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A REFINEMENT FOR ORDERED LABELED TREES

Seunghyun Seo and Heesung Shin*

Abstract. Let $\mathcal{O}_n$ be the set of ordered labeled trees on $\{0,\dots,n\}$. A maximal decreasing subtree of an ordered labeled tree is defined by the maximal ordered subtree from the root with all edges being decreasing. In this paper, we study a new refinement $\mathcal{O}_{n,k}$ of $\mathcal{O}_n$, which is the set of ordered labeled trees whose maximal decreasing subtree has $k+1$ vertices.

1. Introduction

An *ordered tree* is a rooted tree in which children of each vertex are ordered. Figure 1 shows all the ordered tree with 4 vertices. It is well known (see [7, Exercise 6.19]) that the number of ordered trees with $n+1$ vertices is given by the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

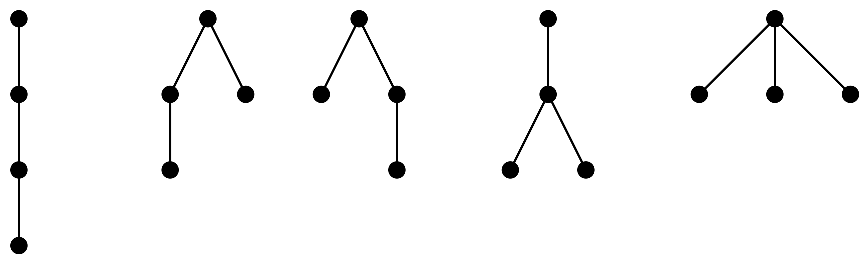

Figure 1. All ordered trees with 4 vertices

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*Corresponding author.

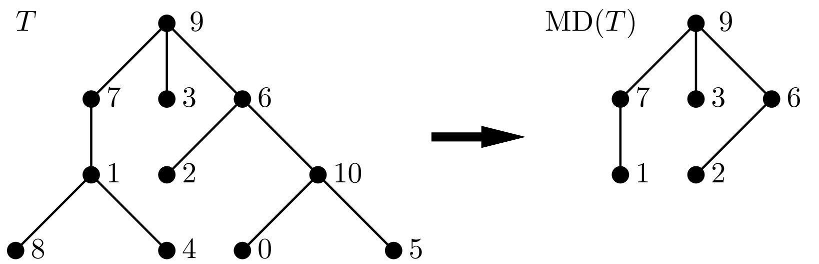

FIGURE 2. The maximal decreasing subtree of the ordered labeled tree T

An *ordered labeled tree* is an ordered tree whose vertices are labeled by distinct nonnegative integers. In most cases, an ordered labeled tree with $n+1$ vertices is identified with an ordered tree on the vertex set $[0,n] := \{0,\dots,n\}$. Let $\mathcal{O}_n$ be the set of ordered labeled trees on $[0,n]$. Clearly the cardinality of $\mathcal{O}_n$ is given by

$$|\mathcal{O}_n| = (n+1)!\,C_n = (n+1)^{(n)}, \tag{1}$$

where $m^{(k)} := m(m+1)\cdots(m+k-1)$ is a rising factorial.

For a given ordered labeled tree T, a *maximal decreasing subtree* of T is defined by the maximal ordered subtree from the root with all edges being decreasing, denoted by $\mathrm{MD}(T)$. Figure 2 illustrates the maximal decreasing subtree of a given tree T. Let $\mathcal{O}_{n,k}$ be the set of ordered labeled trees on $[0,n]$ with its maximal decreasing subtree having k edges.

In this paper we present a formula for $|\mathcal{O}_{n,k}|$, which makes a refined enumeration of $\mathcal{O}_n$, or a generalization of equation (1). Note that a similar refinement for the rooted (unordered) labeled trees was done before (see [5]), but the ordered case is more complicated and has quite different features.

2. Main results

From now on we will consider labeled trees only. So we will omit the word "labeled". Recall that $\mathcal{O}_{n,k}$ is the set of ordered trees on $[0,n]$ with its maximal decreasing ordered subtree having k edges. Let $\mathcal{Z}_{n,k}$ be the set of ordered trees on $[0,n]$ attached additional $(n-k)$ increasing leaves to decreasing tree with k edges. Note that the set $\mathcal{Z}_{n,k}$ first appeared in the Ph.D. Thesis [2, p. 46] of Drake. Let $\mathcal{F}_{n,k}$ be the set of forests on $[n] := \{1, 2, \ldots, n\}$ consisting of k ordered trees, where the k roots are not ordered. In Figure 3, the first two forests are the same, but the third one is a different forest in $\mathcal{F}_{4,2}$.

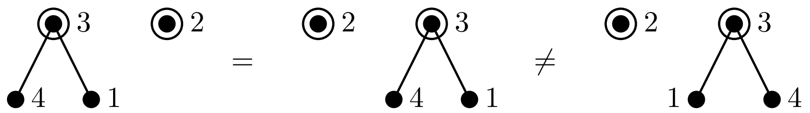

FIGURE 3. Forests in $\mathcal{F}_{4,2}$

Define the numbers

$$\begin{aligned} o(n,k) &= |\mathcal{O}_{n,k}|, \\ z(n,k) &= |\mathcal{Z}_{n,k}|, \\ f(n,k) &= |\mathcal{F}_{n,k}|. \end{aligned}$$

We will show that an ordered tree can be "decomposed" into an ordered tree in $\cup_{n,k}\mathcal{Z}_{n,k}$ and a forest in $\cup_{n,k}\mathcal{F}_{n,k}$. Thus it is crucial to count the numbers $z(n,k)$ and $f(n,k)$.

LEMMA 1. *The numbers $z(n,k)$ satisfy the recursion:*

$$z(n,k) = n \cdot z(n-1,k) + (n+k-1) \cdot z(n-1,k-1) \quad \text{for } 1 \le k < n \tag{2}$$

with the following boundary conditions:

$$z(n,n) = (2n-1)!! \quad \text{for } n \ge 0 \tag{3}$$

$$z(n,k) = 0 \quad \text{for } n < k \text{ or } k < 0, \tag{4}$$

where $(2n-1)!!$ is defined by $(2n-1) = (2n-1)(2n-3)\cdots 3 \cdot 1$.

Proof. Consider a tree Z in $\mathcal{Z}_{n,k}$. The tree Z with $n+1$ vertices consists of its maximal decreasing tree with $k+1$ vertices and the number of increasing leaves is $n-k$. Note that the vertex 0 is always contained in $\mathrm{MD}(Z)$.

If the vertex 0 is a leaf of Z, consider the tree Z' by deleting the leaf 0 from Z. The number of vertices in Z' and $\mathrm{MD}(Z')$ are n and k, respectively. So the number of possible trees Z' is $z(n-1,k-1)$. Since we cannot attach the vertex 0 to $n-k$ increasing leaves in recovering Z, there are $(2n-1)-(n-k)$ ways of recovering Z. Thus the number of Z with the leaf 0 is

$$(n+k-1)\cdot z(n-1,k-1).$$

If the vertex 0 is not a leaf of Z, then the vertex 0 has at least one increasing leaf. Let the vertex ℓ be the leftmost leaf of the vertex 0 and consider the tree Z'' obtained by deleting the leaf ℓ from Z. The number of vertices in Z'' and $\mathrm{MD}(Z'')$ are n and $k+1$, respectively. So the number of possible trees Z'' is $z(n-1,k)$. To recover Z is to relabel Z'' with $[0,n]\setminus\{\ell\}$ and to attach the vertex ℓ to the vertex 0. Since the number ℓ may be the number from 1 to n, the number of Z without the leaf 0 is

$$n\cdot z(n-1,k),$$

which completes the proof of recursion (2).

Since $\mathcal{Z}(n,n)$ is the set of decreasing ordered trees on $[0,n]$, the equation (3) holds [3] with the convention $(-1)!!=1$. For $n<k$ or $k<0$, $\mathcal{Z}_{n,k}$ should be empty, so the equation (4) also holds. $\square$

Lemma 2. *For $0\le k\le n$, we have*

$$f(n,k)=\binom{n}{k}k\,(n+1)(n+2)\cdots(2n-k-1) \tag{5}$$

with $f(0,0)=1$.

Proof. Consider a forest F in $\mathcal{F}_{n,k}$. The forest F consists of (non-ordered) k ordered trees $O_1,\dots,O_k$ with roots $r_1,r_2,\dots,r_k$, where $r_1<r_2<\cdots<r_k$. The number of ways for choosing roots $r_1,r_2,\cdots,r_k$ from $[n]$ is equal to $\binom{n}{k}$. From the *reverse Prüfer algorithm (RP Algorithm)* in [4], the number of ways for adding $n-k$ vertices successively to k roots $r_1,r_2,\cdots,r_k$ is equal to

$$k(n+1)(n+2)\cdots(2n-k-1)$$

for $0<k<n$, thus the equation (5) holds. By definition, $\mathcal{F}(0,0)$ is the set of the empty forest. So $f(0,0)=1$. $\square$

Since the number $z(n,k)$ is determined by the recurrence relation (2) in Lemma 1, we can count the number $o(n,k)$ with the following theorem.

Theorem 3. *We have*

$$o(n,k) = \sum_{k\le m\le n} \binom{n+1}{m+1} z(m,k)\,\frac{m-k}{n-k}(n-k)^{(n-m)} \quad \textit{for} \quad 0\le k<n, \tag{6}$$

and $o(n,n) = (2n-1)!!$, *where* $n^{(k)}$ *is a rising factorial.*

Proof. Given an ordered tree T in $\mathcal{O}_{n,k}$, let Z be the subtree of T consisting of $\mathrm{MD}(T)$ and its increasing edges. If the number of vertices of Z is $m+1$, then Z is a subtree of T with $(m-k)$ increasing leaves. Also, the induced subgraph Y of T generated by the $(n-k)$ vertices not belonging to $\mathrm{MD}(T)$ is a (non-ordered) forest consisting of $(m-k)$ ordered trees whose roots are only increasing leaves of Z.

Now let us count the number of ordered trees $T\in\mathcal{O}_{n,k}$ with $|V(Z)| = m+1$ where $V(Z)$ is the set of vertices in Z. First of all, the number of ways for selecting a set $V(Z)\subset[0,n]$ is equal to $\binom{n+1}{m+1}$. By attaching $(m-k)$ increasing leaves to a decreasing tree with k edges, we can make an ordered trees on $V(Z)$. There are exactly $z(m,k)$ ways for making such an ordered subtree on $V(Z)$. By the definition of $\mathcal{F}_{n,k}$ and Lemma 2, the number of ways for constructing the other parts on $V(T)\setminus V(Z)$ is equal to

$$f(n-k,m-k)\Big/\binom{n-k}{m-k} = \frac{m-k}{n-k}(n-k)^{(n-m)}.$$

Since the range of m is $k\le m\le n$, the equation (6) holds.

Finally, $\mathcal{O}(n,n)$ is the set of decreasing ordered trees on $[0,n]$, so

$$o(n,n) = z(n,n) = (2n-1)!!$$

holds for $n\ge 0$. □

3. Remark

Due to Theorem 3, we can calculate $o(n,k)$ for all n, k. However a closed form, a recurrence relation, or a generating function of $o(n,k)$ have not been found yet. The following might be a direction for solving the problem:

Shor [6] showed that the number $r(n,k)$, which is the number of rooted trees on $[n]$ with k improper edges, satisfies

$$r(n,k) = (n-1)\,r(n-1,k) + (n+k-2)\,r(n-1,k-1),$$

where an edge (u, v) is called *improper* if u is the endpoint closer to root and u has a larger label than some descendant of v. Zeng [1, 8] found that the generating function for $\{r(n,k)\}_{k=0}^{n}$ is the Ramanujan polynomial $R_n(x)$, which is defined by

$$R_{n+1}(x) = n(1+x)R_n(x) + x^2 R_n'(x); \quad R_1(x) = 1.$$

Drake [2, p. 46] observed that $z(n,k) = r(n+1,k)$ for all $k \le n$, by using the generating function method. Actually, $z(n,k)$ and $r(n+1,k)$ satisfy the same recursion and initial conditions, so we are able to construct a recursive bijection between these two objects. With this point of view, it would be interesting to find a certain set of rooted trees of cardinality $o(n,k)$.

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Department of Mathematics Education
Kangwon National University
Chuncheon 200-701, Korea
E-mail: shyunseo@kangwon.ac.kr

Department of Mathematics
Inha University
Incheon 402-751, Korea
E-mail: shin@inha.ac.kr