N_0(R[x]) \\ &= N(R)[x] = N^*(R)[x] = N_*(R)[x] = N_0(R)[x]. \end{aligned}$$

□

Armendariz rings are Abelian by the proof of [1, Theorem 6] or [9, Corollary 8]. Symmetric rings are obviously reversible, however the converse need not be true by [1, Example I.5] or [18, Examples 5 and 7].

By Proposition 1.1, reversibly Armendariz rings are strongly reversible. The following shows that Armendariz and strongly reversible are independent of each other.

Example 2.2. (1) The argument here is essentially due to [16, Example 3.2]. Let $A = \mathbb{Z}_3[x, y]$ be the polynomial ring with two indeterminates x, y over $\mathbb{Z}_3$. Note that I is the ideal of $\mathbb{Z}_3[x, y]$ generated by x^3, x^2y^2 and y^3. Next consider $R = A/I$ and identify x and y with their images in R for simplicity. Then R is commutative and so strongly reversible. Let $f(t) = x + yt, g(t) = x^2 + 2xyt + y^2t^2 \in R[t]$, where $R[t]$ is the polynomial ring with an indeterminate t over R. Then $f(t)g(t) = 0$, but $xy^2 \neq 0$. This implies that R is not Armendariz.

(2) Let K be a field and $A = K\langle a, b\rangle$ be the free algebra with noncommuting indeterminates a, b over K. Let I be the ideal of A generated by ab. Then the factor ring A/I is Armendariz but not reversible by [3, Example 4.10].

(3) $T = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} | a, b, c, d \in R \right\}$ where R is a reduced ring is an Armendariz ring but not reversibly Armendariz. T is Armendariz by [11, Proposition 2], but not reversible by the following computation:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0 \text{ and } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Symmetric rings are reversible. We see a symmetric ring but not (reversibly) Armendariz in the following.

Example 2.3. We refer to the argument in [9, Example 2] and [12, Example 2.1]. Let $A = \mathbb{Z}_2\{a_0, a_1, a_2, b_0, b_1, b_2, c\}$ be the free algebra of polynomials with zero constant terms in noncommuting indeterminates

$a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$. Note that A is a ring without identity and consider an ideal of the ring $\mathbb{Z}_2 + A$, say I, generated by

$$a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_1b_2 + a_2b_1, a_2b_2, a_0rb_0, a_2rb_2,$$

$$b_0a_0, b_0a_1 + b_1a_0, b_0a_2 + b_1a_1 + b_2a_0, b_1a_2 + b_2a_1, b_2a_2, b_0ra_0, b_2ra_2,$$

$$(a_0 + a_1 + a_2)r(b_0 + b_1 + b_2), (b_0 + b_1 + b_2)r(a_0 + a_1 + a_2), \text{and}$$

$$r_1r_2r_3r_4,$$

where $r, r_1, r_2, r_3, r_4 \in A$. Then clearly $A^4 \in I$. Next let $R = (\mathbb{Z}_2 + A)/I$ and identify a_i, b_j and c with their images in R for simplicity. Note that R is symmetric by the method of [8, Example 3.1]. But since $(a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^2) = 0$ and $b_1a_0 \neq 0$, R is not reversibly Armendariz.

In the following we see the basic examples of reversibly Armendariz rings. Given a ring R and $n \geq 2$, consider the following two kinds of subrings of $U_n(R)$:

$$D_n(R) = \{(m_{ij}) \in U_n(R) \mid m_{11} = \cdots = m_{nn}\};$$

and

$$V_n(R) = \{(m_{ij}) \in D_n(R) \mid m_{st} = m_{(s+1)(t+1)} \text{ for } s = 1, \ldots, n-2 \text{ and } t = 2, \ldots, n-1\}.$$

Proposition 2.4. *Let R be a reduced ring.*

(1) *$D_2(R)$ is reversibly Armendariz.*

(2) *$R[x]/(x^n)$ is reversibly Armendariz, where (x^n) is the ideal of $R[x]$ generated by x^n.*

(3) *$V_n(R)$ is reversibly Armendariz.*

Proof. (1) $D_2(R)$ is both Armendariz and reversible by [11, Proposition 2] and [12, Proposition 1.6]. Thus $D_2(R)$ is reversibly Armendariz by Proposition 1.1.

(2) $R[x]/(x^n)$ is both Armendariz and reversible by [1, Theorem 5] and [12, Proposition 2.5]. Thus $R[x]/(x^n)$ is reversibly Armendariz by Proposition 1.1.

(3) It is well-known that $V_n(R)$ is isomorphic to $R[x]/(x^n)$ as rings. So (2) implies (3). □

By Proposition 2.4(1), one may conjecture that if R is reversibly Armendariz then so is $D_2(R)$. However the following example eliminates the possibility.

EXAMPLE 2.5. We use the ring in [12, Example 1.7]. Let $\mathbb{H}$ be the Hamilton quaternions over the real number field and $R = D_2(\mathbb{H})$. Then R is reversibly Armendariz by Proposition 2.4(1). However $D_2(R)$ is not reversible by the computation of [12, Example 1.7].

We consider several basic properties of reversibly Armendariz rings as follows.

PROPOSITION 2.6. (1) *Any direct product of reversibly Armendariz rings is reversibly Armendariz.*

(2) *Any subdirect product of reversibly Armendariz rings is reversibly Armendariz.*

(3) *The class of reversibly Armendariz rings is closed under direct limits.*

Proof. (1) Let R be the direct product of R_i where $\{R_i \mid i \in I\}$ is a set of reversibly Armendariz rings. Suppose $f(x)g(x) = 0$ with $f(x) = \sum_{j=0}^{m}(a(j)_i)x^j$, $g(x) = \sum_{k=0}^{n}(b(k)_i)x^k \in R[x]$. Letting $f_i(x) = \sum_{j=0}^{m} a(j)_i x^j$ and $g_i(x) = \sum_{k=0}^{n} b(k)_i x^k$ we can write $f(x) = (f_i(x))$ and $g(x) = (g_i(x))$. Note that $f_i(x)g_i(x) = 0$ for all i. Since R_i is reversibly Armendariz for all i, $b(k)_i a(j)_i = 0$ for all i, j, k, hence $(b(k)_i)(a(j)_i) = 0$ for all i, j, k. Therefore R is reversibly Armendariz.

(2) A subdirect product is a subring of a direct product, and so the result comes from (1).

(3) Let $D = \{R_i, \alpha_{ij}\}$ be a direct system of reversibly Armendariz rings R_i for $i \in I$ and ring homomorphisms $\alpha_{ij} : R_i \to R_j$ for each $i \le j$ satisfying $\alpha_{ij}(1) = 1$, where I is a directed partially ordered set. Set $R = \varinjlim R_i$ be the direct limit of D with $\iota_i : R_i \to R$ and $\iota_j \alpha_{ij} = \iota_i$. Take $a, b \in R$. Then $a = \iota_i(a_i)$, $b = \iota_j(b_j)$ for some $i, j \in I$ and there is $k \in I$ such that $i \le k$, $j \le k$. Define

$$a + b = \iota_k(\alpha_{ik}(a_i) + \alpha_{jk}(b_j)) \text{ and } ab = \iota_k(\alpha_{ik}(a_i)\alpha_{jk}(b_j)),$$

where $\alpha_{ik}(a_i)$ and $\alpha_{jk}(b_j)$ are in R_k. Then R forms a ring with $0 = \iota_i(0)$ and $1 = \iota_i(1)$. We have to show that R is a reversibly Armendariz ring.

Now let $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ be two polynomials such that $f(x)g(x) = 0$. There is $k \in I$ such that $f(x), g(x) \in R_k[x]$

via ι_i's and α_{ij}'s; hence all a_i and b_j are in R_k. Since R_k is reversibly Armendariz, $b_j a_i = 0$, entailing R being reversibly Armendariz. $\square$

The class of reversibly Armendariz rings is not closed under homomorphic images by the following example.

Example 2.7. Let R be the ring of quaternions with integer coefficients. Then R is a domain, so reversibly Armendariz. However for any odd prime integer q, the ring R/qR is isomorphic to $Mat_2(\mathbb{Z}_q)$ by the argument in [7, Exercise 2A]. Thus R/qR is not reversible, hence R/qR is not reversibly Armendariz when $q \in \{3, 5, 7, 11, \ldots\}$.

Proposition 2.8. *Let e be a central idempotent of a ring R and let M be a multiplicative monoid in R consisting of central regular elements. Then the following conditions are equivalent:*

(1) *R is reversibly Armendariz;*

(2) *eR and $(1-e)R$ are both reversibly Armendariz; and*

(3) *$M^{-1}R$ is reversibly Armendariz.*

Proof. (1)$\Rightarrow$(2) is obvious since eR and $(1-e)R$ are subrings of R.

(2)$\Rightarrow$(1) is done by Proposition 2.6(1).

(3)$\Rightarrow$(1) is obvious.

(1)$\Rightarrow$(3): Let R be reversibly Armendariz and $S = M^{-1}R$. Put $f(x)g(x) = 0$ where $f(x) = \sum_{i=0}^{m} \alpha_i x^i, g(x) = \sum_{j=0}^{n} \beta_j x^j \in S[x]$. We can assume that $\alpha_i = a_i u^{-1}, \beta_j = b_j v^{-1}$ with $a_i, b_j \in R$ for all i, j and $u, v \in M$. Then we have

$$\begin{aligned} 0 &= f(x)g(x) = \sum_{i=0}^{m}\sum_{j=0}^{n} \alpha_i \beta_j x^{i+j} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_i b_j u^{-1} v^{-1} x^{i+j} \\ &= (\sum_{i=0}^{m}\sum_{j=0}^{n} a_i b_j x^{i+j})(uv)^{-1}; \end{aligned}$$

hence $\sum_{i=0}^{m}\sum_{j=0}^{n} a_i b_j x^{i+j} = 0$ in $R[x]$. Since R is reversibly Armendariz, $b_j a_i = 0$ for all i, j and so $\beta_j \alpha_i = b_j v^{-1} a_i u^{-1} = b_j a_i v^{-1} u^{-1} = 0$ for all i, j. Thus S is reversibly Armendariz. $\square$

The ring of *Laurent polynomials* in x, coefficients in a ring R, consists of all formal sum $\sum_{i=k}^{n} r_i x^i$ with obvious addition and multiplication, where $r_i \in R$ and k, n are (possibly negative) integers. We denote this ring by $R[x; x^{-1}]$.

COROLLARY 2.9. *Given a ring R the following conditions are equivalent:*

(1) *R is reversibly Armendariz;*
(2) *$R[x]$ is reversibly Armendariz; and*
(3) *$R[x; x^{-1}]$ is reversibly Armendariz.*

Proof. (1)⇒(2) comes from Proposition 1.1.
(2)⇒(3) follows from Proposition 2.8, letting $M = \{1, x, x^2, \ldots\}$.
(3)⇒(1) is obvious since R is a subring of $R[x, x^{-1}]$. □

A ring R is called right *Ore* if given $a, b \in R$ with b regular there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. It is well-known that R is a right Ore ring if and only if the classical right quotient ring of R exists.

PROPOSITION 2.10. *Let R be a right Ore ring with the classical right quotient ring Q. Then R is reversibly Armendariz if and only if Q is reversibly Armendariz.*

Proof. It is enough to show that if R is reversibly Armendariz then Q is reversibly Armendariz. Suppose that R is reversible and Armendariz. Then Q is reversible and Armendariz by [12, Theorem 2.6] and [9, Theorem 12], respectively. □

It is well-known that a ring R is a semiprime right Goldie ring if and only if there exists the classical right quotient ring of R which is semisimple Artinian. Thus we obtain the following with the help of [9, Corollary 13].

COROLLARY 2.11. *Suppose that a ring R is a semiprime right Goldie ring with the classical right quotient ring Q. Then the following conditions are equivalent:*

(1) *R is Armendariz;*
(2) *R is reversibly Armendariz;*
(3) *R is reversible;*
(4) *R is reduced;*
(5) *R is symmetric; and*
(6) *Q is a finite direct product of division rings.*

The following provides a kind of reversibly Armendariz rings without identity.

Proposition 2.12. *Any infinite direct sum of reversibly Armendariz rings is reversibly Armendariz.*

Proof. Let $\{R_i \mid i \in I\}$ is the set of infinitely many reversibly Armendariz rings. Since the direct sum of R_i's is a subring of the direct product of R_i's. Thus the result comes from Proposition 2.6(1). $\square$

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Department of Mathematics
Pusan Science High School
Pusan 609-735, Korea
E-mail: `kjsccc@hanmail.net`

Department of Mathematics
Pusan National University
Pusan 609-735, Korea
E-mail: `cilee@pusan.ac.kr`

Department of Mathematics Education
Pusan National University
Pusan 609-735, Korea
E-mail: `ylee@pusan.ac.kr`