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# TWO-LAYER MUTI-PARAMETERIZED SCHWARZ ALTERNATING METHOD FOR TWO-DIMENSIONAL PROBLEMS†

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Abstract. The convergence rate of a numerical procedure based on Schwarz Alternating Method(SAM) for solving elliptic boundary value problems depends on the selection of the interface conditions applied on the interior boundaries of the overlapping subdomains. It has been observed that the mixed interface condition, controlled by a parameter, can optimize SAM's convergence rate. In [8], one introduced the *two-layer multi-parameterized SAM* and determined the optimal values of the multi-parameters to produce the best convergence rate for *one-dimensional* elliptic boundary value problems. In this paper, we present a method which utilizes the *one-dimensional* result to get the optimal convergence rate for the *two-dimensional* problem.

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*Key words and phrases* : elliptic partial differential equations, Schwarz alternating method, Jacobi iterative methods.

## 1. Introduction

Schwarz-type alternating methods have become some of the most important approaches in domain decomposition techniques for solution of the boundary value problems (BVP's). These methods are based on a decomposition of the BVP domain into overlapping subdomains. The original BVP is reduced to a set of *smaller* BVP's on a number of subdomains with appropriate *interface conditions* on the interior boundaries of the overlapping areas, whose solutions are coupled through some iterative scheme to produce an approximation of the solution of the original BVP. It is known [1], [6] that under certain conditions the sequence of the solutions of the subproblems converges to the solution of the original problem.

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One of the objectives of this research is to study a class of Schwarz alternating methods (SAM's) whose interface conditions are parameterized and estimate the values of the parameters involved that speed up the convergence of these methods for a class of BVP's. In the context of elliptic BVP's the most commonly used *interface conditions* are of Dirichlet type. For this class of numerical SAM, there are several studies about the convergence, which include [11], [13], [16], [17], [14], [2], [12], [19]. The effect of parameterized mixed interface conditions has been considered by a number of researchers [3], [15], [5], [20]. Among them, Tang proposed a generalized Schwarz splitting [20]. The main part of his approach to the solution of a BVP is to use the mixed boundary condition, known as *Robin condition*,

$$B_i(u) = \omega_i u + (1-\omega_i)\frac{\partial u}{\partial n} \tag{1}$$

on the artificial boundaries. They adopted one same parameter $\omega$ for each $i$-th boundary. In [5], a multi-parameter SAM is formulated in which the mixed boundary conditions are controlled by a distinct parameter $\omega_i$ for the $i$-th overlapping area. Fourier analysis is applied to determine the values of $\omega_i$ parameters that make the convergence factor of SAM be zero.

In [7], one formulated a multi-parameter SAM at the matrix level where the parameters $\alpha_i$ are used to impose mixed interface conditions. The relation between the parameters $\alpha_i$ and $\omega_i$ is given by

$$\alpha_i = \frac{1-\omega_i}{1-\omega_i+\omega_i h} \tag{2}$$

(Refer to [7]), where $h$ is the grid size. One determined analytically the optimal values of $\alpha_i$'s for one-dimensional(1-dim) boundary value problems, which minimize the spectral radius of the block Jacobi iteration matrix associated with the SAM matrix.

In [18], they proposed the over-determined interface condition for 1-dim problem which adopted two-layer interface with one global parameter $\alpha$. In [8], two-layer multi-parameterized SAM was presented where the multi-parameter $\alpha_i$'s were used. In [9], one introduced a new scheme of the double-indexed multi-parameterized SAM for two-dimensional(2-dim) problems, where one used double-indexed multi-parameter $\alpha_{j,i}$ for each $j$-th grid point of the $i$-th interfaces of the subdomains to get the best convergence for 2-dim BVP's.

In this paper, we applied the double-indexed multi-parameter $\alpha_{j,i}$ scheme for the two-layer multi-parameterized SAM for 2-dim BVP's.

In the following section, we summarize the result of the two-layer multi-parameterized SAM on 1-dim problem, which has been presented in [8]. In section 3, we formulate the two-layer multi-parameterized SAM on 2-dim problem where we impose distinct parameters on each grid point on the interfaces of the subdomains. We show that the 2-dim case can be reduced to the 1-dim ones and obtain the optimal values of the two-layer multi-parameters which minimize the spectral radius of the block Jacobi iteration matrix associated with the SAM matrix of 2-dim problem.

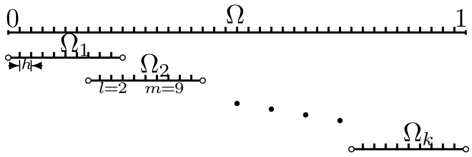


FIGURE 1. An example of the $k$-way decomposition of the domain of 1-dim boundary value problem (3).

## 2. Two-Layer Multi-Parameterized SAM for 1-dim problem

We consider the two-point boundary value problem:

$$\begin{aligned} -u''(t) + q\,u(t) &= f(t),\ t \in (0,1) \\ u(0) = a_0,&\ u(1) = a_1, \end{aligned} \tag{3}$$

with $q \geq 0$ being a constant. We will formulate a numerical instance of SAM based on a $k$-way decomposition ( i.e. the number of subdomains is $k$ ) of the problem domain. An example of $k$-way decomposition is depicted in Figure 1.

Let $T_j(x, y, z)$ be a $j \times j$ tridiagonal matrix such that

$$T_j(x,y,z) = \begin{bmatrix} x & \text{-1} & 0 & \cdots & 0 \\ \text{-1} & y & \text{-1} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \text{-1} & y & \text{-1} \\ 0 & \cdots & 0 & \text{-1} & z \end{bmatrix} \tag{4}$$

and let

$$T_j(x) \equiv T_j(x, x, x). \tag{5}$$

If we discretize the problem (3) by a second order central divided difference discretization scheme with a uniform grid of mesh size $h = \frac{1}{n+1}$, we obtain a linear system

$$A\,x = f \tag{6}$$

where $A = T_n(\beta)$ with $\beta = 2 + qh^2$.

If we consider 3-way ($k = 3$) decomposition, then $Ax = f$ has three overlapping diagonal blocks as follows.

$$\begin{bmatrix} T_{m\text{-}l} & \text{-}F & 0 & 0 & 0 \\ \text{-}E & T_l & \text{-}F & 0 & 0 \\ 0 & \text{-}E & T_{m\text{-}2l} & \text{-}F & 0 \\ 0 & 0 & \text{-}E & T_l & \text{-}F \\ 0 & 0 & 0 & \text{-}E & T_{m\text{-}l} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} \tag{7}$$

where $T_j = T_j(\beta, \beta, \beta)$ in (4) and $m$ and $l$ are the numbers of nodes in each subdomain and the overlapping regions, respectively, such that $l < \frac{m-1}{2}$. In (7), the matrix $E$ have zero elements everywhere except for a 1 at the rightmost top position and the matrix $F$ have zero elements everywhere except for a 1 at the leftmost bottom position. So the matrices $E$ and $F$ have compatible sizes with the following forms.

$$E = \begin{bmatrix} 0 \cdots 0 & 1 \\ 0 \cdots 0 & 0 \\ \vdots \quad \vdots & \vdots \\ 0 \cdots 0 & 0 \end{bmatrix}, \; F = \begin{bmatrix} 0 & 0 \cdots 0 \\ \vdots & \vdots \quad \vdots \\ 0 & 0 \cdots 0 \\ 1 & 0 \cdots 0 \end{bmatrix} \tag{8}$$

The numerical version of SAM [19] for the problem (3) is equivalent to a *block Gauss-Seidel iteration procedure* for a new linear system, called *Schwarz Enhanced Matrix Equation* ,

$$\tilde{A}\tilde{x} = \tilde{f} \tag{9}$$

where

$$\tilde{A} = \tilde{A}(\beta) = \begin{bmatrix} T_{m\text{-}l} & \text{-}F & 0 & 0 & 0 & 0 & 0 \\ \text{-}E & T_l & 0 & \text{-}F & 0 & 0 & 0 \\ \text{-}E & 0 & T_l & \text{-}F & 0 & 0 & 0 \\ 0 & 0 & \text{-}E & T_{m\text{-}2l} & \text{-}F & 0 & 0 \\ 0 & 0 & 0 & \text{-}E & T_l & 0 & \text{-}F \\ 0 & 0 & 0 & \text{-}E & 0 & T_l & \text{-}F \\ 0 & 0 & 0 & 0 & 0 & \text{-}E & T_{m\text{-}l} \end{bmatrix}, \; \tilde{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_2' \\ x_3 \\ x_4 \\ x_4' \\ x_5 \end{bmatrix}, \; \tilde{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_2 \\ f_3 \\ f_4 \\ f_4 \\ f_5 \end{bmatrix}. \tag{10}$$

$\tilde{A}(\beta)$ means that $\tilde{A}$ is a function of $\beta$. Note that the solution $x$ of (6) is obtained from the solution $\tilde{x}$ of (9), vice versa. In [20], it is shown that a good choice of the splitting of $T_l$'s can significantly improve the convergence of SAM. Applying for some splittings of $T_l$'s into $\tilde{A}$ in (10), we have a new equation

$$A'\tilde{x} = \tilde{f} \tag{11}$$

with

$$A' = \begin{bmatrix} T_{m\text{-}l} & \text{-}F & 0 & 0 & 0 & 0 & 0 \\ \text{-}E & B_1 & C_1 & \text{-}F & 0 & 0 & 0 \\ \text{-}E & C_1' & B_1' & \text{-}F & 0 & 0 & 0 \\ 0 & 0 & \text{-}E & T_{m\text{-}2l} & \text{-}F & 0 & 0 \\ 0 & 0 & 0 & \text{-}E & B_2 & C_2 & \text{-}F \\ 0 & 0 & 0 & \text{-}E & C_2' & B_2' & \text{-}F \\ 0 & 0 & 0 & 0 & 0 & \text{-}E & T_{m\text{-}l} \end{bmatrix} \tag{12}$$

where $B_i, C_i', \; i = 1,2$ are some matrices such that $(B_i - C_i')$ is non-singular and

$$T_l = B_i + C_i = B_i' + C_i', \; i = 1,2. \tag{13}$$

Note that two linear system (9) and (11) are equivalent in the sense that they have the same solutions. If $C_i'$ and $C_i$ are chosen such that they are the $l \times l$ matrices with all zero entries except for an $\alpha_i$ in the positions $(1,1)$ and $(l,l)$, respectively, as follows,

$$C_i' = \begin{bmatrix} 1\text{-}\alpha_i & 0 \cdots 0 \\ 0 & 0 \cdots 0 \\ \vdots & \vdots \quad \vdots \\ 0 & 0 \cdots 0 \end{bmatrix}, \quad C_i = \begin{bmatrix} 0 \cdots 0 & 0 \\ \vdots \quad \vdots & \vdots \\ 0 \cdots 0 & 0 \\ 0 \cdots 0 & 1\text{-}\alpha_i \end{bmatrix}, \tag{14}$$

the resulting matrix $A'$ is given as follows

$$A' = A'(\beta, \alpha_1, \alpha_2) = \begin{bmatrix} T_m(\beta,\beta,\beta_1') & \text{-}F_1{}' & 0 \\ \text{-}E_1{}' & T_m(\beta_1',\beta,\beta_2') & \text{-}F_2{}' \\ 0 & \text{-}E_2{}' & T_m(\beta_2',\beta,\beta) \end{bmatrix} \tag{15}$$

where $T_m(x, y, z)$'s are $m \times m$ matrices defined in (4) and $\beta_i' = \beta$-(1-$\alpha_i$) and $E_i{}'$ is the $m \times m$ matrix with zero elements everywhere except that

$$\begin{aligned} (1, m\text{-}l\ \ )\text{-th entry} &= \ 1 \\ (1, m\text{-}l\text{+}1)\text{-th entry} &= \text{-}(1\text{-}\alpha_i) \end{aligned}$$

and $F_i{}'$ is the $m \times m$ matrices with zero elements everywhere except that

$$\begin{aligned} (m, l\ \ )\text{-th entry} &= \text{-}(1\text{-}\alpha_i) \\ (m, l\text{+}1)\text{-th entry} &= \ 1. \end{aligned}$$

So far, $A'$ is corresponding to the one-layer multi-parameterized SAM. Now we add one more layer to get two-layer multi-parameterized SAM (Refer to the over-determined SAM in [18]) as follows.

$$A'' = A''(\beta, \alpha_1, \alpha_2) = \begin{bmatrix} T_m(\beta,\beta,\beta_1'') & \text{-}F_1{}'' & 0 \\ \text{-}E_1{}'' & T_m(\beta_1'',\beta,\beta_2'') & \text{-}F_2{}'' \\ 0 & \text{-}E_2{}'' & T_m(\beta_2'',\beta,\beta) \end{bmatrix} \tag{16}$$

where $\beta_i'' = \beta$-(1-$\alpha_i$)-$\frac{\alpha_i}{\beta}$ and the matrix $E_i''$ is the $m \times m$ matrix with zero elements everywhere except that

$$\begin{aligned} (1, m\text{-}l\text{-}1)\text{-th entry} &= \ \tfrac{\alpha_i}{\beta} \\ (1, m\text{-}l\ \ )\text{-th entry} &= \ (1\text{-}\alpha_i) \\ (1, m\text{-}l\text{+}1)\text{-th entry} &= \text{-}(1\text{-}\alpha_i) \end{aligned}$$

and the matrix $F_i''$ is the $m \times m$ matrix with zero elements everywhere except that

$$\begin{aligned} (m, l\ \ )\text{-th entry} &= \text{-}(1\text{-}\alpha_i) \\ (m, l\text{+}1)\text{-th entry} &= \ (1\text{-}\alpha_i) \\ (m, l\text{+}2)\text{-th entry} &= \ \tfrac{\alpha_i}{\beta}. \end{aligned}$$

If the number of subdomains $k$ is more than 3, the matrix $A''$ is a block $k \times k$ matrix of the form

$$A'' = A''(\beta, \mathbf{a}) = \begin{bmatrix} G_1 & \text{-}F_1{}'' & 0 & 0 & \cdots & 0 \\ \text{-}E_1{}'' & G_2 & \text{-}F_2{}'' & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \text{-}E''_{k\text{-}2} & G_{k\text{-}1} & \text{-}F''_{k\text{-}1} \\ 0 & \cdots & 0 & 0 & \text{-}E''_{k\text{-}1} & G_k \end{bmatrix} \tag{17}$$

where $\mathbf{a} = (\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_k)$ with $\alpha_0 = \alpha_k = \frac{\beta}{\beta-1}$ and $G_i$'s are defined as

$$G_i = T_m(\beta_{i\text{-}1}, \beta, \beta_i), \tag{18}$$

with $\beta_i = \beta\text{-}(1\text{-}\alpha_i)\text{-}\frac{\alpha_i}{\beta}$ for $i = 1, 2, \cdots, k$. We call the matrix $A''$ as *Two-layer Multi-Parameterized Enhanced Matrix.* If we define

$$M = M(\beta, \mathbf{a}) = \begin{bmatrix} G_1 & 0 & \cdots & 0 \\ 0 & G_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & G_k \end{bmatrix}$$
$$N = N(\beta, \mathbf{a}) = \begin{bmatrix} 0 & F_1'' & 0 & 0 & \cdots & 0 \\ E_1'' & 0 & F_2'' & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & E_{k\text{-}2}'' & 0 & F_{k\text{-}1}'' \\ 0 & \cdots & 0 & 0 & E_{k\text{-}1}'' & 0 \end{bmatrix} \tag{19}$$

with $\mathbf{a} = (\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_k)$, then we can write the two-layer multi-parameterized enhanced matrix $A''$ as

$$A'' = M - N \tag{20}$$

which is called *a Two-lyaer Multi-Parameterized Schwarz Splitting (2MPSS)* .

The convergence behavior of 2MPSS depends on the spectral radius of the following block Jacobi matrix

$$J = M^{-1}N = \begin{bmatrix} 0 & G_1^{-1}F_1' & 0 & 0 & \cdots & 0 \\ G_2^{-1}E_1'' & 0 & G_2^{-1}F_2'' & 0 & \cdots & 0 \\ 0 & G_3^{-1}E_2'' & 0 & G_3^{-1}F_3'' & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & G_{k\text{-}1}^{-1}E_{k\text{-}2}'' & 0 & G_{k\text{-}1}^{-1}F_{k\text{-}1}'' \\ 0 & \cdots & 0 & 0 & G_k^{-1}E_{k\text{-}1}'' & 0 \end{bmatrix}. \tag{21}$$

Note that $J$ is a function of the parameters $\alpha_i$'s, which correspond to the parameters $\omega_i$'s in the mixed interface condition (1), respectively. The convergence rate of SAM can be optimized by controlling these parameters $\alpha_i$'s. In [8], one determined the optimal values of the multi-parameter $\alpha_i$'s that make the spectral radius of the block Jacobi matrix $J$ in (21) to be zero. The result of [8] is presented in the following theorem.

**Theorem 1.** *Let* $\theta = \cosh^{-1}(\frac{\beta}{2})$ *with* $\beta = 2 + qh^2$ *and let* $p \in \{1, 2, \cdots, k-1\}$ *and let*

$$\Theta(x) = \begin{cases} \sinh(x\,\theta)\ , & \text{for } \theta > 0 \\ x\ , & \text{for } \theta = 0 \end{cases}$$
$$F_1(\alpha) = \Theta(m\text{-}l - 1) - (\alpha/\beta + 1\text{-}\alpha)\,\Theta(m\text{-}l\text{-}2)$$
$$F_2(\alpha) = \Theta(m\text{-}l) - (\alpha/\beta + 1\text{-}\alpha)\,\Theta(m\text{-}l\text{-}1)$$
$$F_3(\alpha) = \Theta(m\text{-}l + 1) - (\alpha/\beta + 1\text{-}\alpha)\,\Theta(m\text{-}l).$$

*If the values $\alpha_i,\ i = 0, 1, \cdots, k,$ be given by*

$$\alpha_0 = \tfrac{\beta}{\beta-1}$$

$$\alpha_i = \frac{-F_2(\alpha_{i\text{-}1}) + F_3(\alpha_{i\text{-}1})}{F_1(\alpha_{i\text{-}1})/\beta - F_2(\alpha_{i\text{-}1}) + F_3(\alpha_{i\text{-}1})}\ ,\ i = 1, 2, \cdots, p$$

$$\alpha_i = \frac{-F_2(\alpha_{i+1}) + F_3(\alpha_{i+1})}{F_1(\alpha_{i+1})/\beta - F_2(\alpha_{i+1}) + F_3(\alpha_{i+1})},\ i = p+1, \cdots, k-1$$

$$\alpha_k = \tfrac{\beta}{\beta-1}$$

*then the block Jacobi matrix $J$ in (21) is zero, too.*

## 3. Two-Layer Multi-Parameterized SAM for 2-dim Problem

Consider the 2-dim boundary value problem

$$\begin{aligned} -\nabla^2 u(x, y) + q\, u(x, y) = f(x, y),\ \ (x, y) \in \Omega, \\ u(x, y) = g(x, y),\ \ (x, y) \in \Gamma \end{aligned} \tag{22}$$

where $\Gamma$ is the boundary of $\Omega \equiv (0, 1) \times (0, 1)$ and $q \geq 0$ is a constant. We formulate a SAM based on a $k$-way splitting of the domain $\Omega$, i.e., we decompose our domain into $k$ overlapping subdomains $\Omega_i$ along the $x$-axis and make a strip-type decomposition of the rectangular domain $\Omega$ (for instance, see Figure 2). Next we apply the interface conditions on the two interior boundaries between subdomains $\Omega_i$ and $\Omega_{i+1}$. Let $\ell$ be the length of the overlap in $x$-direction and $\eta$ be the length of each subdomain in the same direction. Figure 2 depicts such a 3-way splitting of the unit square $\Omega$.

To begin our analysis we use a 5-point finite difference discretization scheme with uniform grid of mesh size $h = \frac{1}{n+1}$ on both $x$- and $y$-axes and discretize the BVP in (22) to obtain a linear system of the form

$$Bx = f. \tag{23}$$

The natural ordering of the nodes is adopted starting from the origin and going in the $y$-direction first so that the resulting matrix $A$ can be partitioned into block matrices corresponding to the subdomains, respectively. Using tensor product notation $\otimes$ (See [4], and [10] in which tensor products in connection with BVP's were introduced.), the matrix $B$ in (23) can be written as

$$B = T_n(\beta) \otimes I_n + I_n \otimes T_n(2) \tag{24}$$

where $T_j(x)$ is defined in (5) and $\beta = 2 + qh^2$.

Define $l+1 = \frac{\ell}{h}$ and $m+1 = \frac{\eta}{h}$ such that $n = mk - l(k-1)$ and $l < \frac{m-1}{2}$. The numerical version of SAM for the problem (22) is equivalent to a *block Gauss-Seidel iteration procedure* for a new linear system, called the *Schwarz Enhanced Matrix Equation* ,

$$\tilde{B}\tilde{x} = \tilde{f} \tag{25}$$

with

$$\tilde{B} = \tilde{A}(\beta) \otimes I_n + I_{km} \otimes T_n(2) \tag{26}$$

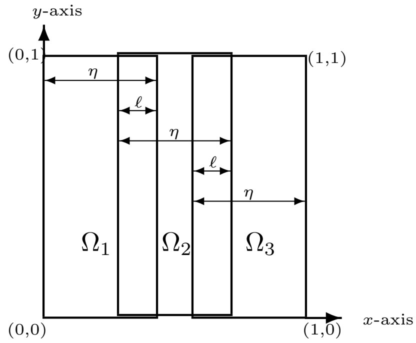


FIGURE 2. A 3-way splitting of the unit square $\Omega$.

where $I_{km}$ is the $km \times km$ identity matrix and $\tilde{A}(\beta)$ is the $k \times k$ block matrix as that defined in (10), which is the case of $k = 3$. Note that each diagonal block in $\tilde{A}(\beta)$ is $m \times m$ matrix. .

Let $X_n$ be the $n \times n$ orthogonal matrix whose columns are the eigenvectors of the matrix $T_n(2)$. Since the eigenvalues of the matrix $T_n(2)$ are known to be $\gamma_i = 2 + 2\cos(\frac{i\pi}{n+1}),\ i = 1, 2, \cdots, n$, we can write

$$X_n^T \, T_n(2) \, X_n = D_n \equiv diag(\gamma_1, \gamma_2, \cdots, \gamma_n). \tag{27}$$

Let $X = I_{km} \otimes X_n$, then its inverse is given by $X^{-1} = I_{km} \otimes X_n^T$, so we have

$$\begin{aligned} X^{-1}\tilde{B}X &= (I_{km} \otimes X_n^T)(\tilde{A}(\beta) \otimes I_n)(I_{km} \otimes X_n) \\ &\quad + (I_{km} \otimes X_n^T)(I_{km} \otimes T_n(2))(I_{km} \otimes X_n) \\ &= (I_{km}\, \tilde{A}(\beta)\, I_{km}) \otimes (X_n^T \, I_n \, X_n) + I_{km} \otimes (X_n^T \, T_n(2) \, X_n) \\ &= \tilde{A}(\beta) \otimes I_n + I_{km} \otimes D_n. \end{aligned}$$

If $P$ is the permutation matrix that maps

$$\text{row } (i-1)n + j \quad \text{into} \quad \text{row } (j-1)km + i$$

for $i = 1, 2, \cdots, km$ and for $j = 1, 2, \cdots, n$, then we have

$$\begin{aligned} \hat{B} \equiv P^{-1}X^{-1}\tilde{B}XP &= P^{-1}(\tilde{A}(\beta) \otimes I_n)P + P^{-1}(I_{km} \otimes D_n)P \\ &= I_n \otimes \tilde{A}(\beta) + D_n \otimes I_{km} \\ &= diag(\tilde{A}(\beta + \gamma_1), \tilde{A}(\beta + \gamma_2), \cdots, \tilde{A}(\beta + \gamma_n)) \\ &= diag(\tilde{A}(\zeta_1), \tilde{A}(\zeta_2), \cdots, \tilde{A}(\zeta_n)) \end{aligned} \tag{28}$$

where

$$\zeta_j = \beta + \gamma_j, \ j = 1, 2, \cdots, n. \tag{29}$$

Note that the solution $\tilde{x}$ of linear system (25) is obtained by $\tilde{x} = XP\,\hat{x}$ if we solve the linear system

$$\hat{B}\hat{x} = \hat{f} \tag{30}$$

where $\hat{f} = P^{-1}X^{-1}\tilde{f}$ with $\tilde{f}$ in (25).

From (28) and (30), we know that the 2-dim problem (25) is reduced to $n$ number of 1-dim problems

$$\tilde{A}(\zeta_j) = \hat{f}_j,\ j = 1, 2, \cdots, n,$$

where $\hat{f}_j$ is the corresponding sub-vector of $\hat{f}$. Based on (28), the *Two-layer Multi-Parameterized Schwarz Enhanced Matrix* for $\hat{B}$ in (28) is defined as

$$B'' = diag(\ A''(\zeta_1, \mathbf{a}), A''(\zeta_2, \mathbf{a}), \cdots, A''(\zeta_n, \mathbf{a})\ ) \tag{31}$$

where $A''(x, \mathbf{a})$ is defined in (17). If we let

$$\begin{aligned} M &= diag(\ M(\zeta_1, \mathbf{a}), M(\zeta_2, \mathbf{a}), \cdots, M(\zeta_n, \mathbf{a})\ ) \\ N &= diag(\ N(\zeta_1, \mathbf{a}),\ N(\zeta_2, \mathbf{a}),\ \cdots,\ N(\zeta_n, \mathbf{a})\ ) \end{aligned} \tag{32}$$

where $M(x, \mathbf{a})$ and $N(x, \mathbf{a})$ are defined in (19), then we can write the two-layer multi-parameterized enhanced matrix $B''$ in (31) as

$$B'' = M - N \tag{33}$$

which is called a *Two-layer Multi-Parameterized Schwarz Splitting (2MPSS)* . The convergence behavior of 2MPSS depends on the spectral radius of the following block Jacobi matrix

$$J = M^{-1}N = diag(\ L_1(\mathbf{a}), L_2(\mathbf{a}), \cdots, L_n(\mathbf{a})\ ) \tag{34}$$

where

$$L_j(\mathbf{a}) = M(\zeta_j, \mathbf{a})^{-1} N(\zeta_j, \mathbf{a}),\ j = 1, 2, \cdots, n.$$

In [8], one failed to determine a parameter vector $\mathbf{a}$ such that the spectral radius of the block Jacobi matrix $J$ in (34) is zero because it is not possible to find such a parameter vector $\mathbf{a}$ that makes all of the spectral radii of the diagonal blocks $L_j(\mathbf{a})$'s in (34) zero simultaneously.

Now we adopt distinct parameter vector $\mathbf{a}_j$ for each diagonal block as follows

$$J = M^{-1}N = diag(\ L_1(\mathbf{a}_1), L_2(\mathbf{a}_2), \cdots, L_n(\mathbf{a}_n)\ ) \tag{35}$$

where

$$\mathbf{a}_j = (\alpha_{j,0}, \alpha_{j,1}, \alpha_{j,2}, \cdots, \alpha_{j,k}),\ j = 1, 2, \cdots, n.$$

Note these 2-dim indices $(j, i)$ in multi-parameter $\alpha_{j,i}$ are related to the idea that one adopts variable parameter $\omega_i(x, y)$ instead of constant parameter $\omega_i$ in (1) along the $i$-th interface, i.e., we have

$$B_i(u) = \omega_i(x, y)u + (1 - \omega_i(x, y))\frac{\partial u}{\partial n} \tag{36}$$

as the mixed interface condition on the $i$-th interface boundary.

Now, using these double-indexed multi-parameter $\alpha_{j,i}$'s, we have the following theorem for double-indexed multi-parameterized Schwarz splitting $B'' = M - N$ in (33).

**Theorem 2.** *For $j = 1, 2, \cdots, n$, let $\theta_j = \cosh^{-1}(\frac{\zeta_j}{2})$ with $\zeta_j$ in (29) and let $p \in \{1, 2, \cdots, k-1\}$ and let*

$$\Theta_j(x) = \begin{cases} \sinh(x\,\theta_j) & , \text{ for } \theta_j > 0 \\ x & , \text{ for } \theta_j = 0 \end{cases}$$
$$F_1(j,\alpha) = \Theta_j(m\text{-}l-1) - (\alpha/\zeta_j + 1\text{-}\alpha)\,\Theta_j(m\text{-}l\text{-}2)$$
$$F_2(j,\alpha) = \Theta_j(m\text{-}l\quad) - (\alpha/\zeta_j + 1\text{-}\alpha)\,\Theta_j(m\text{-}l\text{-}1)$$
$$F_3(j,\alpha) = \Theta_j(m\text{-}l+1) - (\alpha/\zeta_j + 1\text{-}\alpha)\,\Theta_j(m\text{-}l\quad).$$

*If the values $\alpha_{j,i},\ j = 1, 2, \cdots, n,\ i = 0, 1, \cdots, k$ are given by*

$$\alpha_{j,0} = \frac{\zeta_j}{\zeta_j - 1}$$
$$\alpha_{j,i} = \frac{-F_2(j,\alpha_{j,i\text{-}1}) + F_3(j,\alpha_{j,i\text{-}1})}{F_1(j,\alpha_{j,i\text{-}1})/\zeta_j - F_2(j,\alpha_{j,i\text{-}1}) + F_3(j,\alpha_{j,i\text{-}1})}\ ,\ i = 1, 2, \cdots, p$$
$$\alpha_{j,i} = \frac{-F_2(j,\alpha_{j,i+1}) + F_3(j,\alpha_{j,i+1})}{F_1(j,\alpha_{j,i+1})/\zeta_j - F_2(j,\alpha_{j,i+1}) + F_3(j,\alpha_{j,i+1})}\ ,\ i = p+1, \cdots, k-1$$
$$\alpha_{j,k} = \frac{\zeta_j}{\zeta_j - 1}$$

*then the spectral radius of the block Jacobi matrix $J$ in (35) is zero.*

## 4. Numerical Experiments

In this section, we present a numerical experiment to prove the result of the previous section. we will compare the results of *Two-layer Multi-Parameterized SAM (2MPSAM)* with those of *One-layer Multi-Parameterized SAM (1MP-SAM)* [9]. Consider the following model problem

$$\begin{aligned} -\nabla^2 u(x,y) + q\,u(x,y) &= f(x,y), \quad (x,y) \in \Omega = (0,1)\times(0,1), \\ u(x,y) &= g(x,y), \quad (x,y) \in \Gamma, \end{aligned} \tag{37}$$

where $q \geq 0$ and $\Gamma$ is the boundary of $\Omega$, with solution

$$u(x,y) = \sin(2\pi x)\cos(2\pi y).$$

In all the experiments, the vector with all its components 0.0 was used as initial guess of the solution vector. The *relative residual* $r_p$ is computed as the ratio of $\ell_2$-norms of the residuals of the corresponding system of equations after $p$ iterations, i.e.,

$$r_p = \frac{||Bx^{(p)} - f||_2}{||Bx^{(0)} - f||_2}.$$

The convergence rate is very sensitive to the computed optimal value of parameter $\alpha_{j,i}$'s and the symmetric choice of them (i.e. Take $p = [k/2]$ in Theorem (2)) reduces the error propagation when we compute the optimal value of parameters $\alpha_{j,i}$'s.

Table 1 shows the relative residuals of *1PSAM* computed and the maximum errors after $k$ iterations for $k = 3, 4$ and 8 subdomains, $m = 10$ and 20 local grids and minimum ($l = 1$) and half ($l = [m\text{-}1]/2$) overlaps. Although the spectral

Table 1. The *1MPSAM* is applied to the BVP (37).

| $m$ | $l$ | relative residual $r_k$ max(error) $k=3$ | $k=4$ | $k=8$ |
|---|---|---|---|---|
| 10 | 1 | 6.9103E-16 | 1.0715E-15 | 1.2971E-15 |
| | | 4.2362E-03 | 2.4712E-03 | 6.5235E-04 |
| 10 | 4 | 7.0662E-16 | 6.8898E-16 | 1.3027E-15 |
| | | 6.7225E-03 | 4.2362E-03 | 1.2710E-03 |
| 20 | 1 | 1.4107E-15 | 1.6217E-15 | 2.1030E-15 |
| | | 1.0259E-03 | 5.8718E-04 | 1.5067E-04 |
| 20 | 9 | 1.1846E-15 | 1.2108E-15 | 1.5845E-15 |
| | | 1.9301E-03 | 1.2246E-03 | 3.7202E-04 |

Table 2. The *2MPSAM* is applied to the BVP (37).

| $m$ | $l$ | relative residual $r_k$ max(error) $k=3$ | $k=4$ | $k=8$ |
|---|---|---|---|---|
| 10 | 1 | 6.3429E-16 | 9.9953E-16 | 1.2436E-15 |
| | | 4.2362E-03 | 2.4712E-03 | 6.5235E-04 |
| 10 | 4 | 6.3026E-16 | 6.3753E-16 | 1.2167E-15 |
| | | 6.7225E-03 | 4.2362E-03 | 1.2710E-03 |
| 20 | 1 | 1.3874E-15 | 1.5482E-15 | 2.0754E-15 |
| | | 1.0259E-03 | 5.8718E-04 | 1.5067E-04 |
| 20 | 9 | 1.1614E-15 | 1.1497E-15 | 1.5502E-15 |
| | | 1.9301E-03 | 1.2246E-03 | 3.7202E-04 |

radius of the linear system is zero, it takes at least $k$ iterations for the boundary information to reach the whole interior points. We observe that the results of the half overlapping cases is almost same as those of the minimum overlapping cases. Table 2 shows the performance of *2MPSAM* under the same conditions. The results of *2PSAM* are a little better than those of *1PSAM* although they are not quite different presumably because the spectral radii of both cases are zero.

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