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CLASSIFICATION OF TWO-REGULAR DIGRAPHS WITH MAXIMUM DIAMETER

Byeong Moon Kim, Byung Chul Song* and Woonjae Hwang

Abstract. The Klee-Quaife problem is finding the minimum order $\mu(d,c,v)$ of the (d,c,v) graph, which is a c-vertex connected v-regular graph with diameter d. Many authors contributed finding $\mu(d,c,v)$ and they also enumerated and classified the graphs in several cases. This problem is naturally extended to the case of digraphs. So we are interested in the extended Klee-Quaife problem. In this paper, we deal with an equivalent problem, finding the maximum diameter of digraphs with given order, focused on 2-regular case. We show that the maximum diameter of strongly connected 2-regular digraphs with order n is $n-3$, and classify the digraphs which have diameter $n-3$. All 15 nonisomorphic extremal digraphs are listed.

1. Introduction

Let G be a connected graph. G is c-vertex connected if the graph obtained by deleting arbitrary $c-1$ vertices from G remains connected. G is v-regular if each vertex of G is adjacent to exactly v vertices. A (d,c,v) graph (resp. $\langle d,c,v\rangle$ graph) is a c-vertex connected v-regular (resp. minimum degree v) graph with diameter d. The number $\mu(d,c,v)$ (resp. $\mu\langle d,c,v\rangle$) is the minimum order of the (d,c,v) (resp. $\langle d,c,v\rangle$) graphs and a minimum (d,c,v) graph is a (d,c,v) graph on $\mu(d,c,v)$ vertices.

In 1960's, there have been some early results which are equivalent to computing $\mu(d,c,v)$ and $\mu\langle d,c,v\rangle$ for special cases [4, 6, 10]. In 1976,

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*Corresponding author.

concerning with armed connection network, Klee and Quaife [8] raised a problem of finding $\mu(d,c,v)$ and $\mu\langle d,c,v\rangle$. In the same paper, $\mu\langle d,c,v\rangle$ was computed completely. For $\mu(d,c,v)$, there are some partial results and it was lastly computed in 1989 for all d,c,v [3].

Classifying and enumerating all minimum (d,c,v) graphs is a more complicated problem. This problem was solved only in two cases. The first case is $v=3$ and $c=1,2$ done by Klee and Quaife [9] in 1977. The second case is $v=c=3$ and d is odd. This was achieved by Klee [7] in 1980. It is also notable Mayers [11] found a method to construct all the $(d,3,3)$ graphs for all d in 1980, whereas he couldn't enumerate all $(d,3,3)$ graphs. Bhattacharya found a method to construct some minimum (d,n,n) graphs in 1985 [1]. But his method didn't cover all minimum (d,n,n) graphs.

Now we consider the corresponding problem for digraph $D=(V,A)$. A digraph D is strongly connected if for each pair of vertices x,y of D there is a directed walk from x to y. The ingegree $\delta^+(x)$(respectively, the outdegree $\delta^-(x)$) of a vertex x in D is the number of vertices y in D such that $(y,x)\in A$(respectively, $(x,y)\in A$). A digraph D is eulerian if $\delta^+(x)=\delta^-(x)$ for each vertex x in D. D is oriented if there is no pair of vertices x,y in D such that $(x,y)\in A$ and $(y,x)\in A$. The (d,c,δ) digraphs and $\langle d,c,\delta\rangle$ eulerian digraphs are defined similarly as the case of graphs. Finding minimum order of (d,c,δ) digraphs and $\langle d,c,\delta\rangle$ eulerian digraphs is equivalent to determining maximum diameter d such that there is (d,c,v) and $\langle d,c,v\rangle$ eulerian diagraphs on μ vertices, respectively. In [5], Knyazev proved that if D is an eulerian oriented digraph on n vertices, then $\frac{4n}{2\delta+1}-4\leq \mathrm{diam}(D)\leq \frac{5}{2\delta+n}n$. Dankelmann [2] improved the upper bound of $\mathrm{diam}(D)$ to $\frac{4}{2\delta+1}n+2$. Their results imply that the minimum order μ of an eulerian oriented $\langle d,1,\delta\rangle$ digraph satisfies $\frac{(2\delta+1)(d-2)}{4}\leq\mu\leq\frac{(2\delta+1)(d+4)}{4}$.

In this paper, we show that the maximum diameter of strongly connected 2-regular digraphs on n vertices is $n-3$ when $n\geq 9$. As a consequence we have the maximum number of vertices in a strongly connected oriented eulerian $(d,1,2)$ digraph is $d+3$ when $d\geq 6$. Moreover, in this case we classify all 15 digraphs of diameter $n-3$.

2. Main theorems

Let $D = (V, A)$ be a strongly connected digraph on n vertices. Assume that D is 2-regular. For a vertex v in D, we define $A^+(v) = \{w \in V | (v, w) \in A\}$ and $A^-(v) = \{w \in V | (w, v) \in A\}$. We also define $v \xrightarrow{k} w$ for each pair of vertices v, w in D if there is a walk of length k from v to w. We use $v \longrightarrow w$ instead of $v \xrightarrow{1} w$. Since D is 2-regular, the outdegree $\delta^+(v)$ and indegree $\delta^-(v)$ are 2 for every vertex $v \in V$.

THEOREM 1. *If D is a strongly connected 2-regular digraph on $n(\geq 5)$ vertices, then diam$(D) \leq n - 3$.*

Proof. Suppose that $\text{diam}(D) \geq n - 2$. There are vertices v, w in D such that $\text{dist}(v, w) = n - 2$. Since $v \xrightarrow{n-2} w$, there are $v_0, v_1, \ldots, v_{n-2}$ such that $v = v_0 \to v_1 \to v_2 \to \cdots \to v_{n-2} = w$. Since $\delta^+(v) = 2$ and $\delta^-(w) = 2$, there are vertices $v_1'(\neq v_1), v_{n-3}'(\neq v_{n-3})$ such that $v_1' \in A^+(v)$ and $v_{n-3}' \in A^-(w)$. Since $v_1' \neq v_0, v_1$ and $\text{dist}(v, w) = n - 2$, $v_1' \notin \{v_0, v_1, \ldots, v_{n-2}\}$. Similarly, we can show that $v_{n-3}' \notin \{v_0, v_1, \ldots, v_{n-2}\}$. If $v_1' = v_{n-3}'$, then $v \xrightarrow{2} w$ and $\text{dist}(v, w) = 2 < n - 2$, which is a contradiction. So the vertex set V includes $\{v_0, v_1, \ldots, v_{n-2}, v_1', v_{n-3}'\}$, which contradicts $|V| = n$. So $\text{diam}(D) \leq n - 3$. □

By the above theorem, $\text{diam}(D) \leq n - 3$. From now on we assume that $\text{diam}(D) = n - 3$. Then there are vertices v, w such that $\text{dist}(v, w) = n - 3$. So there are vertices $v_0, v_1, \ldots, v_{n-3}$ such that $v = v_0 \to v_1 \to v_2 \to \ldots \to v_{n-3} = w$. And there are vertices $v_1'(\neq v_1), v_{n-4}'(\neq v_{n-4})$ such that $v_0 \longrightarrow v_1'$ and $v_{n-4}' \longrightarrow v_{n-3}$. In this case, $V = \{v_0, v_1, \ldots, v_{n-3}, v_1', v_{n-4}'\}$. Using these notations, we have the following Lemma.

LEMMA 1. *Let $n \geq 7$ and $1 \leq i \leq n - 6$. If $x \in \{v_0, v_1, \ldots, v_i, v_1'\}$ and $y \in \{v_{i+2}, \ldots, v_{n-3}, v_{n-4}'\}$, then $(x, y) \notin A$.*

Proof. If $(x, y) \in A$, then

$$\begin{aligned} n - 3 = \text{dist}(v_0, v_{n-3}) &\leq \text{dist}(v_0, x) + \text{dist}(x, y) + \text{dist}(y, v_{n-3}) \\ &\leq i + 1 + n - i - 5 = n - 4. \end{aligned}$$

This is a contradiction. □

COROLLARY 1. *If $n \geq 10$ and $3 \leq i \leq n - 7$, then $v_{i+1} \longrightarrow v_i$.*

Proof. Let $A_i = \{v_0, v_1, \dots, v_i, v'_1\}$ and $B_i = \{v_{i+1}, \dots, v_{n-3}, v'_{n-4}\}$. Since $A_i \cup B_i = V$, $A_i \cap B_i = \phi$ and D is 2-regular, the number of arcs from A_i to B_i and from B_i to A_i are equal. By Lemma 1, there are no arc from A_{i-1} to B_i and from A_i to B_{i+1}. So (v_i, v_{i+1}) is the only arc from A_i to B_i. Thus there is only one arc from B_i to A_i. Let $A^+(v_{i+1}) = \{v_{i+2}, x\}$ and $A^-(v_i) = \{v_{i-1}, y\}$. Since there is no arc from $\{v_{i+1}\}$ to B_{i+2}, $x \in A_i$. Since (v_{i+1}, x) is the only arc from B_i to A_i, there is no arc from B_{i+1} to $\{v_i\}$. So $y \in A_{i+1}$. By Lemma 1, there is no arc from A_{i-2} to $\{v_i\}$, which implies $y \notin A_{i-2}$. Since $y \neq v_{i-1}, v_i$ and $y = v_{i+1}$. Thus, $v_{i+1} \longrightarrow v_i$. □

By Corollary 1, $v_{i+1} \to v_i \to v_{i-1}$ for $n \geq 11$ and $4 \leq i \leq n-7$. Since D is 2-regular we have the following corollary.

COROLLARY 2. *If* $n \geq 11$ *and* $4 \leq i \leq n-7$, *then* $A^-(v_i) = A^+(v_i) = \{v_{i-1}, v_{i+1}\}$.

If $V' \subset V$, then we use $\langle V' \rangle$ to be the directed subgraph of $D = (V, A)$ which is induced by V'. For $n \geq 2$, let P_n be the path on n vertices and P_1 be trivial graph.

LEMMA 2. *If* $n \geq 9$, *then* $\langle v_3, \dots, v_{n-6} \rangle$ *is isomorphic to the path* P_{n-8}.

Proof. It is trivial when $n = 9, 10$. When $n \geq 11$, by Lemma 1, $v_3 \longrightarrow v_{n-6}$. Since $A^+(v_{n-6}) = \{v_{n-5}, v_{n-7}\}$, $v_{n-6} \longrightarrow v_3$. So by Corollary 2, $\langle v_3, \dots, v_{n-6} \rangle$ is isomorphic to the path P_{n-8}. □

LEMMA 3. *If* $n \geq 9$, *then we have the followings.*
(1) $\langle v_0, \dots, v_3, v'_1 \rangle$ *is isomorphic to one of* $H_1, \dots, H_5$ *in Figure 1.*
(2) $\langle v_{n-6}, \dots, v_{n-3}, v'_{n-4} \rangle$ *is isomorphic to one of* $T_1, \dots, T_5$ *in Figure 1.*

Proof. We divide the proof into the cases according to how the sets $A^+(v_1), A^+(v'_1)$ are given.
Case1. $A^+(v_1) = \{v_0, v_2\}, A^+(v'_1) = \{v_0, v_1\}$
Since $A^-(v_0) = \{v'_1, v_1\}$ and $A^-(v_1) = \{v_0, v'_1\}$, $v_2 \to v'_1$. Since $A^-(v'_1) = \{v_0, v_2\}$, by Lemma 1, $v_3 \to v_2$. So $< v_0, v_1, v_2, v_3, v'_1 >$ is isomorphic to H_1 in Figure 1.
Case2. $A^+(v_1) = \{v'_1, v_2\}, A^+(v'_1) = \{v_0, v_1\}$
Since $A^-(v'_1) = \{v_0, v_1\}$ and $A^-(v_1) = \{v_0, v'_1\}$, by Lemma 1 $v_2 \to v_0$. Lemma 1 and the fact that $A^-(v_0) = \{v'_1, v_2\}$ imply $v_3 \to v_2$. We can

see $\langle v_1', v_1, v_2, v_3, v_0\rangle$ is isomorphic to H_1 in Figure 1.
Case3. $A^+(v_1) = A^+(v_1') = \{v_0, v_2\}$
Since $A^-(v_0) = \{v_1, v_1'\}$, $v_2 \longrightarrow v_0$. By Lemma 1, $v_2 \to v_1$ or $v_2 \to v_1'$. If $v_2 \to v_1$, $A^-(v_1) = \{v_0, v_2\}$. Since $A^-(v_2) = A^-(v_0) = \{v_1, v_1'\}$, by Lemma 1 $v_3 \to v_1'$. So $\langle v_0, v_1, v_2, v_3, v_1'\rangle = H_2$ in Figure 1. If $v_2 \to v_1'$, then we have $v_3 \to v_1$. And $\langle v_0, v_1', v_2, v_3, v_1\rangle$ is isomorphic to H_2 in Figure 1.
Case4. $A^+(v_1) = \{v_1', v_2\}, A^+(v_1') = \{v_0, v_2\}$
Since $A^-(v_1') = \{v_0, v_1\}$, $v_2 \longrightarrow v_1'$. So $v_2 \to v_1$ or $v_2 \to v_0$. If $v_2 \to v_0$, since $A^-(v_0) = \{v_1', v_2\}$ and $A^-(v_2) = \{v_1', v_1\}$, by Lemma 1 $v_3 \to v_1$. So $< v_0, v_1, v_2, v_3, v_1' >$ is isomorphic to H_3 in Figure 1. If $v_2 \to v_1$, then $A^-(v_1') = \{v_0, v_1\}, A^-(v_1) = \{v_0, v_2\}$ and $A^-(v_2) = \{v_1, v_1'\}$. By Lemma 1, $v_3 \to v_0$. So $\langle v_0, v_1, v_2, v_3, v_1'\rangle$ is isomorphic to H_4 in Figure 1.
Case5. $A^+(v_1) = \{v_0, v_2\}, A^+(v_1') = \{v_1, v_2\}$
Since $A^-(v_1) = \{v_0, v_1'\}$, by Lemma 1 $v_2 \to v_0$ or $v_2 \to v_1'$. If $v_2 \to v_0$, since $A^-(v_0) = \{v_1, v_2\}, A^-(v_1) = \{v_0, v_1'\}$ and $A^-(v_2) = \{v_1, v_1'\}$, by Lemma 1 $v_3 \to v_1'$. So $\langle v_0, v_1', v_2, v_3, v_1\rangle$ is isomorphic to H_3 in Figure 1. If $v_2 \to v_1'$, we must have $v_3 \to v_0$. We can see $\langle v_0, v_1', v_2, v_3, v_1\rangle$ is isomorphic to H_4 in Figure 1. Case6. $A^+(v_1) = \{v_1', v_2\}, A^+(v_1') = \{v_1, v_2\}$
Lemma 1, $A^-(v_1) = \{v_0, v_1'\}, A^-(v_1') = \{v_0, v_1\}$ and $A^-(v_2) = \{v_1, v_1'\}$ imply $v_2 \to v_0$ and $v_3 \to v_0$. So $\langle v_0, v_1, v_2, v_3, v_1'\rangle$ is isomorphic to H_5 in Figure 1. So (1) holds. Similarly we can prove (2) by substituting v_0, v_1, v_2, v_3, v_1' with $v_{n-3}, v_{n-4}, v_{n-5}, v_{n-6}, v_{n-4}'$ respectively. □

We call $H_1, \ldots, H_5$ in Lemma 3 as heads and $T_1, \ldots, T_5$ in Lemma 3 as tails. We can see that the union of $\langle v_0, v_1, v_2, v_3, v_1'\rangle, \langle v_3, \ldots, v_{n-6}\rangle$ and $\langle v_{n-6}, v_{n-5}, v_{n-4}, v_{n-3}, v_{n-4}'\rangle$ is a 2-regular digraph on V. So D is the union of subgraphs $\langle v_0, v_1, v_2, v_3, v_1'\rangle$, $\langle v_3, \ldots, v_{n-6}\rangle$ and $\langle v_{n-6}, v_{n-5}, v_{n-4}, v_{n-3}, v_{n-4}'\rangle$. Let $D_{i,j}$ be the union $H_i \cup P_{n-8} \cup T_j$ of digraphs H_i, P_{n-8} and T_j for $1 \le i, j \le 5$.

Theorem 2. *If D is a strongly connected 2-regular digraph on n vertices and diam$(D) = n-3$, then D is isomorphic to one of $\{D_{i,j} | 1 \le i \le j \le 5\}$ in Figure 2.*

Proof. Let $D_{i,j} = H_i \cup P_{n-8} \cup T_j$. The functions f_i defined by

$$(f_i(v_0), f_i(v_1), f_i(v_2), f_i(v_3), f_i(v_1')) = F_i$$

where $F_1 = (v_{n-3}, v_{n-4}', v_{n-5}, v_{n-6}, v_{n-4}), F_2 = (v_{n-4}, v_{n-3}, v_{n-4}', v_{n-6}, v_{n-5})$, $F_3 = (v_{n-3}, v_{n-5}, v_{n-4}, v_{n-6}, v_{n-4}')$, $F_4 = (v_{n-5}, v_{n-4}', v_{n-4}, v_{n-6}, v_{n-4})$,

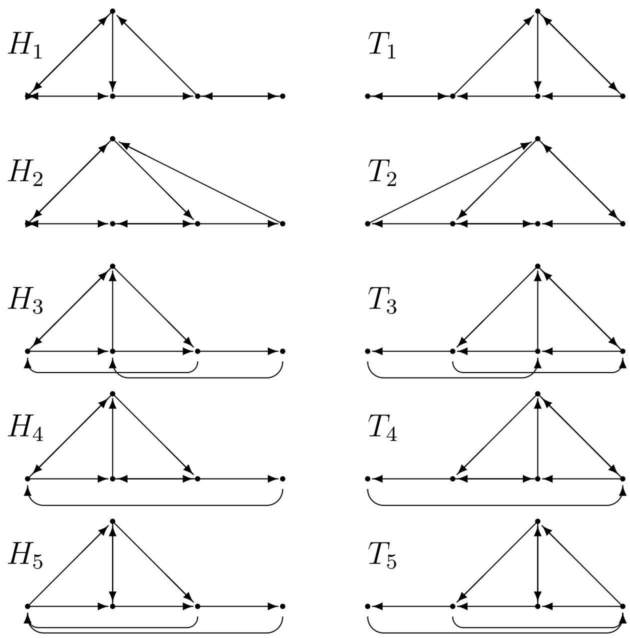

Figure 1. Heads and tails

and $F_5 = (v_{n-5}, v_{n-4}, \ v_{n-3}, v_{n-6}, v'_{n-4})$ give isomorphisms from H_i to T_i for all $i = 1, 2, \ldots, 5$. Thus $D_{i,j}$ and $D_{j,i}$ are isomorphic for all $i = 1, 2, \ldots, 5$. So D is isomorphic to one of

$$\{D_{i,j} | 1 \leq i \leq j \leq 5\}.$$

We can see these 15 digraphs are not isomorphic. □

By Theorem 2, we can conclude that if D is a strongly connected 2-regular digraph and $\mathrm{diam}(D) = d \geq 6$, then D has at least $d+3$ vertices and the extremal cases are given in Figure 2.

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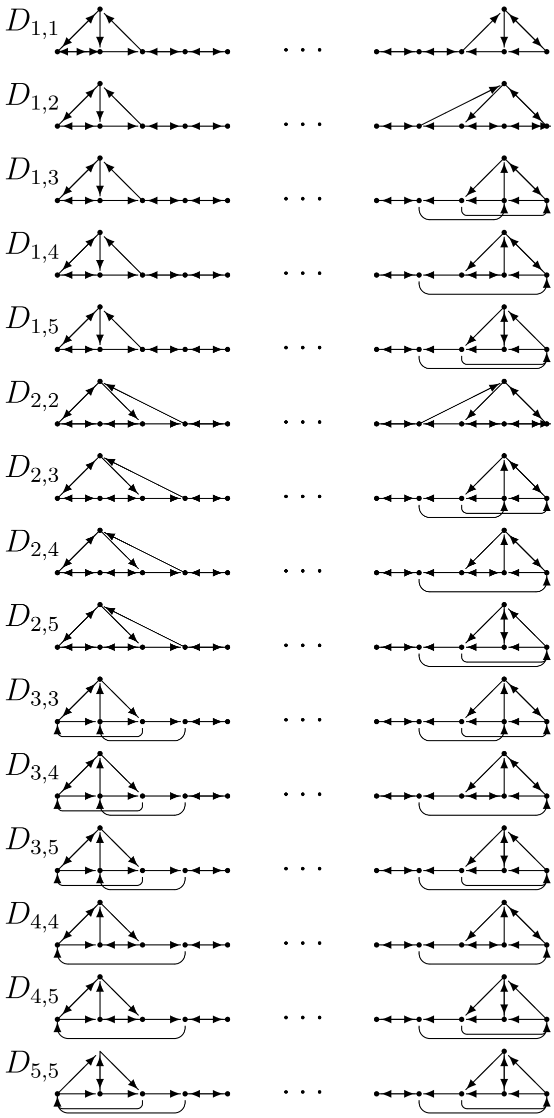

Figure 2. Extremal digraphs

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Department of Mathematics
Gangneung-Wonju National University
Gangneung 210-702, Korea
E-mail: kbm@gwnu.ac.kr

Department of Mathematics
Gangneung-Wonju National University
Gangneung 210-702, Korea
E-mail: bcsong@gwnu.ac.kr

Department of Information and Mathematics
Korea University
Jochiwon 339-700, Korea
E-mail: woonjae@korea.ac.kr