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A NOTE ON FOUR TYPES OF REGULAR RELATIONS

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Abstract. In this paper, we study the four different types of relations, $\mathcal{P}(X,T)$, $\mathcal{R}(X,T)$, $\mathcal{L}(X,T)$, and $\mathcal{S}(X,T)$ in a transformation (X,T), and obtain some of their properties. In particular, we give a relationship between $\mathcal{R}(X,T)$ and $\mathcal{S}(X,T)$.

1. Introduction

The proximal relation were first studied by Ellis and Gottschalk in [6]. The syndetically proximal relation were introduced by Clay in [3]. In [1], Auslander defined the regular minimal sets which may be described as minimal subsets of enveloping semigroups. In [8], Shoenfeld introduced the regular homomorphisms which are defined by extending regular minimal sets to homomorphisms with minimal range. Also Yu introduced the regular relation and the syndetically regular relation (see [9], [10]).

In this paper, we study the four different types of relations in a transformation and give some of their properties.

2. Preliminaries

A *transformation group* (X,T) will consist of a jointly continuous action of the topological group T on the compact Hausdorff space X. The group T, with identity e, is assumed to be topologically discrete and remain fixed throughout this paper, so we may write X instead of (X,T).

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A flow is said to be *minimal* if every point has dense orbit. Minimal flows are also referred to as minimal sets.

A *homomorphism* of transformation groups is a continuous, equivariant map. A one-one homomorphism of X onto X is called an automorphism of X. We denote the group of automorphisms of X by $A(X)$.

The compact Hausdorff space X carries a natural uniformity whose indices are the neighborhoods of the diagonal in $X \times X$. Two points $x, x' \in X$ are said to be *proximal* if, given any index α, there exists $t \in T$ such that $(xt, x't) \in \alpha$. The proximal relation in X, denoted by $\mathcal{P}(X,T)$, is the set of all proximal pairs in X. X is said to be *distal* if $\mathcal{P}(X,T) = \triangle_X$, the diagonal of $X \times X$ and is said to be *proximal* if $\mathcal{P}(X,T) = X \times X$.

Given a transformation group (X,T), we may regard T as a set of self-homeomorphisms of X. We define $E(X)$, the *enveloping semigroup* of X to be the closure of T in X^X, taken with the product topology. $E(X)$ is at once a transformation group and a sub-semigroup of X^X. The minimal right ideals of $E(X)$, considered as a semigroup, coincide with the minimal sets of $E(X)$. A subset A of T is said to be *syndetic* if there exists a compact subset K of T with $T = AK$.

Two points $x, x' \in X$ are said to be *syndetically proximal* if, given any index α, there exists a syndetic subset A of T such that $(xt, x't) \in \alpha$ for all $t \in A$. The set of syndetically proximal pairs in X is called the *syndetically proximal relation* and is denoted by $\mathcal{L}(X,T)$.

Two points $x, x' \in X$ are said to be *regular* if there exists $h \in A(X)$ such that $(h(x), x') \in \mathcal{P}(X,T)$. The set of regular pairs in X is called the *regular relation* and is denoted by $\mathcal{R}(X,T)$.

Two points $x, x' \in X$ are said to be *syndetically regular* if there exists $h \in A(X)$ such that $(h(x), x') \in \mathcal{L}(X,T)$. The set of syndetically regular pairs in X is called the *syndetically regular relation* and is denoted by $\mathcal{S}(X,T)$.

X is said to be *almost periodic* if, given any index α, there exists a syndetic subset A of T such that $xA \subset x\alpha$ for all $x \in X$, where $x\alpha = \{y \in X \mid (x,y) \in \alpha\}$. X is said to be *locally almost periodic* if, given $x \in X$ and U a neighborhood of x, there exists a neighborhood V of x and a syndetic subset A of T with $VA \subset U$.

Remark 2.1. *If $E(X)$ contains just one minimal right ideal, then $\mathcal{P}(X,T)$ and $\mathcal{R}(X,T)$ are invariant equivalence relations on X (see [4], [9]).*

Lemma 2.2. *([2]) Suppose (X,T) is locally almost periodic. Then $\mathcal{P}(X,T) = \mathcal{L}(X,T)$.*

Lemma 2.3. *([4]) (X,T) is almost periodic iff it is locally almost periodic and distal.*

3. Some results on $\mathcal{P}(X,T)$, $\mathcal{R}(X,T)$, $\mathcal{L}(X,T)$ and $\mathcal{S}(X,T)$

The following lemma is an immediate consequence of the definitions.

Lemma 3.1. *Given a transformation group (X,T), the following statements are true :*

(1) $\mathcal{L}(X,T) \subset \mathcal{P}(X,T) \subset \mathcal{R}(X,T)$.
(2) $\mathcal{L}(X,T) \subset \mathcal{S}(X,T) \subset \mathcal{R}(X,T)$.
(3) $\triangle_X \subset \mathcal{L}(X,T)$.
(4) *If* $\mathcal{P}(X,T) = \mathcal{L}(X,T)$, *then* $\mathcal{R}(X,T) = \mathcal{S}(X,T)$.

The next lemma leads to a useful characterization of $\mathcal{L}(X,T)$.

Lemma 3.2. *([5]) Given a transformation group (X,T), the following statements are true :*

(1) $\mathcal{L}(X,T) = \{(x,y) \in X \times X \mid \overline{(x,y)T} \subset \mathcal{P}(X,T)\}$.
(2) $\mathcal{L}(X,T)$ *is an invariant equivalence relation on* X.

Lemma 3.3. *Given a transformation group (X,T), the following statements are true :*

(1) $\mathcal{S}(X,T) = \{(x,y) \in X \times X \mid \overline{(x,y)T} \subset \mathcal{R}(X,T)\}$.
(2) *If $E(X)$ contains just one minimal right ideal, then $\mathcal{S}(X,T)$ is an invariant equivalence relation on X.*

Proof. (1) Use lemma 3.2(1). Assume that $(x,y) \in X \times X$. Then $(x,y) \in \mathcal{S}(X,T)$ iff there exists $h \in A(X)$ such that $(h(x),y) \in \mathcal{L}(X,T)$ iff there exists $h \in A(X)$ such that $\overline{(h(x),y)T} \subset \mathcal{P}(X,T)$ iff there exists $h \in A(X)$ such that $(h(xp),yp) \in \mathcal{P}(X,T)$ for all $p \in E(X)$ iff $(x,y)p \in \mathcal{R}(X,T)$ for all $p \in E(X)$ iff $\overline{(x,y)T} \subset \mathcal{R}(X,T)$. This completes the proof of (1).

(2) It follows immediately from (1) that $\mathcal{S}(X,T)$ is a reflexive, symmetric and invariant relation. To see that $\mathcal{S}(X,T)$ is transitive, assume that $(x,y) \in \mathcal{S}(X,T)$ and $(y,z) \in \mathcal{S}(X,T)$. Then $\overline{(x,y)T} \subset \mathcal{R}(X,T)$ and $\overline{(y,z)T} \subset \mathcal{R}(X,T)$ and hence $(xp,yp) \in \mathcal{R}(X,T)$ and $(yp,zp) \in$

$\mathcal{R}(X,T)$ for all $p \in E(X)$. Since $E(X)$ contains just one minimal right ideal, we have from Remark 2.1 that $(xp, zp) \in \mathcal{R}(X,T)$ for all $p \in E(X)$. Therefore $\overline{(x,z)T} \subset \mathcal{R}(X,T)$ and hence $(x,z) \in \mathcal{S}(X,T)$. $\square$

Remark 3.4. *$\mathcal{P}(X,T)$, $\mathcal{R}(X,T)$, and $\mathcal{S}(X,T)$ are not equivalence relations on X. However, if $E(X)$ contains just one minimal right ideal, then they are invariant equivalence relations on X (see Remark 2.1 and Lemma 3.3).*

Lemma 3.5. *If $\mathcal{P}(X,T)$ is closed, then $\mathcal{R}(X,T)$ is also closed.*

Proof. Let $(x,y) \in \mathcal{R}(X,T)$ and let $q \in E(X)$. Then there exists $h \in A(X)$ such that $(h(x),y) \in \mathcal{P}(X,T)$. Since $\mathcal{P}(X,T)$ is closed, we have that $(h(x),y)q \in \mathcal{P}(X,T)$ and therefore $(h(xq),yq) \in \mathcal{P}(X,T)$. This implies that $(xq,yq) \in \mathcal{R}(X,T)$. Thus $\mathcal{R}(X,T)$ is closed. $\square$

Theorem 3.6. *Let $\mathcal{P}(X,T)$ be closed. Then*
(1) $\mathcal{P}(X,T) = \mathcal{L}(X,T)$
(2) $\mathcal{R}(X,T) = \mathcal{S}(X,T)$.

Proof. To see that (1) holds, assume that $(x,y) \in \mathcal{P}(X,T)$. Since $\mathcal{P}(X,T)$ is closed, it follows that $\overline{(x,y)T} \subset \mathcal{P}(X,T)$. By Lemma 3.2(1), it follows that $\mathcal{P}(X,T) \subset \mathcal{L}(X,T)$ and therefore $\mathcal{P}(X,T) = \mathcal{L}(X,T)$.

The proof of (2) is exactly analogous to that of (1) by Lemma 3.5. $\square$

Ellis' result [4, Lemma 5.17] is a corollary to the above theorem.

Corollary 3.7. *Let $\mathcal{P}(X,T)$ be closed. Then it is an invariant equivalence relation on X.*

Remark 3.8. *Let (X,T) is distal. Since $\mathcal{P}(X,T) = \triangle_X$, it follows that $\mathcal{L}(X,T) = \mathcal{P}(X,T)$ and therefore $\mathcal{P}(X,T)$ is a closed invariant equivalence relation on X (see [4, Lemma 5.12]).*

We can prove Ellis' result [4, Lemma 5.27] as follows :

Theorem 3.9. *Suppose (X,T) is locally almost periodic. Then the following statements are true :*
(1) $\mathcal{L}(X,T) = \mathcal{P}(X,T) \subset \mathcal{R}(X,T) = \mathcal{S}(X,T)$.
(2) *$\mathcal{P}(X,T)$ and $\mathcal{R}(X,T)$ are closed invariant equivalence relations on X.*

Proof. (1) This follows from Lemma 2.2 and Lemma 3.1(4).

(2)] The fact that $\mathcal{P}(X,T)$ is an invariant equivalence relation on X follows from (1) and Lemma 3.2(2). Since $\mathcal{P}(X,T)$ is transitive, it follows from [4, Proposition 5.16] that $E(X)$ contains just one minimal right ideal and therefore $\mathcal{R}(X,T)$ is an invariant equivalence relation on X by Remark 2.1. The closed property of $\mathcal{P}(X,T)$ follows from [4, Proposition 5.26]. The closed property of $\mathcal{R}(X,T)$ follows from Lemma 3.5.

□

The proof of the following corollary follows immediately from Lemma 2.3.

COROLLARY 3.10. *Suppose (X,T) is almost periodic. Then the following statements are true :*

(1) $\mathcal{L}(X,T)=\mathcal{P}(X,T)\subset\mathcal{R}(X,T)=\mathcal{S}(X,T)$.
(2) *$\mathcal{P}(X,T)$ and $\mathcal{R}(X,T)$ are closed invariant equivalence relations on X.*

THEOREM 3.11. *Suppose $A(X)=\{1_X\}$, where $\{1_X\}$ is the identity homomorphism of X. Then $\mathcal{L}(X,T)=\mathcal{S}(X,T)\subset\mathcal{P}(X,T)=\mathcal{R}(X,T)$.*

Proof. Let $(x,y)\in\mathcal{S}(X,T)$. Then $\overline{(x,y)T}\subset\mathcal{R}(X,T)$ by Lemma 3.3(1). Since $A(X)=\{1_X\}$, it follows that $\mathcal{P}(X,T)=\mathcal{R}(X,T)$ and hence $(x,y)\in\mathcal{L}(X,T)$ by Lemma 3.2(1). Therefore $\mathcal{S}(X,T)=\mathcal{L}(X,T)$. □

COROLLARY 3.12. *Suppose (X,T) is minimal and proximal. Then $\mathcal{L}(X,T)=\mathcal{S}(X,T)\subset\mathcal{P}(X,T)=\mathcal{R}(X,T)$.*

Proof. The proof uses [7, (8) of Section 1] to show that if (X,T) is minimal and proximal, then the only homomorphism $(X,T)\to(X,T)$ is the identity. □

LEMMA 3.13. *Let $h\in A(X)$ and let $\check{h}:X\times X\to X\times X$ be the map induced by h. Then the following statements are true :*

(1) $\check{h}\mathcal{P}(X,T)\subset\mathcal{P}(X,T)$.
(2) $\check{h}\mathcal{R}(X,T)\subset\mathcal{R}(X,T)$.
(3) $\check{h}\mathcal{L}(X,T)\subset\mathcal{L}(X,T)$.
(4) $\check{h}\mathcal{S}(X,T)\subset\mathcal{S}(X,T)$.

Proof. The proof of (1) is analogous to that of [4, Proposition 5.22]. Let $(x,y)\in\mathcal{R}(X,T)$. Then there exists $\psi\in A(X)$ with $(\psi(x),y)\in$

$\mathcal{P}(X,T)$. By (1) $(h\circ\psi(x), h(y)) = (h\circ\psi\circ h^{-1}\circ h(x), h(y)) \in \mathcal{P}(X,T)$. Since $h\circ\psi\circ h^{-1} \in A(X)$, it follows that $(h(x), h(y)) = \check{h}(x,y) \in \mathcal{R}(X,T)$. Now let $(x,y) \in \mathcal{L}(X,T)$. Then $\overline{(x,y)T} \subset \mathcal{P}(X,T)$ by Lemma 3.2(1), which means that $(x,y)p \in \mathcal{P}(X,T)$ for all $p \in E(X)$. By (1) $\check{h}(x,y)p \in \mathcal{P}(X,T)$ for all $p \in E(X)$. Therefore $\overline{(h(x),h(y))T} \subset \mathcal{P}(X,T)$ and hence $(h(x),h(y)) \in \mathcal{L}(X,T)$. This proves that $\check{h}\mathcal{L}(X,T) \subset \mathcal{L}(X,T)$. The proof of (4) is analogous to that of (3). □

THEOREM 3.14. *Let $h \in A(X)$ and let $\check{h} : X \times X \to X \times X$ be the map induced by h. Then the following statements are true :*

(1) *If (X,T) is minimal, then $\check{h}\mathcal{P}(X,T) = \mathcal{P}(X,T)$.*
(2) *If (X,T) is minimal and $A(X)$ is abelian, then $\check{h}\mathcal{R}(X,T) = \mathcal{R}(X,T)$.*

Proof. If (X,T) is minimal, then it is pointwise almost periodic. Thus (1) follows from [4, Proposition 5.22]. To see (2), let $(y_1,y_2) \in \mathcal{R}(X,T)$. Then there exists $\psi \in A(X)$ with $(\psi(y_1), y_2) \in \mathcal{P}(X,T)$. By (1) there exists $(x_1,x_2) \in \mathcal{P}(X,T)$ such that $\check{h}(x_1,x_2) = (\psi(y_1), y_2)$. Therefore we have that $(\psi^{-1}(h(x_1)), h(x_2)) = (y_1,y_2)$ and $\psi^{-1} \in A(X)$. Since $A(X)$ is abelian, it follows that $(h(\psi^{-1}(x_1)), h(x_2)) = \check{h}(\psi^{-1}(x_1), x_2) = (y_1,y_2)$, which proves that $\check{h}\mathcal{R}(X,T) = \mathcal{R}(X,T)$. □

For each $h \in A(X)$, we define the subsets $S_h(X)$ and $R_h(X)$ of $X \times X$ as follows:

$$S_h(X) = \{(x,x') \in X \times X \mid (h(x),x') \in \mathcal{L}(X,T)\}$$

$$R_h(X) = \{(x,x') \in X \times X \mid (h(x),x') \in \mathcal{P}(X,T)\}.$$

Note that $S_{1_X}(X) = \mathcal{L}(X,T)$ and $R_{1_X}(X) = \mathcal{P}(X,T)$.

If $\mathcal{V}$ and $\mathcal{H}$ are relations in X, then $\mathcal{V}\circ\mathcal{H}$ is the relation in X defined by as follows :

$(x,y) \in \mathcal{V}\circ\mathcal{H}$ if and only if for some element z, $(x,z) \in \mathcal{H}$ and $(z,y) \in \mathcal{V}$.

LEMMA 3.15. *Let (X,T) be a transformation group and let $h \in A(X)$. Then $S_h(X) \neq \emptyset$ and $R_h(X) \neq \emptyset$.*

Proof. Let $h,k \in A(X)$ and let $x' = h(x)$. Then $(h(x),x') \in \triangle_X \subset \mathcal{L}(X,T) \subset \mathcal{P}(X,T)$ by Lemma 3.1. Therefore $(x,x') \in S_h(X)$ and $(x,x') \in R_h(X)$. This proves that $S_h(X) \neq \emptyset$ and $R_h(X) \neq \emptyset$. □

THEOREM 3.16. *Suppose that (X,T) is a transformation group and that $E(X)$ contains just one minimal right ideal. Then $R_h(X)\circ R_k(X) = R_{h\circ k}(X)$ for all $h,k \in A(X)$.*

Proof. Let $h,k \in A(X)$ and $(x,y) \in R_h(X)\circ R_k(X)$. Then there exists $z \in X$ such that $(x,z) \in R_k(X)$ and $(z,y) \in R_h(X)$. Hence $(k(x),z) \in \mathcal{P}(X,T)$ and $(h(z),y) \in \mathcal{P}(X,T)$. Therefore by Theorem 3.13(1) $(h(k(x)),h(z)) \in \mathcal{P}(X,T)$. Since $E(X)$ contains just one minimal right ideal, it follows from Remark 2.1 that $\mathcal{P}(X,T)$ is transitive and therefore $(h(k(x)),y) \in \mathcal{P}(X,T)$. Since $h\circ k \in A(X)$, we have that $(x,y) \in R_{h\circ k}(X)$.

Let $(x,y) \in R_{h\circ k}(X)$. By Theorem 3.13(1), $(h(k(x)),y) \in \mathcal{P}(X,T)$ shows that $(k(x),h^{-1}(y)) \in \mathcal{P}(X,T)$. Now let $h^{-1}(y) = z$. Then $(k(x),z)) \in \mathcal{P}(X,T)$ and $h(z) = y$. Since $(y,y) \in \mathcal{P}(X,T)$, it follows that $(h(z),y) \in \mathcal{P}(X,T)$. Hence $(x,z) \in R_k(X)$ and $(z,y) \in R_h(X)$. Thus $(x,y) \in R_h(X)\circ R_k(X)$. □

The next corollary states that if $E(X)$ contains just one minimal right ideal , then $(\{R_h(X) \mid h \in A(X)\},\circ)$ forms a group.

COROLLARY 3.17. *Suppose that (X,T) is a transformation group and that $E(X)$ contains just one minimal right ideal. For arbitrary $h,k,r \in A(X)$, the following properties hold :*

(1) $(R_h(X)\circ R_k(X))\circ R_r(X) = R_h(X)\circ(R_k(X)\circ R_r(X))$.
(2) *There exists $1_X \in A(X)$ such that* $\mathcal{P}(X,T)\circ R_h(X) = R_h(X)\circ\mathcal{P}(X,T) = R_h(X)$.
(3) *For each $h \in A(X)$ there exists $h^{-1} \in A(X)$ such that* $R_h(X)\circ R_{h^{-1}}(X) = R_{h^{-1}}(X)\circ R_h(X) = \mathcal{P}(X,T)$.

Proof. This follows from Lemma 3.15, Theorem 3.16, and the fact that $A(X)$ is a group. □

COROLLARY 3.18. *Let (X,T) be a transformation group. Then the following statements are true :*

(1) $S_h(X)\circ S_k(X) = S_{h\circ k}(X)$ *for all* $h,k \in A(X)$.
(2) $(S_h(X)\circ S_k(X))\circ S_r(X) = S_h(X)\circ(S_k(X)\circ S_r(X))$ *for all* $h,k,r \in A(X)$.
(3) $\mathcal{L}(X,T)\circ S_h(X) = S_h(X)\circ\mathcal{L}(X,T) = S_h(X)$ *for all* $h \in A(X)$.
(4) $(S_h(X))^{-1} = S_{h^{-1}}(X)$ *for all* $h \in A(X)$.

Proof. The proof of (1) is analogous to that of Theorem 3.16. Note that $S_h(X) \neq \emptyset$ and $\check{h}\mathcal{L}(X,T) \subset \mathcal{L}(X,T)$ for all $h \in A(X)$, and $\mathcal{L}(X,T)$ is an invariant equivalence relation on X. □

REMARK 3.19. *(1)The collection* $(\{S_h(X) \mid h \in A(X)\}, \circ)$ *is a group by Corollary 3.18.*

(2) Suppose (X,T) *is distal. The collection* $(\{R_h(X) \mid h \in A(X)\}, \circ)$ *forms a group because* (X,T) *is distal iff* $E(X)$ *is a minimal right ideal (see [4, Proposition 5.3]).*

References

[1] J. Auslander, *Regular minimal sets 1*, Trans. Amer. Math. Soc. **123** (1966), 469–479.

[2] I.U. Bronstein, *Extension of minimal transformation groups*, Sijthoff and Nordhoff Inter. Publ., Netherlands, 1979.

[3] J.P. Clay, *Proximity relations in transformation groups*, Trans. Amer. Math. Soc. **108** (1963), 88–96.

[4] R. Ellis, *Lectures on topological dynamics*, Benjamin, New York, 1969.

[5] ______, *A semigroup associated with a transformation group*, Trans. Amer. Math. Soc. **94** (1960), 272–281.

[6] R. Ellis and W.H. Gottschalk, *Homomorphisms of transformation groups*, Trans. Amer. Math. Soc. **94** (1960), 258–271.

[7] S. Glasner, *Compressibility properties in topological dynamics*, Amer. J. Math. **97** (1975), 148–171.

[8] P.S. Shoenfeld, *Regular homomorphisms of minimal sets*, Doctoral Dissertation, Univ. of Maryland (1974).

[9] J.O. Yu, *Regular relations in transformation group*, J. Korean Math. Soc. **21** (1984), 41–48.

[10] ______, *The regionally regular relations*, J. Chungcheong Math. Soc. **19** (2006), 365–373.

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