J. Appl. Math. & Informatics Vol. **30**(2012), No. 3 - 4, pp. 455 - 462
Website: http://www.kcam.biz

COMBINATORIAL PROOF FOR THE POSITIVITY OF THE ORBIT POLYNOMIAL $O_d^{n,3}(q)$†

JAEJIN LEE

Abstract. The cyclic group $C_n = \langle(12\cdots n)\rangle$ acts on the set $\binom{[n]}{k}$ of all k-subsets of $[n]$. In this action of C_n the number of orbits of size d, for $d \mid n$, is

$$O_d^{n,k} = \frac{1}{d}\sum_{\frac{n}{d}|s|n} \mu\left(\frac{ds}{n}\right)\binom{n/s}{k/s}.$$

Stanton and White[7] generalized the above identity to construct the orbit polynomials

$$O_d^{n,k}(q) = \frac{1}{[d]_{q^{n/d}}}\sum_{\frac{n}{d}|s|n} \mu\left(\frac{ds}{n}\right)\begin{bmatrix} n/s \\ k/s \end{bmatrix}_{q^s}$$

and conjectured that $O_d^{n,k}(q)$ have non-negative coefficients. In this paper we give a combinatorial proof for the positivity of coefficients of the orbit polynomial $O_d^{n,3}(q)$.

AMS Mathematics Subject Classification : 05E10.
Key words and phrases : q-binomial coefficient, cyclic group, action, orbit, orbit polynomial.

1. Introduction

When n is a positive integer, we write as $[n] = \{1, 2, \dots, n\}$. Let C_n be the cyclic group generated by a permutation $\sigma = (12\cdots n)$. If $\binom{[n]}{k}$ is the set of all k-subsets of $[n]$, C_n acts on $\binom{[n]}{k}$ via

$$(\tau, \{x_1, x_2, \dots, x_k\}) \mapsto \{x_{\tau(1)}, x_{\tau(2)}, \dots, x_{\tau(k)}\}.$$

The number of orbits in this action of C_n is given

$$O^{n,k} = \frac{1}{n}\sum_{d|\gcd(n,k)} \varphi(d)\binom{n/d}{k/d}, \tag{1}$$

Received October 6, 2010. Revised March 25, 2011. Accepted September 26, 2011.
†This research was supported by Hallym University Research Fund, 2010(HRF-2010-000).

and the number of orbits of size d, for $d \mid n$, is

$$O_d^{n,k} = \frac{1}{d} \sum_{\frac{n}{d}|s|n} \mu\left(\frac{ds}{n}\right) \binom{n/s}{k/s}. \tag{2}$$

See [2]. Here φ is the Euler phi-function and μ is the Möbius function. Stanton and White [7] constructed orbit polynomials $O_d^{n,k}(q)$, a q-version of (2), and conjectured the following.

Conjecture 1. *Fix $d \mid n$, and any non-negative integer k. Polynomials*

$$O_d^{n,k}(q) = \frac{1}{[d]_{q^{n/d}}} \sum_{\frac{n}{d}|s|n} \mu\left(\frac{ds}{n}\right) \begin{bmatrix} n/s \\ k/s \end{bmatrix}_{q^s}$$

have non-negative coefficients.

Here, $[n]_q = 1 + q + \cdots + q^{n-1}$, $[n]!_q = [1]_q[2]_q \cdots [n]_q$ and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q[n-k]!_q}.$$

Möbius inversion implies

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{d|n} [d]_{q^{n/d}} O_d^{n,k}(q). \tag{3}$$

Andrews[1] and Haiman[4] independently verified the above Conjecture 1 when $(n, k) = 1$. In [5] Reiner, Stanton and White defined the cyclic sieving phenomenon, generalization of Stembridge's $q = -1$ phenomenon [8], and use it to prove several enumeration problems involving q-binomial coefficients, non-crossing partitions, polygon dissections and some finite field q-analogues. Drudge [3] has proven that $O^{n,k}(q) = \sum_{d|n} O_d^{n,k}(q)$ is the number of orbits of the Singer cycle on the k-dimensional subspaces of an n-dimensional vector space over a field of order q. Recently Sagan [6] gave combinatorial proofs for several theorems appeared in [5].

In this paper we give a new weight for each 3-subset in $\binom{[n]}{3}$, and show that the sum of weights of all 3-subset in $\binom{[n]}{3}$ is equal to the q-binomial coefficient $\begin{bmatrix} n \\ 3 \end{bmatrix}_q$. This will give a combinatorial proof for the positivity of coefficients of the orbit polynomial $O_d^{n,3}(q)$.

2. Positivity for the orbit polynomial $O_d^{n,3}(q)$

In this section we write as $ijk = \{i, j, k\}$ for convention. We begin with the recurrence relation of q-binomial coefficient $\begin{bmatrix} n \\ 3 \end{bmatrix}_q$. Using the recurrence relations

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \text{ and}$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

several times, we get the following identity.

Proposition 1. *Let $n \geq 3$ be an integer. Then*

$$\begin{bmatrix} n+3 \\ 3 \end{bmatrix}_q = q^6 \begin{bmatrix} n \\ 3 \end{bmatrix}_q + q^{n+6} \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q + (1 + q^2[n-1]_q)[n+3]_q.$$

We now describe the representatives x of orbits in the action of of C_n on $\binom{[n]}{3}$. In each orbit O under C_n we choose $1ij \in O$ as the representative of O, where

$$1 < i \leq \frac{n}{3} + 1 \text{ and } 2i - 1 \leq j \leq n + 1 - i. \tag{4}$$

For example, if $n = 7$, all orbits are given with representatives underlined as follows.

$$O_1 = \langle \underline{123} \rangle = \{\underline{123}, 234, 345, 456, 567, 167, 127\}$$
$$O_2 = \langle \underline{124} \rangle = \{\underline{124}, 235, 346, 457, 156, 267, 137\}$$
$$O_3 = \langle \underline{125} \rangle = \{\underline{125}, 236, 347, 145, 256, 367, 147\}$$
$$O_4 = \langle \underline{126} \rangle = \{\underline{126}, 237, 134, 245, 356, 467, 157\}$$
$$O_5 = \langle \underline{135} \rangle = \{\underline{135}, 246, 357, 146, 257, 136, 247\}.$$

Let $1ij$ be the representative of an orbit under C_n. We define the weight $w_n(1ij)$ as

$$w_n(1ij) = \begin{cases} 1 & \text{if } 3 \mid n \text{ and } i = 1 + \frac{n}{3}, j = 1 + \frac{2n}{3} \\ q^{2n+i-2j-3} & \text{if } 3 \mid n \text{ and } j = n + 1 - i \\ q^{2n+i-2j-4} & \text{else.} \end{cases} \tag{5}$$

The weights for the other elements than the representatives are given using the weights of representatives in (5).

Assume first $\gcd(n, 3) = 1$. Note that all orbits are of size n by (1) and (2). If $O_i = \{x_{i1}, x_{i2}, \ldots, x_{i(n-1)}, x_{in}\}$ is an orbit of size n with the representative x_{i1} and with the action

$$x_{i1} \xrightarrow{\sigma} x_{i2} \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} x_{i(n-1)} \xrightarrow{\sigma} x_{in} \xrightarrow{\sigma} x_{i1},$$

we define

$$w_n(x_{ij+1}) = q w_n(x_{ij}) \text{ for } 1 \leq j \leq n - 1. \tag{6}$$

If $\gcd(n, 3) \neq 1$, there is only one orbit of size $n/3$ and the other orbits are of size n under the action of C_n. The weights for elements in an orbit of size n are defined in the same way as (6). On the other hand, if $O_0 = \{x_{01}, x_{02}, \ldots, x_{0(n/3)}\}$ is the orbit of size $n/3$ with the representative x_{01} and with the action

$$x_{01} \xrightarrow{\sigma} x_{02} \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} x_{0(n/3)} \xrightarrow{\sigma} x_{01},$$

we define

$$w_n(x_{0j+1}) = q^3 w_n(x_{0j}) \text{ for } 1 \leq j \leq \frac{n}{3} - 1.$$

Then the sum of weights of all elements in $\binom{[n]}{3}$ is equal to the q-binomial coefficient $\begin{bmatrix} n \\ 3 \end{bmatrix}_q$ as follows.

Theorem 1. *Let $n \geq 3$ be an integer and let T_n be the set of all 3-subsets of $[n]$, i.e., $T_n = \binom{[n]}{3}$. If we set $w_n(T_n) = \sum_{x \in \binom{[n]}{3}} w(x)$, then we have*

$$w_n(T_n) = \begin{bmatrix} n \\ 3 \end{bmatrix}_q.$$

Proof. We only work out for $n = 3\ell + 1$. The proofs for $n = 3\ell$ and $n = 3\ell + 2$ can be given in the same way with a little modification.

Computing $w_n(T_n)$ and $\begin{bmatrix} n \\ 3 \end{bmatrix}_q$ for $n = 3, 4, 5$ directly, we have

$$w_3(T_3) = 1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}_q, \; w_4(T_4) = 1 + q + q^2 + q^3 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q$$
$$w_5(T_5) = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_q.$$

Suppose now $n = 3\ell+1$ and $w_n(T_n) = \begin{bmatrix} n \\ 3 \end{bmatrix}_q$. Since $\gcd(n, 3) = \gcd(n+3, 3) = 1$, all orbits under C_n are of size n and all orbits under C_{n+3} are of size $n + 3$. Let

$$x_{11}, x_{21}, \ldots, x_{s1}$$

be all representatives of orbits in the action of C_n, where

$$s = |T_n|/|\text{orbit}| = \binom{n}{3}/n = \frac{1}{6}(n-1)(n-2).$$

Let

$$x_{11}, x_{21}, \ldots, x_{s1}, x_{(s+1)1}, \ldots, x_{t1}$$

be all representatives of orbits in the action of C_{n+3}. Here,

$$t = \binom{n+3}{3}/(n+3) = \frac{1}{6}(n+1)(n+2).$$

Then all orbits under C_n are as follows,

$$\begin{aligned} O_1 &= \{x_{11}, x_{12}, \ldots, x_{1(n-1)}, x_{1n}\} \\ O_2 &= \{x_{21}, x_{22}, \ldots, x_{2(n-1)}, x_{2n}\} \\ &\vdots \\ O_s &= \{x_{s1}, x_{s2}, \ldots, x_{s(n-1)}, x_{sn}\} \end{aligned} \tag{7}$$

while

$$\begin{aligned} O'_1 &= \{x_{11}, x_{12}, \ldots, x_{1(n-1)}, x_{1n}, x_{1(n+1)}, x_{1(n+2)}, x_{1(n+3)}\} \\ O'_2 &= \{x_{21}, x_{22}, \ldots, x_{2(n-1)}, x_{2n}, x_{2(n+1)}, x_{2(n+2)}, x_{2(n+3)}\} \\ &\vdots \\ O'_s &= \{x_{s1}, x_{s2}, \ldots, x_{s(n-1)}, x_{sn}, x_{s(n+1)}, x_{s(n+2)}, x_{s(n+3)}\} \\ O'_{s+1} &= \{x_{(s+1)1}, \ldots, x_{(s+1)n}, x_{(s+1)(n+1)}, \ldots, x_{(s+1)(n+3)}\} \\ &\vdots \\ O'_t &= \{x_{t1}, x_{t2}, \ldots, x_{t(n-1)}, x_{tn}, x_{t(n+1)}, x_{t(n+2)}, x_{t(n+3)}\} \end{aligned} \tag{8}$$

are all orbits under C_{n+3}. Let x be the representative of an orbit under the action of C_n. x can be also the representative of an orbit under the action of C_{n+3}. In this case,

$$w_{n+3}(x) = q^6 w_n(x).$$

For example, $x = 123 \in \binom{[n]}{3}$ is the representative of an orbit under the action of C_n. The weight of x is

$$w_n(x) = q^{2n+2-2\cdot 3-4} = q^{2n-8}.$$

Also, $x = 123$ can be considered in $T_{n+3} = \binom{[n+3]}{3}$ and the weight $w_{n+3}(x)$ is

$$w_{n+3}(x) = q^{2(n+3)+2-2\cdot 3-4} = q^{2n-2},$$

so that $w_{n+3}(x) = q^6 w_n(x)$. Another 3-subset $234 = \sigma(123)$ is considered as the element of T_{n+3} as well as T_n. The weight of 234 is

$$w_n(234) = q w_n(123) \quad \text{and} \quad w_{n+3}(234) = q w_{n+3}(123)$$

so that $w_{n+3}(234) = q^6 w_n(234)$. Using this relation we compute $w_{n+3}(T_{n+3})$. From (7) and assumption we have

$$w_n(T_n) = \sum_{i=1}^{s} \sum_{x \in O_i} w_n(x) = \sum_{i=1}^{s} w_n(x_{i1})[n]_q = r_n(q)[n]_q = \begin{bmatrix} n \\ 3 \end{bmatrix}_q,$$

where $r_n(q)$ is the sum of weights of representatives of all orbits of size n. On the other hand, if we use (8), we have

$$w_{n+3}(T_{n+3}) = \sum_{i=1}^{t} \sum_{x \in O'_i} w_{n+3}(x) = \sum_{i=1}^{s} \sum_{x \in O'_i} w_{n+3}(x) + \sum_{i=s+1}^{t} \sum_{x \in O'_i} w_{n+3}(x).$$

Here

$$\begin{aligned}
\sum_{i=1}^{s}\sum_{x\in O_i'} w_{n+3}(x) &= \sum_{i=1}^{s}\sum_{j=1}^{n+3} w_{n+3}(x_{ij}) = \sum_{i=1}^{s} w_{n+3}(x_{i1})[n+3]_q \\
&= \sum_{i=1}^{s} q^6 w_n(x_{i1})([n]_q + q^n[3]_q) \\
&= q^6 r_n(q)[n]_q + q^{n+6} r_n(q)[3]_q \qquad (9)\\
&= q^6 \begin{bmatrix} n \\ 3 \end{bmatrix}_q + q^{n+6}\frac{\begin{bmatrix} n \\ 3 \end{bmatrix}_q}{[n]_q}[3]_q \\
&= q^6 \begin{bmatrix} n \\ 3 \end{bmatrix}_q + q^{n+6}\begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q .
\end{aligned}$$

Using (4) we can find the representatives of all orbits under of C_{n+3}. In particular, for $2 \le a \le \ell + 1$,

$$1a(n-a+2),\ 1a(n-a+3),\ 1a(n-a+4),\ 1(\ell+2)(2\ell+3)$$

are the representatives of orbits in the action of C_{n+3} which are not in orbits of the action of C_n. Using the weights given in (5) and (6)

$$\begin{aligned}
\sum_{i=s+1}^{t}\sum_{x\in O_i'} w_{n+3}(x) &= \left(\sum_{a=2}^{\ell+1}(q^{3a-2}+q^{3a-4}+q^{3a-6}) + q^{3\ell}\right)[n+3]_q \\
&= \left(\sum_{a=1}^{\ell}(q^{3a+1}+q^{3a-1}+q^{3a-3}) + q^{3\ell}\right)[n+3]_q \qquad (10)\\
&= (1+q^2+q^3+\cdots+q^n)[n+3]_q \\
&= (1+q^2[n-1]_q)[n+3]_q.
\end{aligned}$$

Combining (9) and (10), we have

$$\begin{aligned}
w_{n+3}(T_{n+3}) &= q^6 \begin{bmatrix} n \\ 3 \end{bmatrix}_q + q^{n+6}\begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q + (1+q^2[n-1]_q)[n+3]_q \\
&= \begin{bmatrix} n+3 \\ 3 \end{bmatrix}_q \text{ from Proposition 1.}
\end{aligned}$$

Hence we have $w_n(T_n) = \begin{bmatrix} n \\ 3 \end{bmatrix}_q$ for $n \ge 3$. □

Theorem 2. *Orbit polynomials $O_n^{n,3}(q)$ is equal to the sum of weights of representatives of all orbits of size n.*

Proof. Assume first $\gcd(n,3)=1$. Then there are only s orbits of size n under C_n, where $s=\binom{n}{3}/n$. Let $O_1,O_2,\ldots,O_s$ be all orbits of size n under C_n. Then from the proof of Theorem 1 we know that

$$w_n(T_n)=r_n(q)[n]_q, \tag{11}$$

where $r_n(q)$ is the sum of weights of representatives of all orbits of size n.

Assume now $\gcd(n,3)\neq 1$. Then there are s orbits $O_1,O_2,\ldots,O_s$ of size n with where $s=(\binom{n}{3}-\frac{n}{3})/n$, and there is only one orbit

$$O_0=\{x_{01},x_{02},\ldots,x_{0(n/3)}\}$$

of size $n/3$. Hence

$$\begin{aligned} w_n(T_n) &= \sum_{x\in\binom{[n]}{3}} w_n(x)=\sum_{x\in O_0} w_n(x)+\sum_{i=1}^{s}\sum_{x\in O_i} w_n(x)\\ &=(1+q^3+\cdots+q^{n-3})+\sum_{i=1}^{s} w_n(x_{i1})[n]_q\\ &=\begin{bmatrix} n\\ 3\end{bmatrix}_{q^3}+r_n(q)[n]_q, \end{aligned} \tag{12}$$

where $r_n(q)$ is the sum of weights of representatives of all orbits of size n.

From (3), we have

$$\begin{bmatrix} n\\ 3\end{bmatrix}_q=\begin{cases}[n]_q\,O_n^{n,3}(q) & \text{if } \gcd(n,3)=1\\ \left[\frac{n}{3}\right]_{q^3} O_{\frac{n}{3}}^{n,3}(q)+[n]_q\,O_n^{n,3}(q) & \text{if } \gcd(n,3)\neq 1.\end{cases} \tag{13}$$

Note that $O_{\frac{n}{3}}^{n,3}(q)=1$. Comparing (11) and (12) with (13), we have

$$O_n^{n,3}(q)=r_n(q).$$

□

Corollary 1. *Let $d\mid n$. Then orbit polynomials $O_d^{n,3}(q)$ have non-negative coefficients.*

Proof. Since $O_{n/t}^{n,k}(q)=O_{n/t}^{n/t,k/t}(q^t)$, it is sufficient to prove Corollary 1 for $d=n$. Let $r_n(q)$ be the sum of weights of representatives of all orbits of size n. Then $O_n^{n,3}(q)=r_n(q)$ by Theorem 2 and $r_n(q)$ clearly has non-negative coefficients from the definition. □

Acknowledgements

The author thanks Professor Bruce Sagan for encouraging discussions. The author also thanks the referee for his useful comments and suggestions.

References

1. G. Andrews, *The Friedman-Joichi-Stanton monotonicity conjecture at primes*, To appear, Amer.Math.Soc. DIMACS book series.
2. F.Chapoton, http://www.lacim.uqam.ca/chapoton/arbres.html.
3. K. Drudge, *On the orbits of Singer groups and their subgroups*, Elec. J. Comb. **9** (2002), R15.
4. M. Haiman, *Conjectures on the quotient ring by diagonal invariants*, J. Alg. Comb. **3** (1994), 17–76.
5. V. Reiner, D. Stanton and D. White, *The Cyclic Sieving Phenomenon*, J. Combin. Theory Ser. A, **108(1)** (2004), 17–50.
6. B. Sagan, personal communication.
7. D. Stanton and D. White, *Sieved q-Binomial Coefficients*, Preprint.
8. J.R. Stembridge, *Some hidden relations involving the ten symmetry classes of plane partitions*, J. Combin. Theory Ser A **68** (1994), 372–409.

Jaejin Lee received M.Sc. from Seoul National University and Ph.D at University of Minnesota. Since 1991 he has been at Hallym University. His research interests include enumerative combinatorics.

Department of Mathematics, Hallym University, Chunchon 200-702, Korea.
e-mail: `jjlee@hallym.ac.kr`