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INTEGRAL DOMAINS WITH FINITELY MANY STAR OPERATIONS OF FINITE TYPE

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Abstract. Let D be an integral domain and $SF(D)$ be the set of star operations of finite type on D. We show that if $|SF(D)| < \infty$, then every maximal ideal of D is a t-ideal. We give an example of integrally closed quasi-local domains D in which the maximal ideal is divisorial (so a t-ideal) but $|SF(D)| = \infty$. We also study the integrally closed domains D with $|SF(D)| \leq 2$.

1. Introduction

Let D be an integral domain with quotient field K. Let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D. A mapping $I \mapsto I^*$ of $\mathbf{F}(D)$ into $\mathbf{F}(D)$ is called a *star-operation* on D if for all $0 \neq a \in K$ and $I, J \in \mathbf{F}(D)$, the following conditions are satisfied:

(1) $(aD)^* = aD$ and $(aI)^* = aI^*$,
(2) $I \subseteq I^*$; $I \subseteq J$ implies $I^* \subseteq J^*$, and
(3) $(I^*)^* = I^*$.

Given any star operation $*$ on D, one can construct a new star operation $*_f$ by setting $I^{*_f} = \cup\{J^* | J$ is a nonzero finitely generated subideal of $I\}$ for all $I \in \mathbf{F}(D)$. A star operation $*$ on D is said to be of *finite type* if $*_f = *$. Obviously, $(*_f)_f = *_f$, and hence $*_f$ is of finite type. Clearly, $I^* = I^{*_f}$ for all nonzero finitely generated fractional ideals I of D; so if D is a Noetherian domain, then each star operation on D is of finite type. An $I \in \mathbf{F}(D)$ is called a $*$-*ideal* if $I^* = I$, while a $*$-ideal is called

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a *maximal $*$-ideal* if it is maximal among proper integral $*$-ideals. Let $*$-Max(D) denote the set of maximal $*$-ideals of D. It is well known that a maximal $*$-ideal is a prime ideal; each prime ideal minimal over a $*_f$-ideal is a $*_f$-ideal; and $*_f$-Max$(D) \neq \emptyset$ if D is not a field. A star operation $*$ on D is said to be *stable* if $(I \cap J)^* = I^* \cap J^*$ for all $I, J \in \mathbf{F}(D)$. Recall that $*$ is *endlich arithmetisch brauchbar* (*e.a.b.*) if $(AB)^* \subseteq (AC)^*$ for all nonzero finitely generated fractional ideals A, B, C of D implies $B^* \subseteq C^*$.

The most well-known examples of star operations are the d-, v-, and t-operations. The d-operation is just the identity function on $\mathbf{F}(D)$; so $d = d_f$. The v-operation is defined by $I_v = (I^{-1})^{-1}$, where $I^{-1} = \{x \in K | xI \subseteq D\}$, and the t-operation is given by $t = v_f$. We say that a v-ideal is a *divisorial ideal*. For two star operations $*_1$ and $*_2$ on D, we mean by $*_1 \leq *_2$ that $I^{*_1} \subseteq I^{*_2}$ for all $I \in \mathbf{F}(D)$. Clearly, if $*_1 \leq *_2$, then $(*_1)_f \leq (*_2)_f$. We know that if $*$ is any star operation on D, then $d \leq * \leq v$, and hence $d \leq *_f \leq t$. For basic properties of star operations, see [7, Sections 32 and 34].

Let $S(D)$ (resp., $SF(D)$) be the set of star operations (resp., star operations of finite type) on D; so $SF(D) \subseteq S(D)$. In [11, Proposition 2.1], it was shown that if $|S(D)| < \infty$, then each maximal ideal of D is a t-ideal. It is clear that if $|S(D)| < \infty$, then $|SF(D)| < \infty$, but not vice versa (for example, if D is an h-local Prüfer domain that has infinitely many nondivisorial maximal ideals, then $|SF(D)| = 1$ and $|S(D)| = \infty$ [11, Corollary 3.2]). So it is reasonable to ask what happens if $|SF(D)| < \infty$. Specifically, is it true that $|SF(D)| < \infty$ if and only if each maximal ideal of D is a t-ideal ? The purpose of this paper is to give an answer to this question. Precisely, we show that if $|SF(D)| < \infty$, then each maximal ideal of D is a t-ideal. We give an example of integrally closed domains D in which each maximal ideal is a t-ideal but $|SF(D)| = \infty$. We also study the integrally closed domains D with $|SF(D)| \leq 2$.

2. Main Results

Let D be an integral domain with quotient field K. Let $S(D)$ (resp., $SF(D)$) be the set of star operations (resp., star operations of finite type) on D.

We begin this section with a necessary condition for $|SF(D)| < \infty$, which is a simple modification of [11, Proposition 2.1(2)] that if $|S(D)| < \infty$, then each maximal ideal of D is a t-ideal.

LEMMA 1. *Let I be a nonzero finitely generated ideal of D with $I_v = D$. For each integer $n \geq 1$, let $E^{*_n} = (I^n : (I^n : E))$ for all $E \in \mathbf{F}(D)$. Then $*_n$ is a star operation on D such that $(*_n)_f \neq (*_m)_f$ for all positive integers $n \neq m$.*

Proof. Note that $(I^n : I^n) = D$; so $*_n$ is a star operation on D [10, Proposition 3.2]. Also, by the proof of [11, Proposition 2.1], for $0 < m < n$, $(I^n)^{*_n} = I^n$ and $(I^n)^{*_m} = I^m$. Note that $I^n \neq I^m$ for $n \neq m$ [13, Theorem 76] and I^n is finitely generted for all $n \geq 1$. Hence $(I^n)^{(*_n)_f} = (I^n)^{*_n} = I^n \neq I^m = (I^n)^{*_m} = (I^n)^{(*_m)_f}$. Thus $(*_n)_f \neq (*_m)_f$. □

THEOREM 2. *If $|SF(D)| < \infty$, then each maximal ideal of D is a t-ideal.*

Proof. Assume to the contrary that there is a maximal ideal M of D with $M_t = D$. Then there is a nonzero finitely generated subideal I of M such that $I_v = I_t = D$. Hence if we set $E^{*_n} = (I^n : (I^n : E))$ for each $E \in \mathbf{F}(D)$, then $*_n$ is a star operation on D such that $(*_n)_f \neq (*_m)_f$ for all positive integers $m \neq n$ by Lemma 1. Thus $|SF(D)| = \infty$, a contradiction. Thus each maximal ideal of D is a t-ideal. □

Let $SF_s(D)$ be the set of stable star operations of finite type on D; so $SF_s(D) \subseteq SF(D)$. In [3, Theorem 4], it was shown that if Ω is the set of nonzero prime ideals P of D with $P_t = D$, then $|\Omega|+1 \leq |SF_s(D)| \leq 2^{|\Omega|}$. Hence each maximal ideal of D is a t-ideal if and only if $|SF_s(D)| = 1$.

COROLLARY 3. *If $|SF_s(D)| \geq 2$, then $|SF(D)| = \infty$.*

Proof. If $|SF_s(D)| \geq 2$, then D has at least one maximal ideal that is not a t-ideal [3, Theorem 4]. Thus $|SF(D)| = \infty$ by Theorem 2. □

As in [8], we say that a prime ideal P of D is *strongly prime* if $xy \in P$ and $x, y \in K$ imply $x \in P$ or $y \in P$, while D is a *pseudo-valuation domain* (PVD) if every prime ideal of D is strongly prime. It is known that D is a PVD if and only if D is quasi-local whose maximal ideal is strongly prime if and only if there exists a valuation overring V of D such that $\mathrm{Spec}(V) = \mathrm{Spec}(D)$ [8, Theorem 2.7].

We next give an example of integral domains whose maximal ideals are t-ideals but $|SF(D)| = \infty$, which shows that the converse of Theorem 2 does not hold.

Example 4. Let $\mathbb{R}$ be the field of real numbers, y, z be indeterminates over $\mathbb{R}$, $K = \mathbb{R}(y, z)$ be the quotient field of the polynomial ring $\mathbb{R}[y, z]$, X be an indeterminate over K, $V = K[\![X]\!]$ be the power series ring over K (so V is a rank-one DVR), and $D = \mathbb{R} + XK[\![X]\!]$. It is clear that D is an integrally closed PVD, $V/XK[\![X]\!] = K$, and $D/XK[\![X]\!] = \mathbb{R}$ (so $\text{trdeg}(K, \mathbb{R}) = 2$). Hence D has infinitely many *e.a.b.* star operations of finite type [4, Theorem 4.10]. Thus $|SF(D)| = \infty$.

Given an *e.a.b.* star operation on an integrally closed domain D, the *Kronecker function ring of* D with respect to $*$ is defined by

$$Kr(D, *) = \{0\} \cup \{\frac{f}{g} \mid 0 \neq f, g \in D[X] \text{ and } c(f)^* \subseteq c(g)^*\},$$

where $c(h)$ denotes the ideal of D generated by the coefficients of an $h \in D[X]$. It is well known that $Kr(D, *)$ is a Bezout domain and $Kr(D, *) \cap K = D$ [7, Theorem 32.7].

Let $SF_e(D)$ be the set of *e.a.b.* star operations of finite type on D. It is known that $SF_e(D) \neq \emptyset$ if and only if D is integrally closed [7, Corollary 32.8]. Also, there is a bijection between $SF_e(D)$ and the set of Kronecker function rings of D (cf. [7, Remark 32.9]). We next give a lower bound of $|SF_e(D)|$. (Note that Example 4 shows that the equality of Proposition 5 need not hold, but the equality attains when D is a Prüfer domain.)

Proposition 5. *If D is integrally closed, then* $|SF_s(D)| \leq |SF_e(D)|$.

Proof. Let $* \in SF_s(D)$. Then we can construct an *e.a.b.* star operation $*_c$ of finite type on D such that $*$-Max$(D) = *_c$-Max(D) [2, Lemma 3.1]. Recall that if $*' \in SF_s(D)$, then $I^{*'} = \cap_{P \in *'\text{-Max}(D)} ID_P$ for all $I \in \mathbf{F}(D)$ [1, Corollary 4.2]; so if $*_1 \in SF_s(D)$ with $*_1 \neq *$, then $*_1$-Max$(D) \neq *$-Max(D). Hence $*_c$-Max$(D) \neq (*_1)_c$-Max(D), and thus $*_c \neq (*_1)_c$. This completes the proof. □

We next study the integrally closed domains D with $|SF(D)| \leq 2$. To do this, we first need the notion of a b-operation that is an *e.a.b.* star operation of finite type on an integrally closed domain D defined by $E^b = \cap\{EV | V \text{ is a valuation overring of } D\}$ for all $E \in \mathbf{F}(D)$ [7, pp. 397-398]. Clearly, the b-operation is defined on D if and only if D is integrally closed [7, Corollary 32.8]. Also, it is easy to see that $d = b$ if and only if D is a Prüfer domain [7, Theorem 24.7]. This result implies the following theorem.

THEOREM 6. *If D is integrally closed, the following statements are equivalent.*
(1) *D is a Prüfer domain.*
(2) $|SF(D)| = 1$.
(3) $|SF_s(D)| = |SF(D)| < \infty$.

Proof. (1) $\Rightarrow$ (3) If D is a Prüfer domain, then $d = t$, and thus $SF_s(D) = SF(D) = \{d\}$. (3) $\Rightarrow$ (2) This follows directly from Corollary 3. (2) $\Rightarrow$ (1) Note that $\{d, b\} \subseteq SF(D)$; so $d = b$. Thus D is a Prüfer domain. □

Recall that D is a *v-domain* if the v-operation on D is an *e.a.b* star operation; so $Kr(D, v)$ is defined on a v-domain D and $Kr(D, b) \subseteq Kr(D, *) \subseteq Kr(D, v)$ for any *e.a.b.* star operation $*$ on D. It is known that D is a v-domain if and only if each nonzero finitely generated ideal of D is v-invertible [7, Theorem 34.6]. Also, $b = t$ if and only if D is a v-domain [5, Proposition 35]. As in [4], we say that D is a *vacant domain* if D has a unique Kronecker function ring. It is clear that D is a vacant domain if and only if the b-operation is a unique *e.a.b.* star operation of finite type on D.

It is clear that PvMDs are v-domains, but v-domains need not be PvMDs (for example, a one-dimensional completely integrally closed domain that is not a valuation domain is a v-domain but not a PvMD (cf. [6, pp. 157-161])). However, if each maximal t-ideal of D is divisorial, then v-domains are PvMDs. (For if I is a nonzero finitely generated fractional ideal of D, then $(II^{-1})_v = D$, and hence $II^{-1} \nsubseteq P$, because $P_v = P$, for all $P \in t\text{-Max}(D)$. Thus $(II^{-1})_t = D$.)

THEOREM 7. *If D is an integrally closed domain with $|SF(D)| = 2$, then*
(1) *D is not a Prüfer domain,*
(2) *D is a vacant v-domain whose maximal ideals are t-ideals, and*
(3) *D has a nondivisorial maximal t-ideal.*

Proof. (1) This follows directly from Theorem 6.

(2) Recall that $d \leq b \leq t$. If $d = b$, then D is a Prüfer domain, a contradiction. Hence $b = t$ by hypothesis, and thus D is a vacant

v-domain [5, Proposition 35]. Also, by Theorem 2, each maximal ideal of D is a t-ideal.

(3) By the remark before Theorem 7, if each maximal ideal of D is divisorial, then a v-domain is a Prüfer domain. Thus D has at least one maximal t-ideal that is not a divisorial ideal. □

Corollary 8. *Let D be an integrally closed PVD with maximal ideal M.*

(1) *If D is not a valuation domain, then $|SF(D)| \geq 3$.*
(2) *If $|S(D)| = 2$, then D is a valuation domain and $M_v = D$.*
(3) *$|SF(D)| \neq 2$.*

Proof. (1) If $|SF(D)| = 1$, then D is a Prüfer domain by Theorem 6, and since D is quasi-local, D is a valuation domain. Next, if $|SF(D)| = 2$, then D is a v-domain by Theorem 7, and hence D is a Prüfer domain because $M_v = M$. Thus D is a valuation domain.

(2) Note that $d \leq b \leq t \leq v$; so $d \neq v$ and either $d = b$ or $b = v$. If $d = b$, then D is a valuation domain, and since $d \neq v$, we have $M_v = D$ [9, Lemma 5.2]. Next, if $b = v$ (so $t = v$), then $S(D) = SF(D) = \{d, v\}$, and hence D is a valuation domain by (1). But, in this case, $d = t = v$, a contradiction. Moreover, since $d \neq v$, we have $M_v = D$ [9, Lemma 5.2].

(3) If D is a valuation domain, then $|SF(D)| = 1$. Thus $|SF(D)| \neq 2$ by (1). □

Added to the proof. Recently, Houston, Mimouni and Park showed that if D is an integrally closed domain, then $|SF(D)| < \infty$ if and only if D is a Prüfer domain [12, Theorem 5.3]. Thus, there does not exist an integrally closed domain D with $|SF(D)| = 2$ (cf. Theorem 7) and if D is an integrally closed PVD that is not a valuation domain, then $|SF(D)| = \infty$ (cf. Corollary 8(1) and (3)).

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