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SUPERSTABILITY OF THE GENERALIZED PEXIDER TYPE EXPONENTIAL EQUATION IN ABELIAN GROUP

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Abstract. In this paper, we will prove the superstability of the following generalized Pexider type exponential equation

$$f\big(x+y\big)^m \quad = \quad g(x)h(y),$$

where $f, g, h : G \to \mathbb{R}$ are unknown mappings and m is a fixed positive integer. Here G is an Abelian group $(G, +)$, and $\mathbb{R}$ the set of real numbers. Also we will extend the obtained results to the Banach algebra. The obtained results are generalizations of P. Găvruta's result in 1994 and G. H. Kim's results in 2011.

1. Introduction

The stability problem of the functional equation concerned the group homomorphisms was arisen by Ulam [13] during a conference in the university of Wisconsin in 1940. Next year, the problem was affirmatively answered in the case of additive mapping for Banach spaces by Hyers [7], which is called the Hyers-Ulam stability. The result of Hyers was very significantly generalized by Bourgin [3], which is covered with functional variables in C^*-algebras. Unfortunately, since a large portion of the proof have been omitted, his paper was too difficult in researchers. Subsequently, Hyers' result was detailed by Aoki [1] for additive mappings and by Rassias [11] for linear mappings by considering an unbounded Cauchy difference. The paper by Th.M. Rassias has provided a lot of influences in the development of what we now call the generalized Hyers-Ulam stability or Hyers- Ulam-Rassias stability of functional equations. J.M. Rassias [12] considered the Cauchy difference

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controlled by a product of different powers of norm. The above results have been generalized by Forti [4] and Găvruta [5] who permitted the Cauchy difference to become arbitrary unbounded. Páles, Volkmann and Luce [10] also improved pre-results.

In 1979, Baker, Lawrence, and Zorzitto [2] investigated the superstability, which states that if f is a function from a Abelian group to $\mathbb{R}$ satisfying

$$|f(x+y) - f(x)f(y)| \leq \varepsilon$$

for some fixed $\varepsilon > 0$, then either f is bounded or f satisfies the exponential functional equation

$$\text{(E)} \qquad f(x+y) = f(x)f(y).$$

Găvruta [6] proved the superstability of the Lobacevski equation

$$\text{(L)} \qquad f\left(\frac{x+y}{2}\right)^2 = f(x)f(y)$$

under the condition bounded by a constant.

Kim ([8], [9]) improved Găvruta's result under the condition bounded by an unknown function.

Every solution of the functional equation (L) can be represented as an exponential function $f(x) := e^x$ as follows:

$$f\left(\frac{x+y}{2}\right)^2 = \left(e^{\frac{x+y}{2}}\right)^2 = e^x e^y = f(x)f(y).$$

Kim [9] was investigated the superstability of the Pexider type Lobacevski equation

$$\text{(PL)} \qquad f\left(\frac{x+y}{2}\right)^2 = g(x)h(y),$$

which also can be represented as follows :

$$f\left(\frac{x+y}{2}\right)^2 = \left(\alpha\beta e^{\frac{x+y}{2}}\right)^2 = (\alpha^2 e^x)(\beta^2 e^y) = g(x)h(y).$$

Due to the above two functional equations (L) and (PL) and its examples, we can also consider the following exponential type functions:

$$f(x) = a^{\frac{x}{n}}, \quad g(x) = b^x = \left(a^{\frac{m}{n}}\right)^x, \quad h(x) = c^{mx} = \left(a^{\frac{1}{n}}\right)^{mx}$$

for $x, a, b, c \in \mathbb{R}$, and $m, n \in \mathbb{N}$. These functional equations arise the generalized Pexider type exponential equation

$$f(x+y)^m = \left(\sqrt[n]{a^{x+y}}\right)^m = \left(\sqrt[n]{a^x}\right)^m \left(\sqrt[n]{a^y}\right)^m = \left(a^{\frac{m}{n}}\right)^x \left(a^{\frac{1}{n}}\right)^{my} = g(x)h(y),$$

which yields us the target functional equation. In here, by putting $n = m$, two exponential functions g and h imply f, then the above equation implies (E).

The aim of this paper is to prove the superstability of the following generalized Pexider exponential equation

$$\text{(PE)} \qquad f(x+y)^m = g(x)h(y),$$

in Abelian group, where m is a positive integer.

Furthermore, Also we will extend the obtained results to the Banach algebra. The obtained results are generalizations of P. Găvruta's result [6] in 1994 and G. H. Kim's results ([8], [9]) in 2011.

In this paper, let $(G, +)$ be an Abelian group, $\mathbb{C}$ the field of complex numbers, $\mathbb{R}$ the field of real numbers, $\mathbb{R}_+$ the set of positive reals, ε a nonnegative real constant, and m a positive integer. Let $a : G \to \mathbb{R}_+$ be a function such that $a(x) = a^x \quad (0 < a \in R)$. We assume that $f, g, h : G \to \mathbb{C}$ are nonzero and nonconstant functions, and that $\varphi : G$, (or $G \times G) \to \mathbb{R}_+ \cup \{0\}$ be a function.

2. Stability of the generalized Pexider exponential equation (PE)

We will investigate the solution and the superstability of the generalized Pexider type exponential equation (PE).

THEOREM 1. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$\text{(2.1)} \qquad |f(x+y)^m - g(x)h(y)| \le \varepsilon$$

for all $x, y \in G$ and m is a positive integer.

Then, either there exist $C_1, C_2, C_3 > 0$ such that

$$\text{(2.2)} \qquad |g(x)| \le C_1, \quad |h(x)| \le C_2, \quad |f(x)| \le C_3$$

for all $x \in G$, or else each function g and h is represented by scalar times of an exponential function as follows:

$$\text{(2.3)} \qquad g(x) = g(0)a(x), \quad h(x) = h(0)a(x),$$

where $a(x)$ is an exponential.

In particular, if $g(0) = 1 = h(0)$*, then* g *and* h *satisfy* (E) *as same exponential function.*

Proof. Replacing x by y in (2.1), and then subtracting them and using triangle inequality we have

$$|g(x)h(y) - g(y)h(x)| \le 2\varepsilon \quad \forall x, y \in G. \tag{2.4}$$

It follows from the inequality (2.4) that there exist constants $c_1, c_2, d_1, d_2 \ge 0$ such that

$$|g(x)| \le c_1|h(x)| + d_1 \tag{2.5}$$

$$|h(x)| \le c_2|g(x)| + d_2 \tag{2.6}$$

for all $x \in G$. It follows from (2.5) and (2.6) that g is bounded if and only if h is bounded. If either g or h is bounded, then we obtain (2.2) from (2.1).

Now if $h(x)$ is unbounded, then we can choose $(y_n) \in G$ so that $|h(y_n)| \to \infty$ as $n \to \infty$. Letting $y = y_n$ in (2.1), dividing by $|h(y_n)|$, and letting $n \to \infty$, we have

$$g(x) = \lim_{n\to\infty} \frac{f\left(x + y_n\right)^m}{h(y_n)}, \quad \forall x \in G. \tag{2.7}$$

It follows from (2.1) and (2.7) that

$$\begin{aligned} g(x+y)g(z) &= \lim_{n\to\infty} \frac{f\left(x+y+y_n\right)^m g(z)}{h(y_n)} = \lim_{n\to\infty} \frac{g(x)h(y+y_n)g(z) + R_1}{h(y_n)} \\ &= \lim_{n\to\infty} \frac{g(x)f\left(y+z+y_n\right)^m + R_1 + R_2}{h(y_n)} = g(x)g(y+z) + \lim_{n\to\infty} \frac{R_1 + R_2}{h(y_n)}, \end{aligned}$$

where $|R_1| \le \varepsilon|g(z)|, |R_2| \le \varepsilon|g(x)|$, which implies

$$g(x+y)g(z) = g(x)g(y+z) \tag{2.8}$$

for all $x, y, z \in G$.

Letting $z = 0$ in (2.8), we get

$$g(x+y)g(0) = g(x)g(y) \tag{2.9}$$

for all $x, y \in G$, which implies that

$$g(x) = g(0)a_1(x), \tag{2.10}$$

where $g(0) \ne 0$ (since $g(x)$ is a nonzero and nonconstant function) and a_1 is an exponential.

Exchanging the roles of g and h, by the same proceeding, we have

$$h(x) = h(0)a_2(x), \tag{2.11}$$

where $h(0) \neq 0$ and a_2 is an exponential.

Putting (2.10) and (2.11) in (2.4), it implies

$$|a_1(x)a_2(y) - a_1(y)a_2(x)| \leq \frac{2\varepsilon}{|g(0)h(0)|} = M \quad \forall x, y \in G. \tag{2.12}$$

Let $x = 0$ in (2.12). Since a_1, a_2 are exponentials, this implies that $|a_1(y) - a_2(y)| \leq M$ for all $y \in G$. Hence, from this and (2.12), we have

$$\begin{aligned} a_1(y)|a_1(x) - a_2(x)| &= |a_1(x)[a_1(y) - a_2(y)] + a_1(x)a_2(y) - a_1(y)a_2(x)| \\ &\leq |a_1(x)|M + M, \end{aligned}$$

which is

$$|a_1(x) - a_2(x)| \leq \frac{a_1(x)M + M}{|a_1(y)|}, \tag{2.13}$$

for all $x, y \in G$.

Since g is unbounded from (2.2), we can choose $(y_n) \in G$ so that $g(y_n) = g(0)a_1(y_n) \to \infty$ as $n \to \infty$. Letting $y = y_n$ in (2.13), we get that $a_1(x) = a_2(x)$. Let it be denoted by $a(x)$. Then (2.10) and (2.11) state nothing but (2.3).

Finally, if $g(0) = 1$ and $h(0) = 1$ in (2.3), then it is immediate that g and h in (2.3) are same exponential function. □

Corollary 1. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x + y)^m - g(x)g(y)| \leq \varepsilon$$

for all $x, y \in G$.

Then either g is bounded or g is represented by

$$g(x) = g(0)a(x),$$

where $a(x)$ is an exponential. In case $g(0) = 1$, g satisfies (E).

Corollary 2. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x + y)^m - g(x)f(y)| \leq \varepsilon$$

for all $x, y \in G$.

Then either g(or f) is bounded or g and f are represented respectively by

$$g(x) = g(0)a(x) \quad \textit{and} \quad f(x) = f(0)a(x),$$

where $a(x)$ is an exponential. In cases $g(0) = 1 = f(0)$, g and f satisfy (E).

Corollary 3. *Suppose that $f : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y)^m - f(x)f(y)| \leq \varepsilon$$

for all $x, y \in G$.

Then either f is bounded or f is represented by

$$f(x) = f(0)a(x),$$

where $a(x)$ is an exponential. In case $f(0) = 1$, f satisfies (E).

In Corollary 3, it is founded in papers ([6], [8]) that f satisfies (E).

Theorem 2. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y)^m - g(x)h(y)| \leq \varphi(x) \tag{2.14}$$

for all $x, y \in G$.

Then either h is bounded or function g is represented by scalar times of an exponential function as follows:

$$g(x) = g(0)a(x),$$

where $a(x)$ is an exponential. In case $g(0) = 1$, g satisfies (E).

Proof. Suppose that $h(x)$ is unbounded. Then we can choose $(y_n) \in G$ such that $|h(y_n)| \to \infty$ as $n \to \infty$. Letting $y = y_n$ in (2.14), dividing by $|h(y_n)|$, and letting $n \to \infty$, we have

$$g(x) = \lim_{n\to\infty} \frac{f(x+y_n)}{h(y_n)} \quad \forall x \in G. \tag{2.15}$$

Using (2.14) and (2.15), let us runs the same as the proof of Theorem 1, then we arrive the our required results via (2.8), (2.9), and (2.10) in Theorem 1.

Namely, g is represented by

$$g(x) = g(0)a(x),$$

where $a(x)$ is an exponential. In case $g(0) = 1$, g satisfies (E). □

Theorem 3. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y)^m - g(x)h(y)| \leq \varphi(y) \tag{2.16}$$

for all $x, y \in G$.

Then either g is bounded or function h is represented by

$$h(x) = h(0)a(x),$$

where $a(x)$ is an exponential. In case $h(0) = 1$, h satisfies (E).

Proof. The proof runs along a slight change in the step-by-step procedure in Theorem 1 as Theorem 2. $\square$

The following result follows immediately from the above Theorem 2 and Theorem 3 .

THEOREM 4. *Let $\varphi : G \times G \to \mathbb{R}_+ \cup \{0\}$ be a function. Assume that $\varphi(x, y)$ is bounded as a function of y for each $x \in G$ or as a function of x for each $y \in G$, and that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$\|f\,(x + y)^m - g(x)h(y)\| \leq \varphi(x, y) \tag{2.17}$$

for all $x, y \in G$, and $g(0) = 1 = h(0)$.

Then, (i) either h is bounded or g is represented by $g(x) = g(0)a(x)$, where $a(x)$ is an exponential. In case $g(0) = 1$, g satisfies (E).

(ii) either g is bounded or function h is represented by $h(x) = h(0)a(x)$, where $a(x)$ is an exponential. In case $= 1$, h satisfies (E).

Proof. By assumption that $\varphi(x, y)$ is bounded, we can choose (x_n) and $(y_n) \in G$ such that $|g(x_n)|$ and $|h(y_n)| \to \infty$ as $n \to \infty$. Its imply from (2.12)

$$\lim_{n\to\infty} \frac{\varphi(x, y_n)}{h(y_n)} = 0 \quad \text{or} \quad \lim_{n\to\infty} \frac{\varphi(x_n, y)}{h(x_n)} = 0 \quad \forall x, y \in G.$$

Hence, the results hold from Theorem 2 and Theorem 3. $\square$

REMARK 1. *(i) As Corollaries 1 $\sim$ 3 of Theorem 1, by replacing g or h by f, respectively, in Theorem 2, Theorem 3 and Theorem 5, we can obtain a number of corollaries for the following functional equations:*

$$\begin{aligned} f(x + y)^m &= g(x)g(y) \\ f(x + y)^m &= f(x)g(y) \\ f(x + y)^m &= f(x)f(y), \end{aligned} \tag{2.18}$$

in which the results of the equations replaced by $f(\frac{x+y}{2})^2$ to $f(x + y)^m$ in (2.18) *are found in papers* ([8], [9]).

(ii) For the results obtained from each equation of the above (i), by applying $\varphi(y) = \varphi(x) = \varepsilon$, we can obtain the same number of corollaries.

3. Extension to Banach Algebra

All obtained results can be extended to the stability on the Banach algebras. We will illustrate only for the case of Theorem 1 among them.

THEOREM 5. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra with unit I. Assume that $f, g, h : G \to E$ satisfy the inequality*

$$\|f(x+y)^m - g(x)h(y)\| \leq \varepsilon \tag{3.1}$$

for all $x, y \in G$ and m is a positive integer.

Then, for an arbitrary linear multiplicative functional $x^* \in E^*$, either there exist $C_1, C_2, C_3 > 0$ such that

$$|(x^* \circ g)(x)| \leq C_1, \quad |(x^* \circ h)(x)| \leq C_2, \quad |(x^* \circ f)(x)| \leq C_3 \tag{3.2}$$

for all $x \in G$, or else each function g and h is represented by scalar times of an exponential function as follows:

$$g(x) = g(0)a(x), \quad h(x) = h(0)a(x), \tag{3.3}$$

where $a(x)$ is an exponential.

In particular, if $g(0) = I = h(0)$, then g and h satisfy (E), *respectively, as same exponential function.*

Proof. Assume that (3.1) holds, and fix arbitrarily a linear multiplicative functional $x^* \in E^*$. As well known we may assume without loss of generality that $\|x^*\| = 1$ whence, for every $x, y \in G$, we have

$$\begin{aligned}
\varepsilon &\geq \|f(x+y)^m - g(x)h(y)\| \\
&= \sup_{\|y^*\|=1} |y^*\left(f(x+y)^m - g(x)h(y)\right)| \\
&\geq \left| x^*\left(f\left(\frac{x+y}{2}\right)\right) - x^*\left(g(x)\right) x^*\left(h(y)\right)\right|,
\end{aligned}$$

which states that the superpositions $x^* \circ f$, $x^* \circ g$, and $x^* \circ h$ satisfy the inequality (2.1) of Theorem 1. Due to same processing as from (2.4) to (2.6), for any fixed arbitrary linear multiplicative functional $x^* \in E^*$, indeed, we have

$$|(x^* \circ g)(x)(x^* \circ h)(y) - (x^* \circ g)(y)(x^* \circ h)(x)| \leq 2\varepsilon \quad \forall x, y \in G. \tag{3.4}$$

It follows from the inequality (3.4) that there exist constants $c_1, c_2, d_1, d_2 \geq 0$ such that

$$|(x^* \circ g)(x)| \leq c_1 |(x^* \circ h)(x)| + d_1 \tag{3.5}$$
$$|(x^* \circ h)(x)| \leq c_2 |(x^* \circ g)(x)| + d_2 \tag{3.6}$$

for all $x \in G$. Since x^* is an arbitrarily linear multiplicative functional, it follows from (3.5) and (3.6) that g is bounded if and only if h is bounded. Assume that one of g or h is bounded. From (3.1) we arrive at (3.2).

By the assumption (3.2), an appeal to Theorem 1 shows that

$$(x^* \circ g)(x) = (x^* \circ g(0)a_1)(x), \tag{3.7}$$
$$(x^* \circ h)(x) = (x^* \circ h(0)a_2)(x), \tag{3.8}$$

where $a_1, a_2 : G \to \mathbb{R}$ are exponentials. In other words, bearing the linear multiplicativity of x^* in mind, for all $x \in G$, each difference derived from (3.7) and (3.8)

$$D(3.7)(x) := g(x) - (g(0)a_1)(x),$$
$$D(3.8)(x) := h(x) - (h(0)a_2)(x),$$

falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$D(3.7)(x), D(3.8)(x) \in \bigcap \{\ker x^* : x^* \text{ is a multiplicative member of } E^*\}$$

for all $x \in G$. Since the algebra E has been assumed to be semisimple, the last term of the previous formula coincides with the singleton $\{0\}$, i.e.

$$g(x) - g(0)a_1(x) = 0, \quad h(x) - h(0)a_2(x) = 0 \quad x \in G. \tag{3.9}$$

Putting (3.9) in (3.4), following the same proceeding as after (2.11) in Theorem 1, then we arrive that $a_1(x) = a_2(x)$. Indeed, we have

$$|(x^* \circ g(0)a_1)(x)(x^* \circ h(0)a_2)(y) - (x^* \circ g(0)a_1)(y)(x^* \circ h(0)a_2)(x)| \leq 2\varepsilon \tag{3.10}$$

for all $x, y \in G$. This implies that

$$|(x^* \circ a_1)(x)(x^* \circ a_2)(y) - (x^* \circ a_1)(y)(x^* \circ a_2)(x)| \leq \frac{2\varepsilon}{|g(0)h(0)|} = M \quad \forall x, y \in G. \tag{3.11}$$

Letting $x=0$ in (3.11), it implies $|(x^*\circ a_2)(y)-(x^*\circ a_1)(y)| \le \frac{M}{|x^*(1)|} = M'$ for all $y \in G$. Thus, from this and (3.11), we have

$$\begin{aligned}
|(x^* \circ a_1)(y)||(x^* \circ a_1)(x) &- (x^* \circ a_2)(x)| \\
&= |(x^* \circ a_1)(x)[(x^* \circ a_1)(y) - (x^* \circ a_2)(y)] \\
&\quad + (x^* \circ a_1)(x)(x^* \circ a_2)(y) - (x^* \circ a_1)(y)(x^* \circ a_2)(x)| \\
&\le |(x^* \circ a_1)(x)|M' + M,
\end{aligned}$$

which is

$$|(x^* \circ a_1)(x) - (x^* \circ a_2)(x)| \le \frac{|(x^* \circ a_1)(x)|M' + M}{|(x^* \circ a_1)(y)|}, \tag{3.12}$$

for all $x, y \in G$.

Since $x^* \circ g$ is unbounded from (3.2), we can choose $(y_n) \in G$ so that $|(x^* \circ g)(y_n)| = |g(0)(x^* \circ a_1)(y_n)| \to \infty$ as $n \to \infty$. Letting $y = y_n$ in (3.12), which arrive that

$$(x^* \circ a_1)(x) = (x^* \circ a_2)(x). \tag{3.13}$$

Using the same logic as before, i.e., bearing the linear multiplicativity of x^* in mind, the difference derived from (3.13), $D(3.13)(x) := a_1(x) - a_2(x)$ falls into the kernel of x^*. Then, the semisimplicity of E implies that $a_1(x) = a_2(x) = a(x)$, which arrive the claimed (3.3).

In cases $g(0) = I = h(0)$, since $a(x)$ is exponential, it is immediate from (3.3) that each function g and h satisfies (E). $\square$

Remark 2. *All results of Section 2 containing Remark 1 can be extend to the Banach algebra as Theorem 5.*

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