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STRONG CONVERGENCE OF AN ITERATIVE ALGORITHM FOR SYSTEMS OF VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS IN q-UNIFORMLY SMOOTH BANACH SPACES

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Abstract. In this paper, we introduce a new iterative scheme to investigate the problem of finding a common element of nonexpansive mappings and the set of solutions of generalized variational inequalities for a k-strict pseudo-contraction by relaxed extra-gradient methods. Strong convergence theorems are established in q-uniformly smooth Banach spaces.

1. Introduction

Throughout this paper, we assume that E is a real Banach space and E^* the dual space of E. Let C be a subset of E and T be a self mapping of C. Denote by $Fix(T)$ the set of fixed points of T, that is, $Fix(T) = \{x \in C : Tx = x\}$. When $\{x_n\}$ is a sequence in E, $x_n \to x (x_n \rightharpoonup x)$ will denote strong(weak) convergence of the sequence $\{x_n\}$ to x.

Let $q > 1$ be a real number. The duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q, \quad \|f\| = \|x\|^{q-1}\}, \quad \forall x \in E.$$

In particular, $J = J_2$ is called the normalized duality mapping and $J_q(x) = \|x\|^{q-2}J_2(x)$ for $x \neq 0$. If E is a Hilbert space, then $J = I$, where I is the identity mapping.

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Recall that a mapping T is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|$$

for all $x, y \in C$. A mapping T is called a pseudo-contraction if there exists some $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Tx - Ty, j_q(x-y)\rangle \le \|x-y\|^q$$

for all $x, y \in C$. T is said to be a k-strict pseudo-contraction in the terminology of Browder and Petryshyn [1] if there exists a constant $k > 0$ such that

$$\langle Tx - Ty, j_q(x-y)\rangle \le \|x-y\|^q - k\|(I-T)x - (I-T)y\|^q \tag{1.1}$$

for every $x, y \in C$ and for some $j_q(x-y) \in J_q(x-y)$.

Remark 1.1. *From (1.1) we can prove that if T is k-strict pseudo-contraction, then T is Lipschitz continuous with the Lipschitz constant $L = \frac{1+k}{k}$.A Banach space E is called uniformly convex if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \le 1$ and $\|x-y\| \ge \varepsilon$, $\|x+y\| \le 2(1-\delta)$ holds. It is known that a uniformly convex Banach space is reflexive and strictly convex. Let $S(E) = \{x \in E : \|x\| = 1\}$. E is said to be Gâteaux differentiable if the limit*

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S(E)$. In this case, E is called smooth. Let $\rho_E : [0,\infty) \to [0,\infty)$ be the modulus of smoothness of E defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x \in S(E), \quad \|y\| \ge t\right\}.$$

A Banach space E is said to be uniformly smooth if $\lim_{t\to 0} \frac{\rho(t)}{t} = 0$. Let $q > 1$. A Banach space E is said to be q-uniformly smooth if there exists a fixed constant $c > 0$ such that $\rho(t) \le ct^q$. Recall that construction of fixed points for nonexpansive mappings and λ-strict pseudo-contractions via the Mann's iterative algorithm has been extensively investigated by many authors (see [3,6,7,8]). The Mann iteration is extensively and successfully used to approximate fixed points of nonexpansive mappings.

However, iterative methods for strict pseudo-constractions are far less developed than for nonexpansive mappings. On the other hand, strict pseudo-contractions have more powerful applications than nonexpansive mappings do in solving inverse problems (see [11]). Therefore it is interesting to develop the theory of iterative methods for strict pseudo-contractions.In 1967, Halpen [4] introduced the following explicit iteration scheme for a nonexpansive mapping T which was referred to Halpern iteration: for $u, x_0 \in K$, $\alpha_n \in [0,1]$,

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n.$$

Recently, Zhou [17] obtained strong convergence theorem for the following iterative sequence in a 2-uniformly smooth Banach space E: for $u, x_0 \in E$ and a λ-strict pseudo-contraction T,

$$x_{n+1} = \beta_n u + \gamma_n x_n + (1-\beta_n-\gamma_n)[\alpha_n Tx_n + (1-\alpha_n)x_n],$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in (0,1) satisfy

(i) $a \le \alpha_n \le \frac{\lambda}{K^2}$ for some $a > 0$ and for all $n \ge 0$;
(ii) $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$;
(iii) $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$;
(iv) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$.

Very recently, Zhang and Shu [16] extended Zhou's results to q-uniformly smooth Banach space.

Motivated and inspired by the above works, in this paper, we consider the problem of convergence of an iterative algorithm for a system of generalized variational inequalities involving strictly pseudo-contractions and a nonexpansive mapping. We prove the strong convergence of purposed iterative scheme in uniformly convex and q-uniformly smooth Banach spaces.

2. Preliminaries

Let C be a nonempty closed convex subset of a Banach space E and E^* the dual space of E.

Definition 2.1. *Let E be a real Banach space, C a nonempty closed and convex subset of E and K a nonempty subset of C. Let Q be a mapping of C into K. Then Q is said to be:*

(1) *sunny if for each $x \in C$ and $t \in [0,1]$ we have*

$$Q(tx + (1-t)x) = Qx;$$

(2) *a retraction of C onto K if*

$$Qx = x, \quad \forall x \in K;$$

(3) *a sunny nonexpansive retraction if Q is sunny nonexpansive and a retraction onto K.*

The following result describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

PROPOSITION 2.1. *([9]) Let E be a smooth Banach space and let K be a nonempty subset of E. Let $Q : E \to K$ be a retraction and let J be the normalized duality mapping on E. Then the following are equivalent:*

(a) *Q is sunny and nonexpansive;*
(b) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle, \quad \forall x, y \in E;$
(c) $\langle x - Qx, J(y - Qx)\rangle \leq 0, \quad \forall x \in E, y \in K.$

PROPOSITION 2.2. *([5]) Let K be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and S a nonexpansive mapping of C into itself with $Fix(S) \neq \phi$. Then the set $Fix(S)$ is a sunny nonexpansive retract of C. Reich [10], in 1980, proved the following behavior for nonexpansive mappings.*

PROPOSITION 2.3. *Let E be a real uniformly smooth Banach space and C a nonempty closed convex subset of E. Let $T : C \to C$ be a nonexpansive mapping with a fixed point and let $z \in C$. For each $t \in (0, 1)$, let z_t be the unique solution of the equation $x = tz + (1-t)Tx$. Then $\{z_t\}$ converges to a fixed point of T as $t \to 0$ and*

$$Qz = s - \lim_{t \to 0} z_t$$

defines the unique sunny nonexpansive retraction from C onto $Fix(T)$, that is, Q satisfies the property:

$$\langle u - Qu, J(y - Qu)\rangle \leq 0, \quad \forall u \in C, y \in Fix(T).$$

Motivated by Wang and Chen [13], we consider the following general system of variational inequalities in a uniformly smooth Banach space E. Let $S : C \to C$ be a k-strict pseudo-contraction. Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda(I - S)y^* + x^* - y^*, J(x - x^*)\rangle \geq 0, \quad \forall x \in C, \\ \langle \mu(I - S)x^* + y^* - x^*, J(x - x^*)\rangle \geq 0, \quad \forall x \in C. \end{cases} \tag{2.1}$$

In order to prove our main results, we need the following lemmas.

Lemma 2.1. *([14]) Let E be a real q-uniformly smooth Banach space. Then there exists a constant $c_q > 0$ such that*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q$$

for all $x, y \in E$.

Lemma 2.2. *([12]) Let $\{z_n\}$ and $\{w_n\}$ be two bounded sequences in a Banach space E such that*

$$z_{n+1} = (1-\gamma_n)z_n + \gamma_n w_n, \quad n \geq 1,$$

where $\{\gamma_n\}$ satisfies condition: $0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}\gamma_n < 1$. If $\limsup_{n\to\infty}(\|w_{n+1}-w_n\| - \|z_{n+1}-z_n\|) \leq 0$, then $w_n - z_n \to 0$ as $n \to \infty$.

Lemma 2.3. *([2]) Let C be a nonempty closed convex subset of a real Banach space E. Let T_1 and T_2 be nonexpansive mappings from C into itself with a common fixed point. Define a mapping $T : C \to C$ by*

$$Tx = \delta T_1 x + (1-\delta)T_2 x, \quad \forall x \in C,$$

where δ is a constant in $(0,1)$. Then T is nonexpansive and $Fix(T) = Fix(T_1) \cap Fix(T_2)$.

Lemma 2.4. *([15]) Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that
(a) $\sum_{n=1}^{\infty}\gamma_n = \infty$;
(b) $\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.*Then* $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 2.5. *For given $(x^*, y^*) \in C \times C$, where $y^* = Q_C(x^* - \mu(I - S)x^*)$, (x^*, y^*) is a solution of problem (2.1) if and only if x^* is a fixed point of the mapping $D : C \to C$ defined by*

$$D(x) = Q_C[Q_C(x-\mu(I-S)x) - \lambda(I-S)Q_C(x-\mu(I-S)x)], \quad \forall x \in C,$$

where $\lambda, \mu > 0$ are constants and Q_C is a sunny nonexpansive retraction from E onto C.

Proof. Observe that

$$\begin{cases} \langle \lambda(I-S)y^* + x^* - y^*, J(x-x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \mu(I-S)x^* + y^* - x^*, J(x-x^*)\rangle \geq 0, & \forall x \in C. \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} x^* = Q_C(y^* - \lambda(I-S)y^*), \\ y^* = Q_C(x^* - \mu(I-S)x^*). \end{cases}$$

$\Leftrightarrow$

$$x^* = Q_C[Q_C(x^* - \mu(I-S)x^*) - \lambda(I-S)Q_C(x^* - \mu(I-S)x^*)].$$

□

3. Main results

Now, we consider the following main result of this paper.

THEOREM 3.1. *Let C be a nonempty closed convex subset of a uniformly convex and q-uniformly smooth Banach space E and Q_C a sunny nonexpansive retraction from E onto C. Let $S : C \to C$ be a k-strict pseudo-contraction such that $Fix(S) \neq \phi$ and $T : C \to C$ a nonexpansive mapping with $Fix(T) \neq \phi$. Assume that $F = Fix(T) \cap Fix(D) \neq \phi$, where D is defined as Lemma 2.5. Let a sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_1 = u \in C, \\ y_n = Q_C(x_n - \mu(I-S)x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\delta T x_n + (1-\delta)Q_C(y_n - \lambda(I-S)y_n)], n \geq 1, \end{cases} \tag{3.1}$$

where $\delta \in (0,1)$, $\lambda, \mu \in (0, \min\{1, (\frac{qk}{c_q})^{\frac{1}{q-1}}\}]$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ such that

(H1) $\alpha_n + \beta_n + \gamma_n = 1, \quad \forall n \geq 1$,
(H2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(H3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ defined by (3.1) converges strongly to $\overline{x} = Q_F u$ and $(\overline{x}, \overline{y})$, where $\overline{y} = Q_C(\overline{x} - \mu(I-S)\overline{x})$ and Q_F is the unique sunny nonexpansive retraction from C onto F, is a solution of the problem (2.1).

Proof. We divide our proofs into several steps as follows.
(Step 1.) First, we show that F is closed and convex.
It is well known that $Fix(T)$ is closed and convex. Next, we show that $Fix(D)$ is closed and convex. For any $\lambda, \mu \in (0, M]$, $M = \min\{1, (\frac{qk}{c_q})^{\frac{1}{q-1}}\}$,

we have that the mappings $I-\mu(I-S)$ and $I-\lambda(I-S)$ are nonexpansive mappings. Indeed, from Lemma 2.1, we have for all $x, y \in C$,

$$\begin{aligned}
&\|(I-\lambda(I-S))x-(I-\lambda(I-S))y\|^q \\
&= \|x-y-\lambda(x-y-(Sx-Sy))\|^q \\
&\leq \|x-y\|^q - q\lambda\langle x-y-(Sx-Sy), J_q(x-y)\rangle \\
&\quad + c_q\lambda^q\|x-y-(Sx-Sy)\|^q \\
&\leq \|x-y\|^q - q\lambda\|x-y\|^q + q\lambda\langle Sx-Sy, J_q(x-y)\rangle \\
&\quad + c_q\lambda^q\|x-y-(Sx-Sy)\|^q \\
&\leq \|x-y\|^q - q\lambda\|x-y\|^q + q\lambda[\|x-y\|^q - k\|(I-S)x-(I-S)y\|^q] \\
&\quad + c_q\lambda^q\|x-y-(Sx-Sy)\|^q \\
&= \|x-y\|^q - \lambda(qk - c_q\lambda^{q-1})\|x-y-(Sx-Sy)\|^q \\
&\leq \|x-y\|^q,
\end{aligned}$$

which shows that $I-\lambda(I-S)$ is a nonexpansive mapping. So is $I-\mu(I-S)$. By Lemma 2.5, we can see that

$$\begin{aligned}
D &= Q_C[Q_C(I-\mu(I-S)) - \lambda(I-S)Q_C(I-\mu(I-S))] \\
&= Q_C(I-\lambda(I-S))Q_C(I-\mu(I-S))
\end{aligned}$$

is nonexpansive. Thus, $F = Fix(T) \cap Fix(D)$ is closed and convex.
(Step 2.) The sequences $\{x_n\}$ is bounded.
For $x^* \in F = Fix(T) \cap Fix(D)$, we have that

$$x^* = Q_C[Q_C(x^*-\mu(I-S)x^*) - \lambda(I-S)Q_C(x^*-\mu(I-S)x^*)].$$

Set $y^* = Q_C(x^*-\mu(I-S)x^*)$. We obtain $x^* = Q_C(y^*-\lambda(I-S)y^*)$. Since $y_n = Q_C(x_n-\mu(I-S)x_n)$, we have

$$\begin{aligned}
\|y_n-y^*\| &= \|Q_C(x_n-\mu(I-S)x_n) - Q_C(x^*-\lambda(I-S)x^*\| \\
&\leq \|x_n-x^*\|.
\end{aligned} \tag{3.2}$$

For the sake of simplicity, let $u_n = \delta Tx_n + (1-\delta)Q_C(y_n-\lambda(I-S)y_n)$ for each $n \geq 1$. By (3.2), we have

$$\begin{aligned}
\|u_n-x^*\| &= \|\delta Tx_n + (1-\delta)Q_C(y_n-\lambda(I-S)y_n) - x^*\| \\
&\leq \delta\|Tx_n-x^*\| \\
&\quad + (1-\delta)\|Q_C(y_n-\lambda(I-S)y_n - Q_C(y^*-\lambda(I-S)y^*)\| \\
&\leq \delta\|x_n-x^*\| + (1-\delta)\|y_n-y^*\| \\
&\leq \delta\|x_n-x^*\| + (1-\delta)\|x_n-x^*\| \\
&= \|x_n-x^*\|.
\end{aligned} \tag{3.3}$$

Then we have

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|\alpha_n u+\beta_n x_n+\gamma_n u_n - x^*\|\\ &\le \alpha_n\|u-x^*\|+\beta_n\|x_n-x^*\|+\gamma_n\|u_n-x^*\|\\ &\le \alpha_n\|u-x^*\|+(1-\alpha_n)\|x_n-x^*\|\\ &\le \max\{\|u-x^*\|,\|x_n-x^*\|\}.\end{aligned}$$

By induction, we get

$$\|x_n-x^*\|\le \max\{\|u-x^*\|,\|x_1-x^*\|\}.$$

Thus, $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{u_n\}$.

(Step 3.) $x_{n+1}-x_n\to 0$ as $n\to\infty$.We now define $w_n=\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$. Set $M_1=\|u\|+\sup\{\|u_n\|\}$. By using (3.1), we get

$$\begin{aligned}\|w_{n+1}-w_n\| &= \left\|\frac{\alpha_{n+1}u+\gamma_{n+1}u_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n u+\gamma_n u_n}{1-\beta_n}\right\|\\ &= \left\|\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)u+\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)u_n\right.\\ &\quad\left.-\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)u_n+\frac{\gamma_{n+1}}{1-\beta_{n+1}}u_{n+1}-\frac{\gamma_n}{1-\beta_n}u_n\right\|\\ &\le \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|(\|u\|+\|u_n\|)+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|u_{n+1}-u_n\|\\ &\le M_1\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|+\|u_{n+1}-u_n\|\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}\|u_{n+1}-u_n\| &= \|\delta Tx_{n+1}+(1-\delta)Q_C(y_{n+1}-\lambda(I-S)y_{n+1})\\ &\quad-(\delta Tx_n+(1-\delta)Q_C(y_n-\lambda(I-S)y_n))\|\\ &\le \delta\|Tx_{n+1}-Tx_n\|\\ &\quad+(1-\delta)\|Q_C(y_{n+1}-\lambda(I-S)y_{n+1})-Q_C(y_n-\lambda(I-S)y_n)\|\\ &\le \delta\|x_{n+1}-x_n\|+(1-\delta)\|y_{n+1}-y_n\|\\ &\le \delta\|x_{n+1}-x_n\|+(1-\delta)\|x_{n+1}-x_n\|\\ &= \|x_{n+1}-x_n\|.\end{aligned}\tag{3.5}$$

Substituting (3.5) into (3.4) yields

$$\|w_{n+1}-w_n\|\le M_1\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|+\|x_{n+1}-x_n\|.$$

By the assumptions on $\{\alpha_n\}$ and $\{\beta_n\}$, we get

$$\limsup_{n\to\infty}(\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|)\le 0.$$

By using Lemma 2.2, we conclude that $w_n - x_n \to 0$ as $n \to \infty$. Noting that $x_{n+1} - x_n = (1-\beta_n)(w_n - x_n)$, we get $x_{n+1} - x_n \to 0$ as $n \to \infty$.

(Step 4.) There exists a continuous path $\{x_t\}$ such that $x_t \to \overline{x}$ as $t \to 0$, where $\overline{x} = Q_F u$ and $Q_F : C \to F$ is the unique sunny nonexpansive retraction from C onto F. Define a mapping $T_\delta : C \to C$ by

$$T_\delta x = \delta T x + (1-\delta) Q_C(I - \lambda(I-S)) Q_C(I - \mu(I-S))x, \quad \forall x \in C.$$

Then T_δ is nonexpansive and

$$\begin{aligned} Fix(T_\delta) &= Fix(T) \cap Fix(Q_c(I - \lambda(I-S))Q_C(I - \mu(I-S))) \\ &= Fix(T) \cap Fix(D) \\ &= F \end{aligned}$$

by Lemma 2.3. For $t \in (0,1)$ we define a contraction via

$$T_\delta^t x = tu + (1-t)T_\delta x, \quad \forall x \in C.$$

Then, the Banach contraction mapping principle ensures that there exists a unique path $x_t \in C$ such that

$$x_t = tu + (1-t)T_\delta x_t$$

for every $t \in (0,1)$. By Proposition 2.3, we know that $x_t \to \overline{x} \in Fix(T_\delta)$ as $t \to \infty$. Further, if we define $Q_{Fix(T_\delta)} u = \overline{x}$, then $Q_{Fix(T_\delta)} : C \to Fix(T_\delta)$ is a unique sunny nonexpansive retraction from C onto $Fix(T_\delta)$. Noting that $Fix(T_\delta) = F$, we see that $Q_F : C \to F$ is indeed the unique sunny nonexpansive retraction from C onto F.

(Step 5.) $\limsup_{n\to\infty} \langle u - \overline{x}, J(x_n - \overline{x})\rangle \le 0$, where $\overline{x} = Q_F u$.
We note that

$$\begin{aligned} \|x_n - T_\delta x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_\delta x_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n \|u - T_\delta x_n\| + \beta_n \|x_n - T_\delta x_n\|. \end{aligned}$$

This implies that

$$(1-\beta_n)\|x_n - T_\delta x_n\| \le \|x_n - x_{n+1}\| + \alpha_n \|u - T_\delta x_n\|.$$

It follows from conditions (H2), (H3) and Step 3 that $x_n - T_\delta x_n \to 0$ as $n \to \infty$. Since

$$\begin{aligned} x_t - x_n &= tu + (1-t)T_\delta x_t - x_n \\ &= (1-t)(T_\delta x_t - x_n) + t(u - x_n), \end{aligned}$$

then

$$\begin{aligned}\|x_t - x_n\|^2 &= (1-t)\langle T_\delta x_t - x_n, J(x_t - x_n)\rangle + t\langle u - x_n, J(x_t - x_n)\rangle \\ &= (1-t)[\langle T_\delta x_t - T_\delta x_n, J(x_t - x_n)\rangle + \langle T_\delta x_n - x_n, J(x_t - x_n)\rangle] \\ &\quad + t\langle u - x_t, J(x_t - x_n)\rangle + t\langle x_t - x_n, J(x_t - x_n)\rangle \\ &\le (1-t)(\|x_t - x_n\|^2 + \|T_\delta x_n - x_n\|\|x_t - x_n\|) \\ &\quad + t\langle u - x_t, J(x_t - x_n)\rangle + t\|x_t - x_n\|^2 \\ &= \|x_t - x_n\|^2 + \|T_\delta x_n - x_n\|\|x_t - x_n\| + t\langle u - x_t, J(x_t - x_n)\rangle.\end{aligned}$$

It turns out that

$$\langle x_t - u, J(x_t - x_n)\rangle \le \frac{1}{t}\|T_\delta x_n - x_n\|\|x_t - x_n\|, \quad \forall t \in (0,1).$$

By the above inequality, we have

$$\limsup_{n\to\infty}\langle x_t - u, J(x_t - x_n) \le 0.$$

Since J is strong to weak* uniformly continuous on bounded subset of E, we see that

$$\begin{aligned}&|\langle u - \overline{x}, J(x_n - \overline{x})\rangle - \langle x_t - u, J(x_t - x_n)\rangle| \\ &\le |\langle u - \overline{x}, J(x_n - \overline{x})\rangle - \langle u - \overline{x}, J(x_n - x_t)\rangle| \\ &\quad + |\langle u - \overline{x}, J(x_n - x_t)\rangle - \langle x_t - u, J(x_t - x_n)\rangle| \\ &= |\langle u - \overline{x}, J(x_n - \overline{x}) - J(x_n - x_t)\rangle| + |\langle x_t - \overline{x}, J(x_n - x_t)\rangle| \\ &\le \|u - \overline{x}\|\|J(x_n - \overline{x}) - J(x_n - x_t)\| + \|x_t - \overline{x}\|\|x_n - x_t\| \\ &\to 0 \quad \text{as} \quad t \to 0.\end{aligned}$$

For any $\varepsilon > 0$, there exists $\delta > 0$ such that for every $t \in (0, \delta)$

$$\langle u - \overline{x}, J(x_n - \overline{x})\rangle \le \langle x_t - u, J(x_t - x_n)\rangle + \varepsilon.$$

Therefore

$$\limsup_{n\to\infty}\langle u - \overline{x}, J(x_n - \overline{x})\rangle \le \limsup_{n\to\infty}\langle x_t - u, J(x_t - x_n)\rangle + \varepsilon.$$

This implies that

$$\limsup_{n\to\infty}\langle u - \overline{x}, J(x_n - \overline{x})\rangle \le 0.$$

(Step 6.) $x_n \to \overline{x} \in Q_F u$ as $n \to \infty$. By using (3.3) we have

$$\begin{aligned}\|x_{n+1} - \overline{x}\|^2 &= \langle \alpha_n u + \beta_n x_n + \gamma_n u_n - \overline{x}, J(x_{n+1} - \overline{x})\rangle \\ &= \alpha_n \langle u - \overline{x}, J(x_{n+1} - \overline{x})\rangle + \beta_n \langle x_n - \overline{x}, J(x_{n+1} - \overline{x})\rangle \\ &\quad + \gamma_n \langle u_n - \overline{x}, J(x_{n+1} - \overline{x})\rangle \\ &\leq \alpha_n \langle u - \overline{x}, J(x_{n+1} - \overline{x})\rangle + \beta_n \|x_n - \overline{x}\| \|x_{n+1} - \overline{x}\| \\ &\quad + \gamma_n \|u_n - \overline{x}\| \|x_{n+1} - \overline{x}\| \\ &\leq \alpha_n \langle u - \overline{x}, J(x_{n+1} - \overline{x})\rangle + \beta_n \|x_n - \overline{x}\| \|x_{n+1} - \overline{x}\| \\ &\quad + \gamma_n \|x_n - \overline{x}\| \|x_{n+1} - \overline{x}\| \\ &\leq \alpha_n \langle u - \overline{x}, J(x_{n+1} - \overline{x})\rangle + (1 - \alpha_n) \|x_n - \overline{x}\| \|x_{n+1} - \overline{x}\| \\ &\leq \alpha_n \langle u - \overline{x}, J(x_{n+1} - \overline{x})\rangle + \frac{1 - \alpha_n}{2} (\|x_n - \overline{x}\|^2 + \|x_{n+1} - \overline{x}\|^2),\end{aligned}$$

which implies that

$$\|x_{n+1} - \overline{x}\|^2 \leq (1 - \alpha_n) \|x_n - \overline{x}\|^2 + 2\alpha_n \langle u - \overline{x}, J(x_{n+1} - \overline{x})\rangle$$

and hence $x_n \to \overline{x}$ as $n \to \infty$ by virtue of Lemma 2.4. This completes the proof. □

REMARK 3.1. *Since $L^p (1 < p \leq 2)$ is uniformly convex and p-uniformly smooth, we see that Theorem 3.1 is applicable to L^p for $1 < p \leq 2$.*

4. Applications

In real Hilbert spaces, Lemma 2.3 is reduced to the following.

LEMMA 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H. For given $(\overline{x}, \overline{y}) \in C \times C$, where $\overline{y} = P_C(\overline{x} - \mu(I - S)\overline{x})$, $(\overline{x}, \overline{y})$ is a solution of the following problem:*

$$(4.1) \qquad \begin{cases} \langle \lambda(I - S)\overline{y} + \overline{x} - \overline{y}, x - \overline{x}\rangle \geq 0, & \forall x \in C, \\ \langle \mu(I - S)\overline{x} + \overline{y} - \overline{x}, x - \overline{x}\rangle \geq 0, & \forall x \in C, \end{cases}$$

if and only if $\overline{x}$ is a fixed point of the mapping $\overline{D} : C \to C$ defined by

$$\overline{D}(x) = P_C[P_C(x - \mu(I - S)x) - \lambda(I - S)P_C(x - \mu(I - S)x)],$$

where P_C is a metric projection H onto C. Utilizing Theorem 3.1 we can obtain the following results.

THEOREM 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $S : C \to C$ be a k-strict pseudo-contraction such that $Fix(S) \neq \phi$ and $T : C \to C$ a nonexpansive mapping with $Fix(T) \neq \phi$. Assume that $F = Fix(T) \cap Fix(\overline{D}) \neq \phi$, where $\overline{D}$ is defined as Lemma 4.1. Let a sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_1 = u \in C, \\ y_n = P_C(x_n - \mu(I - S)x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\delta T x_n + (1-\delta)P_C(y_n - \lambda(I-S)y_n)], \quad n \geq 1, \end{cases} \tag{4.2}$$

where $\delta \in (0, 1)$, $\lambda, \mu \in (0, 2k)$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ such that

(H1) $\alpha_n + \beta_n + \gamma_n = 1, \quad \forall n \geq 1$,
(H2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(H3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ defined by (4.2) converges strongly to $\overline{x} = P_F u$ and $(\overline{x}, \overline{y})$ is a solution of problem (4.1), where $\overline{y} = P_C(\overline{x} - \mu(I - S)\overline{x})$.

THEOREM 4.2. *Let C be a nonempty closed convex subset of H. Let $T, S : C \to C$ be a nonexpansive mapping such that $Fix(T) \neq \phi$ and $Fix(S) \neq \phi$. Assume that $F = Fix(T) \cap Fix(\overline{D}) \neq \phi$, where $\overline{D}$ is defined as Lemma 4.1. Let the sequence $\{x_n\}$ generated by (4.2) such that the conditions (H1), (H2), (H3) hold. Then $\{x_n\}$ converges strongly to $\overline{x} = P_F u$ and $(\overline{x}, \overline{y})$ is a solution of problem (4.1), where $\overline{y} = P_C(\overline{x} - \mu(I - S)\overline{x})$.*

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