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A BIFURCATION PROBLEM FOR THE BIHARMONIC OPERATOR

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Abstract. We investigate the number of the solutions for the biharmonic boundary value problem with a variable coefficient nonlinear term. We get a theorem which shows the existence of m weak solutions for the biharmonic problem with variable coefficient. We obtain this result by using the critical point theory induced from the invariant function and invariant linear subspace.

1. Introduction

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. Let Δ be the elliptic operator and Δ^2 be the biharmonic operator. Let $c \in R$, $a : \overline{\Omega} \to R$ be a continuous function and $g : \overline{\Omega} \to R$ be a C^1 function. Assume that $a(x) > 0$ in $\overline{\Omega}$. In this paper we investigate the multiplicity of the weak solutions for the following variable coefficient nonlinear biharmonic equation with Dirichlet boundary condition

$$\begin{aligned} \Delta^2 u + c\Delta u &= \Lambda(a(x)u + g(u)) \qquad \text{in } \Omega, \\ u = 0, \qquad \Delta u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Let λ_j, $j \geq 0$ be the eigenvalues and ϕ_j, $j \geq 1$ be the corresponding eigenfunctions suitably normalized with respect to $L^2(\Omega)$ inner product and each eigenvalue λ_j is repeated as often as its multiplicity, of the eigenvalue problem

$$\Delta u + \lambda u = 0 \qquad \text{in } \Omega,$$

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$$u = 0 \qquad \text{on } \partial\Omega.$$

The eigenvalue problem

$$\Delta^2 u + c\Delta u = \mu a(x) u \qquad \text{in } \Omega,$$
$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega,$$

has also infinitely many eigenvalues $\mu_j = \lambda_j(\lambda_j - c)$, $j \geq 1$ and corresponding eigenfunctions ψ_j, $j \geq 1$. We note that $\mu_1 < \mu_2 \leq \mu_3 \ldots$, $\mu_j \to +\infty$.

We assume that g satisfies the following conditions:

$(g1)$ $g \in C^1(R, R)$ and $g(\xi) = o(|\xi|)$ uniformly with respect to $x \in \overline{\Omega}$.

$(g2)$ $g(\xi) < 0$ for any $\xi \in R$.

$(g3)$ $g(u) = -g(-u)$ for any $u \in \overline{\Omega}$.

Jung and Choi [4] showed the existence of at least two solutions, one of which is bounded solution and large norm solution of (1.1) when $g(u)$ is polynomial growth or exponential growth nonlinear term. The authors proved this result by the variational method and the mountain pass theorem. For the constant coefficient nonlinear case Choi and Jung [3] showed that the problem

$$\Delta^2 u + c\Delta u = bu^+ + s \qquad \text{in } \Omega, \tag{1.2}$$
$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega,$$

has at least two nontrivial solutions when ($c < \lambda_1$, $\Lambda_1 < b < \Lambda_2$ and $s < 0$) or ($\lambda_1 < c < \lambda_2$, $b < \Lambda_1$ and $s > 0$). The authors obtained these results by use of the variational reduction method. The authors [5] also proved that when $c < \lambda_1$, $\Lambda_1 < b < \Lambda_2$ and $s < 0$, (1.2) has at least three nontrivial solutions by use of the degree theory. Tarantello [9] also studied the problem

$$\Delta^2 u + c\Delta u = b((u+1)^+ - 1) \qquad \text{in } \Omega, \tag{1.3}$$
$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega.$$

She show that if $c < \lambda_1$ and $b \geq \Lambda_1$, then (1.3) has a negative solution. She obtained this result by the degree theory. Micheletti and Pistoia [7] also proved that if $c < \lambda_1$ and $b \geq \Lambda_2$, then (1.3) has at least four solutions by the vatiational linking theorem and Leray-Schauder degree theory. The authors [6] investigate the multiple solutions of semilinear elliptic equations. In this paper we are trying to find weak solutions of (1.1), that is,

$$\int_\Omega [\Delta^2 u \cdot v + c\Delta u \cdot v - \Lambda(a(x)u + g(u))v]dx = 0, \qquad \forall v \in H,$$

where H is introduced in section 2.

Our main result is the following.

THEOREM 1.1. *Let $\lambda_j < c < \lambda_{j+1}$. Assume that $a(x) > 0$ and g satisfies the conditions $(g1) - g(3)$. If $\mu_k < \Lambda < \mu_{k+1}$, $k \geq j+1$, then (1.1) has at least k weak solutions.*

We prove Theorem 1.1 by the critical point theory induced from the invariant subspace and invariant functional. The outline of the proof of Theorem 1.1 is as follows: In section 2, we introduce a Hilbert space H and a closed invariant linear subspace X of H which is invariant under the operator $u \mapsto \int_\Omega |\Delta u|^2 - c|\nabla u|^2 dx$, the invariant subspaces of X and the invariant function on X. We obtain some results on the norm $\|\cdot\|$ and the functional $f(u)$, and recall a critical point theory in terms of the invariant functional and invariant subspaces which plays a crucial role for the proof of the main result. In section 3, we prove Theorem 1.1.

2. Critical point theory induced from the invariant subspace and the invariant function

Let $L^2(\Omega)$ be a square integrable function space defined on Ω. Any element u in $L^2(\Omega)$ can be written as

$$u = \sum h_k \phi_k \quad \text{with} \quad \sum h_k^2 < \infty.$$

We define a subspace H of $L^2(\Omega)$ as follows

$$H = \{u \in L^2(\Omega) | \ \sum |\mu_k| h_k^2 < \infty\}.$$

Then this is a complete normed space with a norm

$$\|u\| = [\sum |\mu_k| h_k^2]^{\frac{1}{2}}.$$

Since $\lambda_k \to +\infty$ and c is fixed, we have

(i) $\Delta^2 u + c\Delta u \in H$ implies $u \in H$.

(ii) $\|u\| \geq C\|u\|_{L^2(\Omega)}$, for some $C > 0$.

(iii) $\|u\|_{L^2(\Omega)} = 0$ if and only if $\|u\| = 0$,

which is proved in [2].

Let

$$H^+ = \{u \in H \mid h_k = 0 \text{ if } \ \mu_k < 0\},$$
$$H^- = \{u \in H \mid h_k = 0 \text{ if } \ \mu_k > 0\}.$$

Then $H = H^- \oplus H^+$, for $u \in H$, $u = u^- + u^+ \in H^- \oplus H^+$. Let P_+ be the orthogonal projection on H^+ and P_- be the orthogonal projection on H^-. We can wtite $P_+ u = u^+$, $P_- u = u^-$, for $u \in H$. We are looking for the weak solutions of (1.1). By the following Proposition 2.1, the weak solutions of (1.1) coincide with the critical points of the associated functional

$$I(u) \in C^1(H, R),$$

$$(2.1)\quad I(u) = \int_\Omega [\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - \Lambda \int_\Omega [\frac{1}{2}a(x)u^2 + G(u)]dx$$
$$= \frac{1}{2}(\|P_+ u\|^2 - \|P_- u\|^2) - \Lambda \int_\Omega [\frac{1}{2}a(x)u^2 + G(u)]dx,$$

where $G(\xi) = \int_0^\xi g(\tau)\tau$. By $(g1)$, I is well defined.

Proposition 2.1. *Assume that $\lambda_j < c < \lambda_{j+1}$, $j \geq 1$, and g satisfies $(g1) - (g3)$. Then $I(u)$ is continuous and Fréchet differentiable in H with Fréchet derivative*

$$(2.2)\qquad \nabla I(u)h = \int_\Omega [\Delta u \cdot \Delta h - c\nabla u \cdot \nabla h - \Lambda(a(x)u + g(u))h]dx.$$

If we set

$$F(u) = \Lambda \int_\Omega [\frac{1}{2}a(x)u^2 + G(u)]dx,$$

then $F'(u)$ is continuous with respect to weak convergence, $F'(u)$ is compact, and

$$F'(u)h = \Lambda \int_\Omega (a(x)u + g(u))hdx \qquad \text{for all } h \in H,$$

this implies that $I \in C^1(H, R)$ and $F(u)$ is weakly continuous.

The proof of Proposition 2.1 has the similar process to that of the proof in Appendix B in [8].

Let us define some notations and concepts on Z_2-invariant set and Z_2-invariant function: Let H be a real Hilbert space on which the action Z_2 acts orthogonally. For $u \in H$, we define Z_2- actions on H by

$$Tu = u \qquad \text{or} \;\; Tu = -u.$$

that is, the Z_2 action have the identity map and the antipodal map as an action. Thus Z_2- action acts freely on the subspace $\{u|\; Tu = -u\}$. Let Fix_{Z_2} be the set of fixed points of the action, i.e.,

$$\mathrm{Fix}_{Z_2} = \{u \in H|\; Tu(x) = u(x), \text{ for all } \; x \in \Omega, \; u \in H, \; Z_2 - action\ T\}.$$

We note that $\mathrm{Fix}_{Z_2} = \{0\}$. Let

$$X_1 = \mathrm{Fix}_{Z_2} = \{0\} \qquad X_2 = X_1^{\perp}.$$

Thus Z_2- action has the representation $x \mapsto -x$, for $x \in X_2$ and $H = X_1 \oplus X_2$. We say a subset B of H an Z_2-invariant set if for all $u \in B$, $Tu \in B$. A function $I : H \to R^1$ is called Z_2-invariant if $I(Tu) = I(u)$, $\forall u \in H$. Let $C(B, H)$ be the set of continuous functions from B into H. If B is an invariant set we say $h \in C(B, H)$ is an equivariant map if $h(Tu) = Th(u)$ for all $u \in B$. We note that H is a closed invariant linear subspace of H compactly embedded in $L^2(\Omega, R)$ under the Z_2-action. Let

$$(Lu)h = \int_\Omega [\Delta u \cdot \Delta h - c\nabla u \cdot \nabla h]dx.$$

We can check easily that $L(H) \subseteq H$, $L : H \to H$ is an isomorphism and $\nabla I(H) \subseteq H$. Therefore constrained critical points on H are in fact free critical points on H. Moreover, distinct critical orbits give rise to geometrically distinct solutions. We have the following lemma which can be checked easily since $\mathrm{Fix}_{Z_2} = \{0\}$:

LEMMA 2.1. *Assume that g satisfies the conditions $(g1) - (g3)$. Let $u \in Fix_{Z_2} = \{0\}$ and u be a critical point of the functional of I, i.e., $\nabla I(u) = 0$. Then $I(u) = 0$.*

Now we recall the critical point theory in terms of the invariant subspace and invariant function in Theorem 4.1 of [1] which plays a crucial role for the proof of Theorem 1.1: Let S_ρ be the sphere centered at the origin of radius ρ. Let $I : H \to R$ be a functional of the form

$$I(u) = \frac{1}{2}(Lu)u - F(u), \tag{2.3}$$

where $L : H \to H$ is linear, continuous, symmetric and equivariant, $F : H \to R$ is of class C^1 and invariant and $DF : H \to H$ is compact.

THEOREM 2.1. *Assume that $I \in C^1(H, R^1)$ is Z_2-invariant and there exist two closed invariant linear subspaces V, W of H and $\rho > 0$ with the following properties:*

(a) *$V + W$ is closed and of finite codimension in H;*

(b) *$Fix_{Z_2} \subseteq V + W$;*

(c) *$L(W) \subseteq W$;*

(d) $\sup_{S_\rho \cap V} I < +\infty$ *and* $\inf_W I > -\infty$;

(e) *$u \notin Fix_{Z_2}$ whenever $DI(u) = 0$ and* $\inf_W I \leq I(u) \leq \sup_{S_\rho \cap V} I$;

(f) *I satisfies $(P.S.)_c$ condition whenever* $\inf_W I \leq c \leq \sup_{S_\rho \cap V} I$.

Then I possesses at least

$$dim(V \cap W) - codim_H(V + W)$$

distinct critical orbits in $I^{-1}([\inf_W I, \sup_{S_\rho \cap V} I])$.

3. Proof of Theorem 1.1

To prove Theorem 1.1 we shall prove that the functional I satisfies the assumptions $(g1) - g(3)$ of Theorem 2.1. We assume that g satisfies the conditions $(g1) - (g3)$. Let us set

$$H_1^+ = \{u |\ u \in H,\ u \in \text{span}\{\psi_l,\ l \geq 1\}\},$$

$$H_k^- = \{u |\ u \in H,\ u \in \text{span}\{\psi_l,\ 1 \leq l \leq k\}\}.$$

We have the following lemma which can be checked easily since $\text{Fix}_{Z_2} = \{0\}$:

LEMMA 3.1. *Assume that g satisfies the conditions $(g1) - (g3)$. Then there exist $\rho > 0$ and a sphere S_ρ centered at 0 in H such that the* *functional $I(u)$ is bounded from above on $S_\rho \cap H_k^-$ and from below on H_1^+. That is,*

$$-\infty < \inf_{u \in H_1^+} I(u) \qquad \text{and} \qquad \sup_{u \in S_\rho \cap H_k^-} I(u) < 0.$$

Proof. We note that

$$\begin{aligned} &\forall u \in H_k^- : (Lu)u \le \mu_k \textstyle\int_\Omega a(x)u^2dx, \\ &\forall u \in H_1^+ : (Lu)u \ge \mu_1 \textstyle\int_\Omega a(x)u^2dx. \end{aligned} \tag{3.1}$$

Then for $u \in H_k^-$,

$$\begin{aligned} I(u) &= \frac{1}{2}(Lu)u - \Lambda \int_\Omega [\frac{1}{2}a(x)u^2 + G(u)]dx \\ &\le \frac{1}{2}(\mu_k - \Lambda) \int_\Omega a(x)u^2dx + o(\|u\|^2_{L^2(\Omega)}) \end{aligned} \tag{3.2}$$

since $G(\xi) \in C^2$. Thus we can choose a number $\rho > 0$ and a sphere S_ρ centered at 0 in H such that for any $u \in S_\rho$,

$$\begin{aligned} &\frac{1}{2}(\mu_k - \Lambda) \int_\Omega a(x)u^2dx + o(\|u\|^2_{L^2(\Omega)}) \\ &\le \frac{1}{2}(\mu_k - \Lambda)(\sup a(x))\rho^2 + o(\|u\|^2_{L^2(\Omega)}) < 0 \end{aligned} \tag{3.3}$$

since $\mu_k - \Lambda < 0$. Thus we have $\sup_{S_\rho \cap H_k^-} I(u) < 0$. Let $u \in H_1^+$. Then we have

$$\begin{aligned} I(u) &= \frac{1}{2}(Lu)u - \Lambda \int_\Omega [\frac{1}{2}a(x)u^2 + G(u)]dx \\ &\ge \frac{1}{2}(\mu_1 - \Lambda) \int_\Omega a(x)u^2dx + o(\|u\|^2_{L^2(\Omega)}) \\ &> \frac{1}{2}(\mu_1 - \Lambda)(\sup a(x))\|u\|^2_{L^2} + o(\|u\|^2_{L^2(\Omega)}) \\ &> -\infty \end{aligned}$$

since $\mu_1 - \Lambda < 0$, $G(u) = o(\|u\|^2_{L^2(\Omega)})$.

Thus we have $\inf_{u \in H_1^+} I(u) > -\infty$. $\square$

LEMMA 3.2. *Assume that g satisfies the conditions $(g1) - (g3)$. Then the functional I satisfies $(P.S.)_c$ condition for every $c \in [\inf_W I(u), \sup_{S_\rho \cap V} I(u)]$.*

Proof. Let $u \in H$. Since $H = H_1^+$, the functional

$$
\begin{aligned}
(3.4) \qquad I(u) &= \frac{1}{2}(Lu)u - \Lambda \int_\Omega [\frac{1}{2}a(x)u^2 + G(u)]dx \\
&\geq \frac{1}{2}(\mu_1 - \Lambda)\int_\Omega a(x)u^2 dx - \Lambda \int_\Omega G(u)dx \\
&> \frac{1}{2}(\mu_1 - \Lambda)\sup(a(x))\|u\|^2_{L^2} - o(\|u\|^2_{L^2}) \\
&\geq \frac{1}{2}(\mu_1 - \Lambda)\sup(a(x))\|u\|^2_{L(\Omega)} - o(\|u\|^2_{L^2}) \\
&> -\infty.
\end{aligned}
$$

Thus $I(u)$ is bounded from below since $G(\xi) = o(|\xi|^2)$. Thus $I(u)$ satisfies the $(P.S.)_c$ condition. $\square$

[Proof of Theorem 1.1]

If we set $V = H_k^-$ and $W = H_1^+ = H$, then $V + W$ is closed invariant subspaces of H with $V + W = H$ and of finite codimension in H. We note that $\mathrm{Fix}_{Z_2} = \{0\}$ and $\mathrm{Fix}_{Z_2} = \{0\} \subseteq V + W = H$. We also note that $L(W) \subseteq W$. By Lemma 3.1,

$$
-\infty < \inf_W I \qquad \sup_{H_k^-} \cap S_\rho I < 0.
$$

Thus the condition (d) of Theorem 2.1 is satisfied. Suppose that u be a critical point of the functional of I and $\inf_W I \leq I(u) \leq \sup_{S_\rho \cap V} I$. Then by Lemma 3.1, $-\infty < \inf_W I \leq I(u) \leq \sup_{S_\rho \cap V} I < 0$. We claim that $u \notin \mathrm{Fix}_{Z_2}$. If not, then $u \in \mathrm{Fix}_{Z_2} = \{0\}$ i.e., $u = 0$. Since $u = 0$ is a critcal point of $I(u)$ with $I(0) = 0$ and $0 \notin [\inf_W I, \sup_{S_\rho \cap V} I]$, it leads to a contradiction to the fact that $\inf_W I \leq I(u) \leq \sup_{S_\rho \cap V} I$. Thus $u \notin \mathrm{Fix}_{Z_2}$. Thus the condition (e) is satisfied. By Lemma 3.2, I satisfies $(P.S.)_c$ condition whenever $\inf_W I \leq c \leq \sup_{S_\rho \cap V} I$.
Thus the assumptions $(a) - (e)$ of Theorem 1.1 are satisfied. Thus by the Theorem 2.1, Then I possesses at least

$$
dim(V \cap W) - codim_H(V + W) = k
$$

distinct critical orbits in $I^{-1}([\inf_W I, \sup_{S_\rho \cap V} I])$.

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