

**PROJECTIVE REPRESENTATIONS OF A QUIVER  
WITH THREE VERTICES AND TWO EDGES AS  
 $R[x]$ -MODULES**

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ABSTRACT. In this paper we show that the projective properties of representations of a quiver  $Q = \bullet \rightarrow \bullet \rightarrow \bullet$  as left  $R[x]$ -modules. We show that if  $P$  is a projective left  $R$ -module then  $0 \longrightarrow 0 \longrightarrow P[x]$  is a projective representation of a quiver  $Q$  as  $R[x]$ -modules, but  $P[x] \longrightarrow 0 \longrightarrow 0$  is not a projective representation of a quiver  $Q$  as  $R[x]$ -modules, if  $P \neq 0$ . And we show a representation  $0 \longrightarrow P[x] \xrightarrow{id} P[x]$  of a quiver  $Q$  is a projective representation, if  $P$  is a projective left  $R$ -module, but  $P[x] \xrightarrow{id} P[x] \longrightarrow 0$  is not a projective representation of a quiver  $Q$  as  $R[x]$ -modules, if  $P \neq 0$ . Then we show a representation  $P[x] \xrightarrow{id} P[x] \xrightarrow{id} P[x]$  of a quiver  $Q$  is a projective representation, if  $P$  is a projective left  $R$ -module.

## 1. Introduction

A quiver is just a directed graph with vertices and edges (arrows) [1]. We may consider many different types of quivers. We allow multiple edges and multiple arrows, and edges going from a vertex back to the same vertex. Originally a representation of the quiver assigned a vector space to each vertex - and a linear map to each edge (or arrow) - with the linear map going from the vector space assigned to the initial vertex to

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the one assigned to the terminal vertex. For example, a representation of the quiver  $Q = \bullet \rightarrow \bullet \rightarrow \bullet$  is  $V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$ ,  $V_1, V_2$  and  $V_3$  are vector spaces and  $f, g$  are linear maps (morphisms). Then we extend this representation to the left  $R$ -modules, a representation of the quiver  $Q = \bullet \rightarrow \bullet \rightarrow \bullet$  is  $M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3$ ,  $M_1, M_2$  and  $M_3$  are left  $R$ -modules and  $\alpha, \beta$  are  $R$ -linear maps. Throughout this paper a (the) quiver  $Q$  means a (the) quiver  $Q = \bullet \rightarrow \bullet \rightarrow \bullet$ .

If  $M$  is a left  $R$ -module, then the polynomial  $M[x]$  is a left  $R[x]$ -module defined by

$$r(m_0 + m_1x + m_2x^2 \cdots + m_ix^i) = rm_0 + rm_1x + rm_2x^2 + \cdots + rm_ix^i$$

$$x(m_0 + m_1x + m_2x^2 \cdots + m_ix^i) = m_0x + m_1x^2 + m_2x^3 + \cdots + m_ix^{i+1}.$$

We call  $M[x]$  a polynomial module. Similarly we can define the power series  $M[[x]]$  as a left  $R[[x]]$ -module and we call a power series modules.

The following diagram

$$\begin{array}{ccccc} M_1 & \xrightarrow{g_1} & M_2 & \xrightarrow{g_2} & M_3 \\ \downarrow k_1 & & \downarrow k_2 & & \downarrow k_3 \\ N_1 & \xrightarrow{h_1} & N_2 & \xrightarrow{h_2} & N_3 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

is commutative which means  $h_1k_1 = k_2g_1, h_2k_2 = k_3g_2$ , and  $k_3g_2g_1 = h_2h_1k_1 = h_2k_2g_1$ .

In [3] a homotopy of a quiver was developed and in [2] a cyclic quiver ring was studied. The theory of projective representations were developed in [4] and the theory of injective representations were studied in [5]. Recently, in [7] injective covers and envelopes of representations of linear quivers were studied, and in [6] properties of multiple edges of quivers were studied.

DEFINITION 1.1. [9] A left  $R$ -module  $P$  is said to be *projective* if given any surjective linear map  $\sigma : M' \rightarrow M$  and any linear map  $h : P \rightarrow M$

, there is a linear map  $g : P \rightarrow M'$  such that  $\sigma \circ g = h$ . That is

$$\begin{array}{ccc}
 & P & \\
 g \swarrow & \downarrow h & \\
 M' & \xrightarrow{\sigma} & M \longrightarrow 0
 \end{array}$$

can always be completed to a commutative diagram.

DEFINITION 1.2. Let  $M_1, M_2, M_3, N_1, N_2, N_3, P_1, P_2, P_3$  be left  $R$ -modules, and  $k_1 : M_1 \rightarrow N_1, k_2 : M_2 \rightarrow N_2, k_3 : M_3 \rightarrow N_3$  be onto  $R$ -linear maps.

A representation  $P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3$  of a quiver  $Q = \bullet \rightarrow \bullet \rightarrow \bullet$  is called a *projective representation* if every diagram of representations

$$\begin{array}{ccccccc}
 & & (P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3) & & & & \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
 (M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3) & \longrightarrow & (N_1 \xrightarrow{h_1} N_2 \xrightarrow{h_2} N_3) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0) & & 
 \end{array}$$

with

$$\begin{array}{ccccc}
 M_1 & \xrightarrow{g_1} & M_2 & \xrightarrow{g_2} & M_3 \\
 \downarrow k_1 & & \downarrow k_2 & & \downarrow k_3 \\
 N_1 & \xrightarrow{h_1} & N_2 & \xrightarrow{h_2} & N_3 \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & 0
 \end{array}$$

can be completed to a commutative diagram as follows :

$$\begin{array}{ccccccc}
 & & (P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3) & & & & \\
 & s \swarrow & \downarrow \alpha & u \swarrow & \downarrow \beta & & \downarrow \gamma \\
 (M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3) & \xrightarrow{t} & (N_1 \xrightarrow{h_1} N_2 \xrightarrow{h_2} N_3) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0) & & 
 \end{array}$$

## 2. Projective representation of a quiver $Q$ as $R[x]$ -modules

**THEOREM 2.1.** If  $P$  is a projective left  $R$ -module then  $0 \longrightarrow 0 \longrightarrow P[x]$  is a projective representation of a quiver  $Q$  as  $R[x]$ -modules.

*Proof.* Let  $M_1, M_2, M_3, N_1, N_2, N_3$  be left  $R[x]$ -modules, and  $k_1 : M_1 \rightarrow N_1$ ,  $k_2 : M_2 \rightarrow N_2$ ,  $k_3 : M_3 \rightarrow N_3$  be onto  $R[x]$ -linear maps, and  $f : P[x] \rightarrow N_3$  be an  $R[x]$ -linear map. And consider the following diagrams

$$\begin{array}{ccccc}
 M_1 & \xrightarrow{g_1} & M_2 & \xrightarrow{g_2} & M_3 \\
 \downarrow k_1 & & \downarrow k_2 & & \downarrow k_3 \\
 N_1 & \xrightarrow{h_1} & N_2 & \xrightarrow{h_2} & N_3 \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & 0
 \end{array}$$

and

$$\begin{array}{ccccccc}
 & & (0 \longrightarrow 0 \longrightarrow P[x]) & & & & \\
 & & \downarrow & & \downarrow & & \downarrow f \\
 (M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3) & \longrightarrow & (N_1 \xrightarrow{h_1} N_2 \xrightarrow{h_2} N_3) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0). & & 
 \end{array}$$

Since  $P$  is a projective left  $R$ -module, there exists an  $R$ -linear map  $t : P \rightarrow M_3$  such that  $k_3 t = f|_P$ . Define  $\bar{t} : P[x] \rightarrow M_3$  by  $\bar{t}(p_0 + p_1 x + \cdots + p_n x^n) = t(p_0) + t(p_1)x + \cdots + t(p_n)x^n$ . Then

$$\begin{aligned}
 & k_3 \bar{t}(p_0 + p_1 x + \cdots + p_n x^n) \\
 &= k_3(t(p_0) + t(p_1)x + \cdots + t(p_n)x^n) \\
 &= (k_3 t)(p_0) + (k_3 t)(p_1)x + \cdots + (k_3 t)(p_n)x^n \\
 &= f|_P(p_0) + f|_P(p_1)x + \cdots + f|_P(p_n)x^n \\
 &= f(p_0 + p_1 x + \cdots + p_n x^n).
 \end{aligned}$$

So we have  $k_3 \bar{t} = f$ . Therefore, we can complete the following diagram

$$\begin{array}{ccccccc}
 & & & (0 \longrightarrow 0 \longrightarrow P[x]) & & & \\
 & & \swarrow 0 & \downarrow \bar{\tau} & \downarrow f & & \\
 & & 0 & & & & \\
 (M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3) & \xrightarrow{\quad} & (N_1 \xrightarrow{h_1} N_2 \xrightarrow{h_2} N_3) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0) & & 
 \end{array}$$

as a commutative diagram. Hence,  $0 \rightarrow 0 \rightarrow P[x]$  is a projective representation of a quiver  $Q$  as  $R[x]$ -modules.

□

REMARK 2.2. We see that  $P[x] \longrightarrow 0 \longrightarrow 0$  is not a projective representation of a quiver  $Q$  as  $R[x]$ -modules if  $P \neq 0$ , because the following diagram

$$\begin{array}{ccccccc}
 & & & (P[x] \longrightarrow 0 \longrightarrow 0) & & & \\
 & & & \downarrow id & \downarrow & \downarrow & \\
 (P[x] \xrightarrow{id} P[x] \longrightarrow 0) & \longrightarrow & (P[x] \longrightarrow 0 \longrightarrow 0) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0) & & 
 \end{array}$$

can not be completed as a commutative diagram with

$$\begin{array}{ccccc}
 P[x] & \xrightarrow{id} & P[x] & \longrightarrow & 0 \\
 id \downarrow & & \downarrow & & \downarrow \\
 P[x] & \longrightarrow & 0 & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & 0.
 \end{array}$$

Similarly,  $P[[x]] \longrightarrow 0 \longrightarrow 0$  is not a projective representation of a quiver  $Q$  as  $R[x]$ -modules if  $P \neq 0$ .

COROLLARY 2.3. If  $P$  is a projective left  $R$ -module then  $0 \longrightarrow 0 \longrightarrow P[[x]]$  is a projective representation of a quiver  $Q$  as  $R[x]$ -modules.

Let  $Z_n$  denotes the ring of integers modulo  $n$ .

EXAMPLE 2.4. Let  $R = Z_6$ , then  $P = Z_2$  is a projective  $Z_6$ -module.  $0 \longrightarrow 0 \longrightarrow Z_2[x]$  is a projective representation of a quiver  $Q$  as  $Z_6[x]$ -modules.

THEOREM 2.5. A representation  $0 \longrightarrow P[x] \xrightarrow{id} P[x]$  of a quiver  $Q$  is a projective representation, if  $P$  is a projective left  $R$ -module.

*Proof.* Let  $f : P[x] \longrightarrow N_2$  be an  $R[x]$ -linear map and  $k_1 : M_1 \longrightarrow N_1$ ,  $k_2 : M_2 \longrightarrow N_2$ ,  $k_3 : M_3 \longrightarrow N_3$  be onto  $R[x]$ -linear maps and choose  $h_2 f : P[x] \longrightarrow N_3$  as an  $R[x]$ -linear map. Consider the following diagrams

$$\begin{array}{ccccc}
 M_1 & \xrightarrow{g_1} & M_2 & \xrightarrow{g_2} & M_3 \\
 k_1 \downarrow & & k_2 \downarrow & & k_3 \downarrow \\
 N_1 & \xrightarrow{h_1} & N_2 & \xrightarrow{h_2} & N_3 \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & 0
 \end{array}$$

and

$$\begin{array}{ccccccc}
 & & & (0 \longrightarrow P[x] \xrightarrow{id} P[x]) & & & \\
 & & & \downarrow & \downarrow f & \downarrow h_2 f & \\
 (M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3) & \longrightarrow & (N_1 \xrightarrow{h_1} N_2 \xrightarrow{h_2} N_3) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0). & & 
 \end{array}$$

Then since  $P$  is a projective left  $R$ -module, there exists  $t : P \rightarrow M_2$  such that  $k_2 t = f|_P$ .

Define  $\bar{t} : P[x] \rightarrow M_2$  by  $\bar{t}(n_0 + n_1 x + \cdots + n_i x^i) = t(n_0) + t(n_1)x + \cdots + t(n_i)x^i$ . Then

$$\begin{aligned}
 & k_2 \bar{t}(n_0 + n_1 x + \cdots + n_i x^i) \\
 &= k_2(t(n_0) + t(n_1)x + \cdots + t(n_i)x^i) \\
 &= (k_2 t)(n_0) + (h_2 t)(n_1)x + \cdots + (h_2 t)(n_i)x^i \\
 &= f|_P(n_0) + f|_P(n_1)x + \cdots + f|_P(n_i)x^i \\
 &= f(n_0 + n_1 x + \cdots + n_i x^i).
 \end{aligned}$$

Now let  $g_2 \bar{t} : P[x] \rightarrow M_3$  be an  $R[x]$ -linear map, then  $k_3 g_2 \bar{t} = h_2 k_2 \bar{t} = h_2 f$ . So we have the following commutative diagram

$$\begin{array}{ccccccc}
 & & & & (0 \longrightarrow & P[x] \xrightarrow{id} & P[x]) \\
 & & & & \downarrow & \downarrow & \downarrow \\
 & & & & \bar{t} & g_2 \bar{t} & f \\
 & & & & \downarrow & \downarrow & \downarrow \\
 & & & & 0 & & h_2 f \\
 (M_1 \xrightarrow{g_1} & M_2 \xrightarrow{g_2} & M_3) \xrightarrow{\bar{t}} & (N_1 \xrightarrow{h_1} & N_2 \xrightarrow{h_2} & N_3) \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0).
 \end{array}$$

Hence,  $P$  is a projective left  $R$ -module implies a representation

$$0 \longrightarrow P[x] \xrightarrow{id} P[x]$$

□

REMARK 2.6. We see that  $P[x] \xrightarrow{id} P[x] \longrightarrow 0$  is not a projective representation of a quiver  $Q$  as  $R[x]$ -modules if  $P \neq 0$ , because the following diagram

$$\begin{array}{ccccccc}
 & & & & (P[x] \xrightarrow{id} & P[x] \longrightarrow & 0) \\
 & & & & \downarrow id & \downarrow & \downarrow \\
 (P[x] \xrightarrow{id} & P[x] \xrightarrow{id} & P[x]) \longrightarrow & (P[x] \xrightarrow{id} & P[x] \longrightarrow & 0) \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0)
 \end{array}$$

can not be completed as a commutative diagram with

$$\begin{array}{ccccc}
 P[x] & \xrightarrow{id} & P[x] & \xrightarrow{id} & P[x] \\
 id \downarrow & & \downarrow id & & \downarrow \\
 P[x] & \xrightarrow{id} & P[x] & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & 0.
 \end{array}$$

Similarly,  $P[[x]] \xrightarrow{id} P[[x]] \longrightarrow 0$  is not a projective representation of a quiver  $Q$  as  $R[x]$ -modules if  $P \neq 0$ .

COROLLARY 2.7. If  $P$  is a projective left  $R$ -module then  $0 \longrightarrow P[[x]] \xrightarrow{id} P[[x]]$  is a projective representation of a quiver  $Q$  as  $R[x]$ -modules.

EXAMPLE 2.8. Let  $R = Z_6$ , then  $P = Z_2$  is a projective  $Z_6$ -module.  $0 \longrightarrow Z_2[x] \xrightarrow{id} Z_2[x]$  is a projective representation of a quiver  $Q$  as  $Z_6[x]$ -modules.

THEOREM 2.9. A representation  $P[x] \xrightarrow{id} P[x] \xrightarrow{id} P[x]$  of a quiver  $Q$  is a projective representation, if  $P$  is a projective left  $R$ -module.

*Proof.* Let  $f : P[x] \rightarrow N_1$  be an  $R[x]$ -linear map and  $k_1 : M_1 \rightarrow N_1$ ,  $k_2 : M_2 \rightarrow N_2$ ,  $k_3 : M_3 \rightarrow N_3$  be onto  $R[x]$ -linear maps and choose  $h_1 f : P[x] \rightarrow N_2$  and  $h_2 h_1 f : P[x] \rightarrow N_3$  as  $R[x]$ -linear maps. And consider the following diagrams

$$\begin{array}{ccccc} M_1 & \xrightarrow{g_1} & M_2 & \xrightarrow{g_2} & M_3 \\ k_1 \downarrow & & k_2 \downarrow & & k_3 \downarrow \\ N_1 & \xrightarrow{h_1} & N_2 & \xrightarrow{h_2} & N_3 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

and

$$\begin{array}{ccccccc} & & (P[x] \xrightarrow{id} P[x] \xrightarrow{id} P[x]) & & & & \\ & & \downarrow f & \downarrow h_1 f & \downarrow h_2 h_1 f & & \\ (M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3) & \longrightarrow & (N_1 \xrightarrow{h_1} N_2 \xrightarrow{h_2} N_3) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0). & & \end{array}$$

Then since  $P$  is a projective left  $R$ -module, there exists  $t : P \rightarrow M_1$  such that  $k_1 t = f|_P$ .

Define  $\bar{t} : P[x] \rightarrow M_1$  by  $\bar{t}(n_0 + n_1 x + \cdots + n_i x^i) = t(n_0) + t(n_1)x + \cdots + t(n_i)x^i$ . Then

$$\begin{aligned} & k_1 \bar{t}(n_0 + n_1 x + \cdots + n_i x^i) \\ &= k_1(t(n_0) + t(n_1)x + \cdots + t(n_i)x^i) \\ &= (k_1 t)(n_0) + (k_1 t)(n_1)x + \cdots + (k_1 t)(n_i)x^i \\ &= f|_P(n_0) + f|_P(n_1)x + \cdots + f|_P(n_i)x^i \\ &= f(n_0 + n_1 x + \cdots + n_i x^i). \end{aligned}$$

Now let  $g_1 \bar{t} : P[x] \rightarrow M_2$  be an  $R[x]$ -linear map, then  $k_2 g_1 \bar{t} = h_1 k_1 \bar{t} = h_1 f$ . And let  $g_2 g_1 \bar{t} : P[x] \rightarrow M_3$  be an  $R[x]$ -linear map. Then  $k_3 g_2 g_1 \bar{t} = h_2 k_2 g_1 \bar{t} = h_2 h_1 k_1 \bar{t} = h_2 h_1 f$ . So we have the following commutative diagram



$$\begin{array}{ccccccc}
 & & & (P[x] \xrightarrow{id} P[x] \xrightarrow{id} P[x]) & & & \\
 & & \bar{t} & \swarrow & & & \\
 & & g_1 \bar{t} & \downarrow f & g_2 g_1 \bar{t} & \downarrow h_1 f & \downarrow h_2 h_1 f \\
 (M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3) & \xrightarrow{\quad} & (N_1 \xrightarrow{h_1} N_2 \xrightarrow{h_2} N_3) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0).
 \end{array}$$

Hence,  $P$  is a projective left  $R$ -module implies a representation

$$P[x] \xrightarrow{id} P[x] \xrightarrow{id} P[x]$$

is a projective representation of a quiver  $Q$ . □

**COROLLARY 2.10.** If  $P$  is a projective left  $R$ -module then

$P[[x]] \xrightarrow{id} P[[x]] \xrightarrow{id} P[[x]]$  is a projective representation of a quiver  $Q$  as  $R[x]$ -modules.

**EXAMPLE 2.11.** Let  $R = Z_6$ , then  $P = Z_2$  is a projective  $Z_6$ -module.

$Z_2[x] \xrightarrow{id} Z_2[x] \xrightarrow{id} Z_2[x]$  is a projective representation of a quiver  $Q$  as  $Z_6[x]$ -modules.

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